# A Note on the Condition of the Parallelism of the Nonholonomic Frame in Vn

이를 敎育學碩士學位 論文으로 提出함



濟州大學校敎育大學院數學敎育專攻

提出者 玄 桂 龍

指導敎授 玄 進 五

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# 玄桂龍의 碩士學位 論文을 認准함

濟州大學校教育大學院

主審 ㊞



副審 ㊞

副審 ㊞

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# 감 사 의 글

이 논문이 완성되기까지 바쁘신 가운데도 자상한 마음으로 친절하게 지도하여 주신 현진오 교수님과 제주대학교 수학과 여러교수님께 심심한 사의를 표합니다.

그리고 그동안 어려운 환경속에서도 저에게 사랑과 격려를 아끼지 않았던 아내와 부모님, 주위의 많은 분들께 또한 감사를 드립니다.

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현 계 룡

CONETNTS

# I. INTRODUCTION

Let $e_i^v$ ($i=1,2,\cdots,n$) be a set of n lineary independent vectors in n-dimensional Riemannian space $V_n$ referred to a real coordinate system $X^v$.

There is a unigue reciprocal set of n lineary independent covariant vectors $e_\lambda^i$ ($i=1,2,\cdots,n$) satisfying

(1.1) $e_i^v \; e_\lambda^i = \delta_\lambda^v, e_j^\lambda \; e_\lambda^i = \delta_j^i$ (**)

Within the vectors $e_i^v$ and $e_\lambda^i$, a nonholonomic frame of $v_n$ defined in the following way.

**Definition 1.1.** If $T_{v\cdots}^{\lambda\cdots}$ are holonomic components of a tensor, then its nonholonomic components are defined by

(1.2) $^*T_{j\cdots}^{i\cdots} \; \underline{\underline{def}} \; T_{\lambda\cdots}^{v\cdots} \; e_v^i \; e_j^\lambda$

In this paper, for our further discussion, results obtained in our previous paper will be introduced without proof.



**Theorem 1.2.** We have

(1.3) a $\quad T^v = e_i^v \; {}^*T^i$

(1.3) b $\quad T^{v\lambda} = e_i^v \; {}^*T^{ij} \; e_j^\lambda$

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(**) Throughout the present paper, indices take values $1,2,\cdots,n$ unless explicitly stated otherwise and follow the summation convention, while Roman indices with symbol * are used for the nonholonomic components of a vector or tensor and also follow the summation convension.

**Theorem 1.3.** We have the covariant derivative of the nonholonomic contravariant vector $^*a^i$, as follows

$$(1.4) \quad \nabla_k(^*a^i) = \nabla_\mu(a^\nu)\, e^i_\nu\, e^\mu_k$$

$$= \frac{d\,^*a^i}{dy^k} + {}^*a^j\ {}^*\{^{\,i}_{j\,k}\}$$

, where $^*\{^{\,i}_{j\,k}\}$ is the second kind Christoffel symbol in the nonholonomic frame.

**Theorem 1.4.** The covariant derivative of the nonholonomic covariant vector $^*a_j$ is equivalent to

$$(1.5) \quad \nabla_k(^*a_j) = \nabla_\mu(a_\lambda)\, e^\lambda_j\, e^\mu_k$$

$$= \left[\frac{da_\lambda}{dx^\mu} - a_\nu\{^{\,\nu}_{\lambda\,\mu}\}\right] e^\mu_k\ e^\lambda_j$$

$$= \frac{d\,^*a_j}{dy^k} - {}^*a_i\ {}^*\{^{\,i}_{j\,k}\}$$



## II. PARALLEL DISPLACEMENT OF A NONHOLONOMIC CONTRAVARIANT VECTORS OF CONSTANT MAGNITUDE

In this section, we will study some of the properties that a nonholonomic vector $^*\mathbf{a}$ of constant magnitude is parallel with respect to $v_n$ along the curve C.

Since the coordinates of points on the curve may be expressed in terms of the arc-length S, the condition for parallelism of **a** along C in the holonomic frame is

(2.1) $\frac{dx^{\mu}}{ds}\nabla_{\mu}(a^{v})=0$

**Definition 2.1.** The vectors a satisfying the condition(2.1) is said to undergo a parallel displacement along the curve.

**Theorem 2.2.** The condition for parallelism of nonholonomic voctor ***a** along C is

(2.2) $\frac{dx^{\mu}}{ds}\nabla_{k}({}^{*}a^{i})=0$

**proof**. Using the first class of the right hand (1.4) and (2.1)

(2.3) $\frac{dx^{\mu}}{ds}\left(\nabla_{\mu}(a^{v})e^{j}_{v}\,e^{\mu}_{k}\right)=\left[\frac{dx^{\mu}}{ds}\nabla_{\mu}(a^{v})\right]e^{j}_{v}\,e^{\mu}_{k}$
$=0$

These equations are equivalent to (2.2)

**Theorem 2.3.** We have

(2.4) $\frac{d\,{}^{*}a^{i}}{ds}+{}^{*}a^{j}\,{}^{*}\{{}^{\,i}_{j\,k}\}\frac{dy^{k}}{ds}=0$



**proof**. The condition of parallelism in the holonomic frames is

(2.5) $\frac{dx^{\mu}}{ds}(\nabla_{\mu}a^{v})=\frac{dx^{\mu}}{ds}\left[\frac{da^{v}}{dx^{\mu}}+a^{\lambda}\{{}^{\,v}_{\lambda\,\mu}\}\right]=0$

That is,

(2.6) $\frac{da^{v}}{dx^{\mu}}+a^{\lambda}\{{}^{\,v}_{\lambda\,\mu}\}=0$

By means of the second class of the right hand of (1.4) and (2.5)

(2.7) $\frac{dx^{\mu}}{ds}\nabla_{k}({}^{*}a^{i})=\frac{dx^{\mu}}{ds}\left[\frac{d\,{}^{*}a^{i}}{dy^{k}}+{}^{*}a^{j}\,{}^{*}\{{}^{\,i}_{j\,k}\}\right]$

$$= \frac{d^*a^i}{ds} \; e^\nu_k + {}^*a^j \; {}^*\{{}_j{}^i{}_k\} \frac{dy^k}{ds} \cdot e^\nu_k$$

Since $e^\nu_k \neq 0$, We have the result.

We know this concept of parallelism in the holonomic frame is due to Levi -Civita.

**Corollary 2.3.** The arc-rate of change of the holonomic and nonholonomic contravariant components $a^v$ and $^*a^i$ is given by

$$(2.8)a \qquad da^v = -a^\lambda \{{}_\lambda{}^v{}_\mu\} \, dx^\mu$$

$$(2.8)b \qquad d\, {}^*a^i = -{}^*a^j \; {}^*\{{}_j{}^i{}_k\} \, dy^k$$

**proof.** From (2.4) and (2.6), we obtain direct the result.

Let **a**,**b** be two unit vectors.

Then the cosine of their mutual inclination has the value $g_{v\lambda} a^v b^\lambda$, the derivative of this along the carve is egual to

$$(2.9) \quad \frac{dx^\mu}{ds} \nabla_\mu (g_{v\lambda} a^v b^\lambda)$$

$$= \frac{dx^\mu}{ds} \left[ \nabla_\mu (g_{v\lambda}) \; a^v \; b^\lambda + g_{v\lambda} \nabla_\mu (a^v) \, b^\lambda + g_{v\lambda} \; a^v \nabla_\mu (b^\lambda) \right]$$



by virtue of (2.1),

$$\frac{dx^\mu}{ds} \nabla_\mu (a^v) = 0 = \frac{dx^\mu}{ds} \nabla_\mu (b^\lambda)$$

Hence the eguation (2.9) are vanish.

**Theorem 2.4.** If any two nonholonomic contravariant vectors of constant magnitudes, undergo parallel displacements along a given curve, they are inclined at a constant angle in nonholonomic frame.

**proof**. Making use of (1.2), (1.3)a and (1.3)b, the cosine of two vectors ***a**, ***b** mutual inclination has the value $*g_{ij}\ *a^i\ *b^j$

Hence $\frac{dx^\mu}{ds} \nabla_k ( *g_{ij}\ *a^i\ *b^j )$

$$= \frac{dx^\mu}{ds} [\nabla_k ( *g_{ij} )\ *a^i\ *b^j + *g_{ij} (\nabla_k *a^i )\ *b^j + *g_{ij}\ *a^i \nabla_k (*b^j ) ]$$

by means of (2.2)

$$\frac{dx^\mu}{ds} \nabla_k ( *g_{ij}\ *a^i\ *b^j ) = 0$$

## Ⅲ. PARALLEL DISPLACEMENT OF A NONHOLONOMIC COVARIANT VECTORS OF CONSTANT MAGNITUDE

The condition of parallel displacement alog a curve may be egually well expressed in terms of the covariant components of ***a** along the curve C.

**Theorem 3.1.** We have

$$(3.1) \quad \frac{dx^\mu}{ds} \nabla_k ( a_\lambda ) = 0$$



**proof**. In order to prove (3.1), multiplying both side of(2.1) by $g_{\nu\lambda}$ and summing for $\lambda$,

$$\frac{dx^\mu}{ds} g_{\nu\lambda} \nabla_\mu (a^\nu ) = \frac{dx^\mu}{ds} \nabla_\mu ( g_{\nu\lambda}\ a^\nu )$$

We have (3.1)

**Theorem 3.2.** Any two nonholonomic covariant vectors, of constant magnitudes, undergo parallel displacements along a given curve, they are inclined at a constant angle in nonholonomic frame.

**proof**. By means of (1.5) and (3.1),

$$(3.2)\quad \frac{dx^{\mu}}{ds}\nabla_{k}({}^{*}a_{i}) = \frac{dx^{\mu}}{ds}(\nabla_{\mu}(a_{\lambda})\ e^{\mu}_{k}\ e^{\lambda}_{i})$$

$$= \frac{dx^{\mu}}{ds}(\nabla_{\mu}(a_{\lambda})\ e^{\mu}_{k}\ e^{\lambda}_{i}$$

$$= 0$$

**Theorem 3.3**. Any nonholonomic vectors which undergoes a parallel displacement along a geodesic is inclined at a constant angle to the curve.

**proof**. The condition of parallelism in holonomic frame is

$$\frac{dx^{\mu}}{ds}\nabla_{\mu}(a_{v}) = \frac{dx^{\mu}}{ds}\left[\frac{da_{v}}{dx^{\mu}} - a_{\lambda}\{{}_{v}{}^{\lambda}{}_{\mu}\}\right]$$

$$= 0$$

That is, $\frac{da_{v}}{dx^{\mu}} - a_{\lambda}\{{}_{v}{}^{\lambda}{}_{\mu}\} = 0$

By virtue of (1.5),

$$\frac{dx^{\mu}}{ds}\nabla_{k}({}^{*}a_{i}) = \frac{dx^{\mu}}{ds}\left[\frac{d\,{}^{*}a_{i}}{dy^{k}} - {}^{*}a_{j}\ {}^{*}\{{}_{i}{}^{j}{}_{k}\}\right]$$

$$= \frac{d\,{}^{*}a_{i}}{ds}e^{\mu}_{k} - {}^{*}a_{j}\ {}^{*}\{{}_{i}{}^{j}{}_{k}\}\frac{dy^{k}}{ds}e^{\mu}_{k}$$



$$= \left[\frac{d\,{}^{*}a_{i}}{ds} - {}^{*}a_{j}\ {}^{*}\{{}_{i}{}^{j}{}_{k}\}\frac{dy^{k}}{ds}\right]e^{\mu}_{k}$$

$$= 0$$

Thus

$$(3.3)\quad \frac{d\,{}^{*}a_{i}}{ds} = {}^{*}a_{j}\ {}^{*}\{{}_{i}{}^{j}{}_{k}\}\frac{dy^{k}}{ds}$$

**Corollary 3.4**. Eguation (3.3) is eguivalent to

$$(3.4)\quad d\,{}^{*}a_{i} = {}^{*}a_{j}\ {}^{*}\{{}_{i}{}^{j}{}_{k}\}\ dy^{k}$$

**proof**. We obtain the result from (3.3)

**Theorem 3.5.** Any nonholonomic vector ***a**, which satisfies the conditions of (2.2) and (3.2) has constant magnitude along the curve.

**proof.**

$$(3.5)\quad \frac{d\ {}^{*}\mathbf{a}^2}{ds} = \frac{d}{ds}\left({}^{*}a^i\ {}^{*}a_i\right) = \nabla_\mu\left({}^{*}a^i\ {}^{*}a_i\right)\frac{dx^\mu}{ds}$$

$$= \left(\nabla_\mu({}^{*}a^i)\frac{dx^\mu}{ds}\right){}^{*}a_i + \left(\nabla_\mu({}^{*}a_i)\frac{dx^\mu}{ds}\right){}^{*}a^i$$

The results can be obtained by making use of (2.2) and (3.2).

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(國文抄錄)

**Riemann 空間 Vn에서 Nonholonomic**
**構造의 평행조건에 관한 소고**

**玄 桂 龍**

濟州大學校 教育大學院 數學教育專攻

(指導教授 玄 進 五)

이 논문의 중요한 목적은 Riemann 공간 Vn 에서 Nonholonomic 구조를 갖는 크기가 일정한 Vector 들의 평행조건에 대한 몇가지 성질들을 찾아내고 새로운 방법으로 증명해 보는데 있다.