On the Christoffel Symbols of the Non-holonomic fromes in Vn

Jin-oh Hyun · Hong-gi Kim

RIEMANN공간 Vn에서의 비-호로노미 구조에 대한 CHRISTOFFEL SYMBOL에 관한 소고

玄 進 五 · 金 弘 基

Summary

The purpose of the present paper is primarily to study the relationships between holonomic and nonholonomic components of the christoffel symbols defined by a general symmetric covariant tensor $a_{\lambda\mu}$. Secondarily, we derive a useful representation of the christoffel symbols, formed with respect to the metric tensor $h_{\lambda\mu}$ of V_n, in terms of orthogonal ennuple.

I. INTRODUCTION

Let V_n be a n-dimensional Riemannian space referred to a real coordinate system x^ν and defined by a fundamental metric tensor $h_{\lambda\mu}$, whose determinant

(1.1) $$h \overset{\text{def}}{=} Det(h_{\lambda\mu}) \neq 0.$$

According to (1.1) there is a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ defined by

(1.2) $$h_{\lambda\mu}\, h^{\lambda\nu} \overset{\text{def}}{=} \delta_\mu^\nu.$$

The tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and lowering indices of tensor quantities in V_n in the usual manner.

Let $\underset{i}{e}^\nu, (i=1,\dots,n)$, be a set of n linearly independent vectors. Then there is a unique reciprocal set of n linearly independent covariant vectors $\overset{i}{e}_\lambda$ $(i=1,\dots,n)$, satisfying

(1.3)a $$\underset{i}{e}^\nu\, \overset{i}{e}_\lambda = \delta_\mu^{\nu}\ {}^{(**)}$$

With these vectors $\underset{i}{e}^\nu$ and $\overset{i}{e}_\lambda$ a nonholonomic frame of V_n may be constructed as in the following way: If $T_\mu^\nu{}^{\cdots}_{\cdots}$ are holonomic components of a tensor, then its nonholonomic components are defined by

(1.4)a $$T^{i\cdots}_{j\cdots} \overset{\text{def}}{=} T^{\nu\cdots}_{\mu\cdots}\, \overset{i}{e}_\nu\, \underset{j}{e}^\lambda \dots$$

An easy inspection of (1.3)a and (1.4)a shows

(*) Supported by the research fund of the Ministry of Education, R.O.K., 1979.

(**) Throughout the present paper, all indices take the values 1, 2, ⋯n and follow the summation convention. Greek indices are used for the holonomic compoments of a tensor, while Roman indices are used for the nonholonomic components of a tensor.

that

(1.4)b $$T^{\nu\cdots}_{\lambda\cdots} = T^{i\cdots}_{j\cdots} \underset{i}{e^{\nu}} \overset{j}{e_{\lambda}} \cdots$$

With respect to an orthogonal nonholonomic frame of V_n constructed by an orthogonal emmple e_i^{ν}, $(i=l,\ldots,n)$, it was shown in Chung, K.T & Hyun, J. O. 1976. that

(1.5)a $$h_{ij}=\partial_{ij},\ h^{ij}=\partial^{ij};$$

(1.5)b $$\underset{i}{e^{\nu}}=\overset{i}{e^{\nu}},\quad \overset{j}{e_{\lambda}}=\underset{j}{e_{\lambda}}.$$

The purpose of the present paper is, in the first, to study the relationships between holonomic and nonholonomic components of the Christoffel symbols defined by a general symmetric covariant tensor $a_{\lambda\mu}$. In the second, we derive a useful representation of Christoffel symbols, formed with respect to the metric tensor $h_{\lambda\mu}$ of V_n in terms of orthogonal ennuple.

II. NONHOLONOMIC COMPONENTS OF CHRISTOFFEL SYMBOLS IN V_n.

Consider a symmetric covariant tensor a whose determinant $a \underset{=}{\text{def}} \mathrm{Det}((a_{\lambda\mu}))$ is not zero. It is well known that the quantities $a^{\lambda\nu}$ defined by

$$a^{\lambda\nu} \underset{=}{\text{def}} \frac{\text{cofactor of } a_{\lambda\nu} \text{ in } a}{a}$$

is a symmetric contravariant tensor satisfying

(2.1) $$a_{\lambda\mu}\, a^{\lambda\nu}=\delta_{\mu}^{\nu}.$$

Theorem (2.1). The holonomic and nonholonomic components of Christoffel symbols satisfy

(2.2)a $$[jk,m]_a=[\lambda\mu,\omega]_a \underset{j}{e^{\lambda}}\underset{k}{e^{\mu}}\underset{m}{e^{\omega}}+ a_{\lambda\mu}(\partial_{\gamma}\underset{j}{e^{\lambda}})\, \underset{k}{e^{\gamma}}\underset{m}{e^{\mu}},$$

(2.2)b $$\left\{{i \atop jk}\right\}_a=\left\{{\nu \atop \lambda\mu}\right\} \overset{i}{e_{\nu}}\underset{j}{e^{\lambda}}\underset{k}{e^{\mu}}+\overset{i}{e_{\nu}}\underset{k}{e^{\mu}}\left(\partial_{\mu}\underset{k}{e^{\nu}}\right)$$

Here, $[jk,m]_a$ and $\left\{{j \atop jk}\right\}_a$ are the Christoffel symbols of the first and second kind, respectively, defined by $a_{\lambda\mu}$.

Proof. Take a coordinate system y^i for which we have at a point P of V_n

(2.3) $$\frac{\partial y^i}{\partial x^{\lambda}}=\overset{i}{e_{\lambda}},\quad \frac{\partial x^{\nu}}{\partial y^{i}}=\underset{i}{e^{\nu}}.$$

If $a_{\lambda\mu}$ and a_{ij} are holonomic and nonholonomic components of the tnesor defined above, it follows from (1.4)a that

(2.4) $$a_{jk}=a_{\lambda\mu}\underset{j}{e^{\lambda}}\underset{k}{e^{\mu}}$$

Differentiating with respect to y^m, we have

(2.5) $$\partial_m a_{jk}=(\partial_{\omega}a_{\lambda\mu})\underset{j}{e^{\lambda}}\underset{k}{e^{\mu}}\underset{m}{e^{\omega}}+a_{\lambda\mu}(\partial_m \underset{j}{e^{\lambda}})\underset{k}{e^{\mu}} + a_{\lambda\mu}\underset{j}{e^{\lambda}}(\partial_m \underset{k}{e^{\mu}}).$$

The first of the following equations is obtained from (2.5) by interchanging k and m throughout and the dummy indices ω and μ, the second by interchanging j and m and the dummy indices λ and ω:

(2.6)
$$\partial_k a_{jm}=(\partial_{\mu}a_{\lambda\omega})\underset{j}{e^{\lambda}}\underset{m}{e^{\omega}}\underset{k}{e^{\mu}}+a_{\lambda\omega}(\partial \underset{kj}{e^{\lambda}})\underset{m}{e^{\omega}} +a_{\lambda\omega}\underset{j}{e^{\lambda}}(\partial_k \underset{m}{e^{\omega}}),$$
$$\partial_{\lambda} a_{km}=(\partial^{\lambda}a_{\mu\omega})\underset{k}{e^{\mu}}\underset{m}{e^{\omega}}\underset{j}{e^{\lambda}}+a_{\mu\omega}(\partial_j \underset{m}{e^{\omega}})\underset{k}{e^{\mu}} +a_{\mu\omega}\underset{m}{e^{\omega}}(\partial_j \underset{k}{e^{\mu}}).$$

If from the sum of these two equations in(2.6) we subtract (2.5) and divide by 2, we have in consequence of (2.3) the first relation (2.2)a as in the following way:

$$[jk,m]_a=[\lambda\mu,\omega]_a\underset{j}{e^{\lambda}}\underset{k}{e^{\mu}}\underset{m}{e^{\omega}}+a_{\lambda\mu}\underset{m}{e^{\mu}}(\partial_k \underset{j}{e^{\lambda}})$$
$$=[\lambda\mu,\omega]_a\underset{j}{e^{\lambda}}\underset{k}{e^{\mu}}\underset{m}{e^{\omega}}+a_{\lambda\mu}(\partial_{\gamma}\underset{j}{e^{\lambda}})\underset{k}{e^{\gamma}}\underset{m}{e^{\mu}}.$$

The second relation (2.2)b may be obtained by multiplying

$$a^{im}=a^{\alpha\beta}\overset{i}{e}_{\alpha}\overset{m}{e}_{\beta}$$

to both sides of (2.2)a and by making use of (1.3) and (2.1), as in the following way:

$$\left\{ {i \atop jk} \right\}_{a}=a^{im}[jk,m]_{a}=a^{\alpha\beta}\overset{i}{e}_{\alpha}\overset{m}{e}_{\beta}([\lambda\mu,\omega]_{a}\underset{j}{e}^{\lambda}\underset{k}{e}^{\mu}\underset{m}{e}^{\omega}+$$

$$a_{\lambda\mu}(\partial_{\gamma}\underset{j}{e}^{\lambda})\underset{k}{e}^{\gamma}\underset{m}{e}^{\mu})$$

$$=\left\{ {\alpha \atop \lambda\mu} \right\}_{a}\overset{i}{e}_{\alpha}\underset{j}{e}^{\lambda}\underset{k}{e}^{\mu}+\overset{i}{e}_{\lambda}\underset{k}{e}^{\gamma}(\partial_{\gamma}\underset{j}{e}^{\lambda}).$$

Theorem (2.2). The nonholonomic components of the Christoffel symbols of the second kind may be expressed as

$$(2.7)\qquad \left\{ {i \atop jk} \right\}_{a}=-\underset{j}{e}^{\nu}\underset{k}{e}^{\mu}\nabla_{\mu}\overset{i}{e}_{\nu},$$

where ∇_{μ} is the symbol of the covariant derivative with respect to $\left\{ {\nu \atop \lambda\mu} \right\}_{a}$

Proof. Making use of (1.3)a, we may derive (2.7) from (2.2)b as in the following way:

$$\left\{ {i \atop jk} \right\}_{a}=\overset{i}{e}_{\nu}\underset{k}{e}^{\mu}(\partial_{\mu}\underset{j}{e}^{\nu}+\left\{ {\nu \atop \lambda\mu} \right\}_{a}\underset{j}{e}^{\lambda})=\overset{i}{e}_{\nu}\underset{k}{e}^{\mu}\nabla_{\mu}\underset{j}{e}^{\nu}$$

$$=-\underset{j}{e}^{\nu}\underset{k}{e}^{\mu}(\nabla_{\mu}\overset{i}{e}_{\nu}).$$

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〈초　록〉

RIEMANN 공간 Vn에서의 비-호로노미 구조에 대한 Christoffel Symbol에 관한 소고

현　진　오
김　홍　기

본 논문에서는 Riemann 공간 Vn에서의 일반적인 Symmetric Covariant tensor $a_{\lambda\mu}$에 의하여 정의되어진 Christoffel Symbol의 holonomic과 nonholonomic compoment 사이의 관계를 구명하고 이에 대한 효율적이고 새로운 표현방법을 연구했다.

* 제주대학 교육학부 수학과 전임강사