碩 士 學 位 論 文

Characterizations of idempotent matrices over semirings

指導敎授　宋 錫 準

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

濟州大學校　敎育大學院

數 學 敎 育 專 攻

韓　希　政

2006年　6月

Characterizations of idempotent matrices over semirings

指導教授　宋 錫 準

이 論文을 教育學 碩士學位 論文으로 提出함.

2006年　6月　　日

제주대학교 중앙도서관

濟州大學校　教育大學院　數學教育專攻

提出者　韓　希　政

韓希政의 教育學 碩士學位 論文을 認准함.

2006年　6月　　日

審 查 委 員 長 ＿＿＿＿＿＿＿＿ 印

審　查　委　員 ＿＿＿＿＿＿＿＿ 印

審　查　委　員 ＿＿＿＿＿＿＿＿ 印

CONTENTS

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

<**Abstract**>

Characterizations of idempotent matrices over semirings

In this paper, we extend the characterizations of idempotent matrices over the binary Boolean algebra to those of idempotent matrices over several semirings. In Section 2, relevant definitions and notations are presented. In Section 3, we will give a sufficient condition of idempotent matrices over an arbitrary semiring. In Section 4, we characterize idempotent matrix over the general Boolean algebra. In Section 5, we characterize idempotent matrix over fuzzy semiring. Finally in Section 6, we obtain characterizations of idempotent matrix over nonnegative integer semiring.

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

1 Introduction

A semiring is essentially a ring in which only the zero is required to have an additive inverse (a formal definition is given in Section 2). Thus all rings are semirings. The set of all nonnegative integers, the general Boolean algebra of subsets of a finite set, and the fuzzy scalars are combinatorially interesting examples of semirings. The concepts of algebraic operations on matrices over a semiring are defined as if the underlying scalars were in a field.

There are many papers on the study of semiring matrix theory. In particular, Beasley and Pullman [3] studied on linear operators that preserve idempotent matrices over several semirings. Consequently they showed that the semigroup of linear operators on the semiring matrices *strongly* preserving idempotents (that map idempotents to idempotents and non-idempotents to non-idempotents) is generated by transposition and the similarity operators (those that map X to PXP^T for some permutation matrix P).

But there are few papers on the characterizations of idempotent matrices over a semiring. Recently Bapat et al. [2] obtained characterizations of nonnegative real idempotent matrices, and Beasley et al. [5] characterized all idempotent binary Boolean matrices.

In this paper, we extend the characterizations of idempotent matrices over the binary Boolean algebra to those of idempotent matrices over several semirings. In Section 2, relevant definitions and notations are presented. In Section 3, we will give a sufficient condition of idempotent matrices over an arbitrary semiring. In Section 4, we characterize idempotent matrix over the general Boolean algebra. In Section 5, we characterize idempotent matrix over fuzzy semiring. Finally in Section

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

6, we obtain characterizations of idempotent matrix over nonnegative integer semiring.

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

2 Definitions and Notations

Definition 2.1. [7, 10] A *semiring* $\mathbb{S}$ consists of a set $\mathbb{S}$ and two binary operations, addition $+$, and multiplication $\cdot$, such that

(1) $\mathbb{S}$ is an Abelian monoid under addition (identity denoted by 0);

(2) $\mathbb{S}$ is a monoid under multiplication (identity denoted by 1);

(3) multiplication is distributive over addition on both sides;

(4) $s0 = 0s = 0$ for all $s \in \mathbb{S}$.

Definition 2.2. A semiring $\mathbb{S}$ is called *antinegative* if the zero element is the only element with an additive inverse.

Let $\mathbb{Z}_+$ be the set of all nonnegative integers. Then $\mathbb{Z}_+$ is a commutative antinegative semiring which has no zero-divisors.

Definition 2.3. Let $\mathbb{B} \equiv \mathbb{B}_k$ be the (*general*) *Boolean algebra* of subsets of a k element set S_k and $\sigma_1, \sigma_2, \ldots, \sigma_k$ denote the singleton subsets of S_k. Union is denoted by $+$, and intersection by $\cdot$; 0 denote the null set and 1 the set S_k. Under these two operations, $\mathbb{B}$ is a commutative antinegative semiring; all of its elements, except 0 and 1, zero-divisors.

제주대학교 중앙도서관

In the above Definition, if $k = 1$, then $\mathbb{B}_1$ is just set $\{0, 1\}$, which is called the *binary Boolean algebra*.

Definition 2.4. For $\mathbb{F} = [0, 1] = \{x \in \mathbb{R} \,|\, 0 \leq x \leq 1\}$, we define $x + y$ as $\max(x, y)$ and xy as $\min(x, y)$ for all $x, y \in \mathbb{F}$. Then $\mathbb{F}$ becomes a commutative antinegative semiring that has no zero-divisors, and called a *fuzzy semiring*.

Throughout this paper, we will assume that $\mathbb{S}$ is a commutative antinegative semiring.

Let $\mathcal{M}_n(\mathbb{S})$ denote the set of all $n \times n$ matrices with entries in $\mathbb{S}$. The usual definitions for addition, multiplication by scalars, and the product of matrices over fields are applied to $\mathbb{S}$ as well. The zero matrix is denoted by O_n, the identity matrix by I_n and the matrix with all entries equal to 1 is denoted by J_n.

Definition 2.5. An $n \times n$ matrix with only one $(i, j)^{\text{th}}$ entry equal to 1 is called a *cell*, and denoted by $E_{i,j}$. A matrix $E \in \mathcal{M}_n(\mathbb{S})$ is called a *weighted cell* if there exist a nonzero $a \in \mathbb{S}$ and a cell $E_{i,j}$ such that $E = aE_{i,j}$. We say that the weighted cell $aE_{i,j}$ is in i^{th} row and it is in j^{th} column. When $i \neq j$, we say that the weighted cell $aE_{i,j}$ is *off-diagonal*; $aE_{i,i}$ is *diagonal*.

The following Proposition is an immediate consequence of the rules of matrix multiplication.

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

Proposition 2.6. *For any weighted cells $aE_{i,j}$ and $bE_{u,v}$, we have $(aE_{i,j})(bE_{u,v}) = abE_{i,v}$ or O_n according as $j = u$ or $j \neq u$.*

Definition 2.7. A matrix E in $\mathcal{M}_n(\mathbb{S})$ is called *idempotent* if $E^2 = E$. Otherwise, E is called *non-idempotent*.

The matrices O_n and I_n are clearly idempotents in $\mathcal{M}_n(\mathbb{S})$. By Proposition 2.6, we have all diagonal cells are idempotents, but all off-diagonal cells are non-idempotents. The matrix J_n is idempotent over the general Boolean algebra or the fuzzy semiring while it is not non-idempotent over the nonnegative integers because $J_n^2 = nJ_n$ in $\mathcal{M}_n(\mathbb{Z}_+)$.

Let $A \in \mathcal{M}_n(\mathbb{S})$ be a given matrix. For $i = 1, \ldots, n$, we define an i^{th} *row matrix* $\boldsymbol{R_i}(A)$ of A as a matrix whose i^{th} row is the same as the i^{th} row of A and the other rows are zero. Similarly, we can define a j^{th} *column matrix* $\boldsymbol{C_j}(A)$ of A for $j = 1, \ldots, n$. If the matrix A is clear from the context, we write $\boldsymbol{R_i}(A)$ and $\boldsymbol{C_j}(A)$ as $\boldsymbol{R_i}$ and $\boldsymbol{C_j}$, respectively. Thus we have

$$A = [a_{ij}] = \sum_{i=1}^{n} \boldsymbol{R_i}(\mathrm{A}) = \sum_{j=1}^{n} \boldsymbol{C_j}(A) \quad \text{or} \quad A = \sum_{i=1}^{n} \boldsymbol{R_i} = \sum_{j=1}^{n} \boldsymbol{C_j}.$$

Let $A = [a_{i,j}]$ be any matrix in $\mathcal{M}_n(\mathbb{S})$. The matrix A can be written uniquely as $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} a_{i,j} E_{i,j}$. Thus the matrix A is the sum of some weighted cells. If $a_{i,j} \neq 0$ for some i and j, then we say that the cell $E_{i,j}$ is in the matrix A; $a_{i,j}E_{i,j}$ is a weighted cell of A.

Definition 2.8. A *line matrix* is an i^{th} row matrix or a j^{th} column matrix of a matrix.

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

Definition 2.9. Weighted cells $E_1, \ldots, E_k$ are called *collinear* if $\sum\limits_{i=1}^{k} E_i$ is a line matrix.

Definition 2.10. We say that a matrix $A = [a_{i,j}] \in \mathcal{M}_n(\mathbb{S})$ *dominates* a matrix $B = [b_{i,j}] \in \mathcal{M}_n(\mathbb{S})$ if and only if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \sqsupseteq B$ or $B \sqsubseteq A$.

Let $A = [a_{i,j}]$ be a matrix in $\mathcal{M}_n(\mathbb{S})$. For an arbitrary cell $E_{i,j}$, we have that $E_{i,j} \sqsubseteq A$ if and only if $E_{i,j}$ is in A if and only if $a_{i,j} \neq 0$.

Lemma 2.11. *Let A be idempotent in $\mathcal{M}_n(\mathbb{S})$. If F and G are cells in A, then $FG \sqsubseteq A$.*

Proof. If $FG = O_n$, then $FG \sqsubseteq A$. If $FG \neq O_n$, then by Proposition 2.6, FG is a cell which is a summand for the matrix A^2. By the addition rules in $\mathbb{S}$, there is no element that can cancel a nonzero summand. Thus $FG \sqsubseteq A^2 = A$ since A is idempotent. Thus the result follows. ■

By applying Lemma 2.11 repeatedly, it follows that

Corollary 2.12. *Let A be idempotent in $\mathcal{M}_n(\mathbb{S})$. If $k \geq 2$ and $F_1, \ldots, F_k$ are cells in A, then $F_1 F_2 \cdots F_k \sqsubseteq A$.*

Lemma 2.13. *Let $A \in \mathcal{M}_n(\mathbb{S})$ be idempotent and F be an off-diagonal cell in A. Then there exist distinct cells G and H in A such that $F = GH$. Moreover if both cells G and H are off-diagonal, then the cells F, G and H are mutually distinct.*

Proof. Since A is the sum of some weighted cells, we may assume that $A = \sum_{i=1}^{m} E_i$, where each E_i is a weighted cell of A. Since A is idempotent, we have

$$\sum_{i=1}^{m} E_i^2 + \sum_{i,j=1, i\neq j}^{m} E_i E_j = A^2 = A = \sum_{i=1}^{m} E_i.$$

Thus F is either a square of a cell or a product of two distinct cells. Since F is off-diagonal, it follows from Proposition 2.6 that F is not a square of a cell. Thus F is a product of two distinct cells G and H in A. Furthermore, if G and H are off-diagonal, then F, G and H are mutually distinct by Proposition 2.6. ■

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

Definition 2.14. Let $A = [a_{i,j}] \in \mathcal{M}_n(\mathbb{S})$. For $1 \le i, j \le n$, $\boldsymbol{R_i}$ and $\boldsymbol{C_j}$ are said to be (i, j)-*disjoint* if $XY = O_n$ for any off-diagonal weighted cell X of $\boldsymbol{R_i}$ and for any off-diagonal weighted cell Y of $\boldsymbol{C_j}$.

Definition 2.15. A *weight* of $A \in \mathcal{M}_n(\mathbb{S})$ is the number of nonzero entries of A and is denoted by $|A|$.

Definition 2.16. Let $A \in \mathcal{M}_n(\mathbb{S})$ with $|A| = 4$. Then we say that A is a *frame* if four nonzero entries in A constitute a rectangle with at least one entry on diagonal; A is *pure* if it has only one nonzero diagonal entry.

For example, consider the following two frames in $\mathcal{M}_3(\mathbb{S})$;

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then A is pure, but B is not. If $\mathbb{S} = \mathbb{B}$ or $\mathbb{T}$, then we can easily show that A and B are all idempotent. If $\mathbb{S} = \mathbb{Z}_+$, then A is idempotent, while B is not because $B^2 (= 2B) \neq B$.

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

Definition 2.17. Let $A = [a_{i,j}] \in \mathcal{M}_n(\mathbb{S})$. We say that A has an i^{th} *rectangle part* if the following hold:

(1) there is a frame X in $\mathcal{M}_n(\mathbb{S})$ such that $E_{i,i} \sqsubseteq X$ and $X \sqsubseteq A$;

(2) for any $1 \le k, l \le n$, if $E_{l,i} \sqsubseteq A$ and $E_{i,k} \sqsubseteq A$, then $E_{l,k} \sqsubseteq A$.

Definition 2.18. If $A \in \mathcal{M}_n(\mathbb{S})$ has an i^{th} rectangle part, then the matrix in $\mathcal{M}_n(\mathbb{S})$ with the smallest number of nonzero elements which is dominated by A and dominates all frames of A dominating $E_{i,i}$ is called

i^{th} *rectangle part* of A and is denoted $RP(i)[A]$ or $RP(i)$, if A is clear from the context.

Suppose that $A = [a_{i,j}] \in \mathcal{M}_n(\mathbb{S})$ has the i^{th} rectangle part $RP(i)$. Let

$$\{E_{i,i_1}, \ldots, E_{i,i_t}\} \quad \text{and} \quad \{E_{j_1,i}, \ldots, E_{j_s,i}\}$$

be the sets of all off-diagonal cells that are in $\boldsymbol{R_i}$ and $\boldsymbol{C_i}$, respectively. Then

$$RP(i) = \sum_{k=1}^{s}\sum_{l=1}^{t}(\alpha_1 E_{i,i} + \alpha_2 E_{i,i_l} + \alpha_3 E_{j_k,i} + \alpha_4 E_{j_k,i_l})$$

for some nonzero scalars $\alpha_1, \ldots, \alpha_4 \in \mathbb{S}$.

Let

$$A_1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $A_1 = RP(1)[A_1] = RP(3)[A_1] = RP(1)[A_2] = RP(3)[A_2]$ and $RP(2)[A_3] = E_{1,1} + E_{1,2} + E_{2,1} + E_{2,2}$, however the 1^{st} rectangle part of A_3 does not exist.

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

Definition 2.19. It is said that a matrix $A = [a_{i,j}] \in \mathcal{M}_n(\mathbb{S})$ has an i^{th} *line part* if there exists $i \in \{1, \ldots, n\}$ such that $a_{i,i} \neq 0$ and either $|\boldsymbol{R_i}| = 1$ or $|\boldsymbol{C_i}| = 1$ or both $|\boldsymbol{R_i}| = 1 = |\boldsymbol{C_i}|$. In these cases $\boldsymbol{R_i} + \boldsymbol{C_i}$ is a line matrix dominating $E_{i,i}$ which is called a *line part* of A and is denoted by $LP(i)[A]$. If the matrix A is clear from the context, we write $LP(i)[A]$ as $LP(i)$.

Let $A_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $A_5 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then $LP(1)[A_4] = E_{1,1} + E_{1,2}$, $LP(2)[A_4] = E_{1,2} + E_{2,2}$, while A_5 do not have line parts.

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

3 Some results

In this section, we will give some properties of idempotent matrices in $\mathcal{M}_n(\mathbb{S})$, where $\mathbb{S}$ is a commutative antinegative semiring. For this purpose, we shall analyze the structures of the sums of weighted cells.

For any matrix $A = [a_{i,j}]$ in $\mathcal{M}_n(\mathbb{S})$, define the matrix $A^* = [a^*_{i,j}]$ in $\mathcal{M}_n(\mathbb{B}_1)$ as $a^*_{i,j} = 1$ if and only if $a_{i,j} \neq 0$. If $\mathbb{S}$ is a semiring which has no zero-divisors, then we can easily show that

$$(A+B)^* = A^* + B^*, \quad (AB)^* = A^*B^*, \quad \text{and} \quad (\alpha A)^* = \alpha^* A^* \tag{3.1}$$

for all $A, B \in \mathcal{M}_n(\mathbb{S})$ and for all $\alpha \in \mathbb{S}$. The following Lemma is an immediate consequence of (3.1).

Lemma 3.1. *Let $\mathbb{S}$ be a semiring which has no zero-divisors. If A is idempotent in $\mathcal{M}_n(\mathbb{S})$, then A^* is idempotent in $\mathcal{M}_n(\mathbb{B}_1)$.*

In general, the converse of Lemma 3.1 may be not true. For example, consider a matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ in $\mathcal{M}_2(\mathbb{Z}_+)$. Then A is non-idempotent in $\mathcal{M}_2(\mathbb{Z}_+)$ because $A^2 = 2A \neq A$ while $A^*(= A)$ is idempotent in $\mathcal{M}_2(\mathbb{B}_1)$.

제주대학교 중앙도서관

The following two Lemmas are useful in characterizing idempotent matrix in $\mathcal{M}_n(\mathbb{S})$ and have been proved in [5].

Lemma 3.2. *Let A be a nonzero matrix in $\mathcal{M}_n(\mathbb{B}_1)$. If all cells in A are off-diagonal, then A is non-idempotent .*

Lemma 3.3. *Let A be idempotent in $\mathcal{M}_n(\mathbb{B}_1)$. Assume that there exists an off-diagonal cell $F \sqsubseteq A$ such that for any diagonal cell $E \sqsubseteq A$, E and*

F are not collinear. Then F is in a frame with one diagonal cell and two additional off-diagonal cells in A.

Corollary 3.4. *Let $\mathbb{S}$ be a semiring which has no zero-divisors, and let $A = \sum_{i=1}^{k} \alpha_i E_i$ be a nonzero matrix in $\mathcal{M}_n(\mathbb{S})$, where each E_i is a cell. Then*

(1) *if all E_i are diagonal, then A is idempotent if and only if each α_i is an idempotent element in $\mathbb{S}$,*

(2) *if all weighted cells of A are off-diagonal, then A is non-idempotent.*

Proof. (1) Suppose that A is idempotent in $\mathcal{M}_n(\mathbb{S})$. By Proposition 2.6, we have

$$(\alpha_1 E_1 + \cdots + \alpha_k E_k)^2 = \alpha_1^2 E_1 + \cdots + \alpha_k^2 E_k = \alpha_1 E_1 + \cdots + \alpha_k E_k.$$

Therefore $\alpha_i^2 = \alpha_i$ for all i. The converse is clear.

(2) Suppose that all weighted cells of A are off-diagonal. By (3.1), A^* is just sum of off-diagonal cells in $\mathcal{M}_n(\mathbb{B}_1)$. By Lemma 3.2, A^* is non-idempotent in $\mathcal{M}_n(\mathbb{B}_1)$. It follows from Lemma 3.1 that A is non-idempotent in $\mathcal{M}_n(\mathbb{B}_1)$. ■

Corollary 3.5. *If a diagonal entry of $A = [a_{i,j}] \in \mathcal{M}_n(\mathbb{Z}_+)$ is greater than 1, then A is non-idempotent.*

Proof. Suppose that $a_{i,i} > 1$ for some $i \in \{1, \ldots, n\}$. Then we can easily show that the $(i,i)^{\text{th}}$ entry of A^2 is greater than $a_{i,i}$. Therefore the $(i,i)^{\text{th}}$ entries of A and A^2 are distinct. Hence A is non-idempotent. ■

Corollary 3.6. *Let $\mathbb{S}$ be a semiring which has no zero-divisors, and let A be idempotent in $\mathcal{M}_n(\mathbb{S})$. If A has an off-diagonal weighted cell αE (where E is a cell) such that E is not collinear with any diagonal cell in A, then αE is in a pure frame with an i^{th} diagonal weighted cell and two off-diagonal weighted cells $\beta_i F$ and $\gamma_i G$ in A. Furthermore, if I is the set of all indices such i, then we have $\alpha = \sum_{i \in I} \beta_i \gamma_i$.*

Proof. In the view of Corollary 3.4-(2), without loss of generality we may assume that A has at least one nonzero diagonal entry. Let $E = E_{b,c}$, where $b \neq c$. Since A is idempotent in $\mathcal{M}_n(\mathbb{S})$, it follows from Lemma 3.1 that A^* is idempotent in $\mathcal{M}_n(\mathbb{B}_1)$. By the assumption, $E_{b,c}$ is an off-diagonal cell in A^* such that it is not collinear with any diagonal cell in A^*. By Lemma 3.3,

$$E_{i,i} + E_{b,i} + E_{i,c} + E_{b,c}$$

is a pure frame in A^* for some $i \in \{1, \ldots, n\}$ different from b and c. Therefore we have that $(i,c)^{\text{th}}$ and $(b,i)^{\text{th}}$ off-diagonal entries of A are nonzero elements β_i and γ_i in $\mathbb{S}$, respectively. It follows that αE is in a pure frame with an i^{th} diagonal weighted cell and two off-diagonal weighted cells $\beta_i E_{i,c}$ and $\gamma_i E_{b,i}$ in A. The rest follows from the arithmetic rules in $\mathcal{M}_n(\mathbb{S})$. ■

제주대학교 중앙도서관

Example 3.7. Consider two matrices

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & x & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 0 & 1 \end{bmatrix} \in \mathcal{M}_5(\mathbb{Z}_+)$$

and

$$B = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & 0 & y & \frac{2}{3} & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{5} & 1 & 0 \\ 0 & 0 & \frac{5}{6} & 0 & 1 \end{bmatrix} \in \mathcal{M}_5(\mathbb{F}),$$

where $\mathbb{F}$ is the fuzzy semiring. Then A and B are just sums of three diagonal weighted cells and seven off-diagonal weighted cells. Also, we can easily show that A is idempotent (if and) only if $x = 19$. In fact, we note that the cell $E_{2,3}$ in A is not collinear with all diagonal cells $E_{1,1}, E_{4,4}$ and $E_{5,5}$ in A. It follows from Corollary 3.6 that

$$xE_{2,3} = E_{2,1}E_{1,3} + (2E_{2,4})(3E_{4,3}) + (3E_{2,5})(4E_{5,3})$$

is a necessary condition for A to be idempotent. Similarly, we obtain that B is idempotent (if and) only if

$$y = \frac{1}{4} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{4}{5} + \frac{3}{4} \cdot \frac{5}{6} = \frac{3}{4} \cdot \frac{5}{6} = \frac{3}{4}. \qquad \blacksquare$$

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

Proposition 3.8. *Let A be idempotent in $\mathcal{M}_n(\mathbb{S})$. If $\boldsymbol{R_i}$ and $\boldsymbol{C_j}$ are not (i,j)-disjoint, then $E_{i,j} \sqsubseteq A$.*

Proof. Suppose that $\boldsymbol{R_i}$ and $\boldsymbol{C_j}$ are not (i,j)-disjoint. Then there exist off-diagonal weighted cells $\alpha E_{i,x} \sqsubseteq \boldsymbol{R_i}$ and $\beta E_{y,j} \sqsubseteq \boldsymbol{C_j}$ such that $(\alpha E_{i,x})(\beta E_{y,j}) \neq O_n$. It follows from Proposition 2.6 that $x = y$ and $E_{i,x}E_{y,j} = E_{i,j}$. Since A is idempotent, $E_{i,j} \sqsubseteq A$ by Lemma 2.11. ■

Lemma 3.9. *Let $\mathbb{S}$ be a semiring which has no zero-divisors, and let $A = [a_{i,j}]$ be idempotent in $\mathcal{M}_n(\mathbb{S})$ with $a_{i,i} \neq 0$ for some i. If $|\boldsymbol{R_i}| = s+1$*

and $|\boldsymbol{C_i}| = t+1$, *then there exist exactly* $s \cdot t$ *frames in* A *dominating* $E_{i,i}$. *In particular, if* $\mathbb{S} = \mathbb{Z}_+$, *then all frames are pure.*

Proof. If $s = 0$ or $t = 0$, then the result is straightforward. Thus we can assume that $s, t \geq 1$. Since A is idempotent, Lemma 2.11 and Proposition 2.6 implies that for any cells

$$E_{k,i} \sqsubseteq \boldsymbol{R_i} \sqsubseteq A \quad \text{and} \quad E_{i,l} \sqsubseteq \boldsymbol{C_i} \sqsubseteq A,$$

their product $E_{k,i}E_{i,l} = E_{k,l} \sqsubseteq A$. Therefore, the four cells $E_{i,i}, E_{k,i}, E_{i,l}$ and $E_{k,l}$ are in a frame in A for each k, l such that $E_{k,i} \sqsubseteq A$ and $E_{i,l} \sqsubseteq A$. Thus A has at least $s \cdot t$ frames such that each frame dominates $E_{i,i}$. It follows from the definition of frame that A has at most $s \cdot t$ frames dominating $E_{i,i}$.

Let $\mathbb{S} = \mathbb{Z}_+$. Suppose that A has a frame dominating $E_{i,i}$ such that it is not pure. Then there exists an index j different from i such that $E_{i,i}, E_{j,i}, E_{i,j}, E_{j,j} \sqsubseteq A$ so that $a_{i,j}a_{j,i}a_{i,j}a_{j,j} \neq 0$. Therefore the $(i,j)^{\text{th}}$ entry $b_{i,j}$ of A^2 becomes

$$b_{i,j} = \sum_{k=1}^{n} a_{i,k}a_{k,j} \geq a_{i,i}a_{i,j} + a_{i,j}a_{j,j} = (a_{i,i} + a_{j,j})a_{i,j} \geq 2a_{i,j} > a_{i,j},$$

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

a contradiction. Hence we have that all frames dominating $E_{i,i}$ are pure for $\mathbb{S} = \mathbb{Z}_+$. ■

Example 3.10. Let $\mathbb{B} = \mathbb{B}_2$ be the Boolean algebra of a two element set S_2, and let

$$A = \begin{bmatrix} 1 & \sigma_1 & \sigma_2 \\ \sigma_2 & 0 & \sigma_2 \\ \sigma_1 & \sigma_1 & 0 \end{bmatrix} \in \mathcal{M}_3(\mathbb{B}_2).$$

Then we can easily show that A is idempotent in $\mathcal{M}_3(\mathbb{B}_2)$. Notice that $|\boldsymbol{R_1}| = 2 + 1 = |\boldsymbol{C_1}|$. But A has only two frames dominating $E_{1,1}$. Thus, the condition that $\mathbb{S}$ has no zero-divisors in Lemma 3.9 is needed. ■

Let A be idempotent in $\mathcal{M}_n(\mathbb{S})$, where $\mathbb{S}$ is a semiring which has no zero-divisors. If $a_{i,i} \neq 0$, $|\boldsymbol{R_1}| > 1$ and $|\boldsymbol{C_1}| > 1$, then Lemma 3.9 shows that the i^{th} rectangle part of A exists.

Theorem 3.11. *Let $\mathbb{S}$ be a semiring which has no zero-divisors. If A is idempotent in $\mathcal{M}_n(\mathbb{S})$, then every cell dominated by A is in either a rectangle part or a line part of A.*

Proof. It follows directly from Corollary 3.6 and Lemma 3.9. ■

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

4 The case of Boolean algebra

In this Section, we will characterize all idempotent matrices over the general Boolean algebra $\mathbb{B}$.

Let $A = [a_{i,j}] \in \mathcal{M}_n(\mathbb{S})$. Suppose that A has i^{th} and j^{th} rectangle parts $RP(i)$ and $RP(j)$ for some i and j with $i \neq j$. We say that $RP(i)$ and $RP(j)$ are *disjoint* if either $\boldsymbol{R_i}$ and $\boldsymbol{C_j}$ are (i,j)-disjoint or $\boldsymbol{R_j}$ and $\boldsymbol{C_i}$ are (j,i)-disjoint or both.

In [5], Beasley et al. characterized all idempotent matrices over the binary Boolean algebra $\mathbb{B}_1$ as the following:

Theorem 4.1. *Let A be in $\mathcal{M}_n(\mathbb{B}_1)$. Then A is idempotent if and only if the following two conditions are satisfied*:

(1) *there exist integers $r, l \geq 0$ such that A is a sum of r disjoint rectangle parts and l line parts,*

(2) *if for some $i \neq j$ $\boldsymbol{R_i}$ and $\boldsymbol{C_j}$ are not (i,j)-disjoint, then $E_{i,j} \sqsubseteq A$.*

Let $\mathbb{B} = \mathbb{B}_k$ be the Boolean algebra of all subsets of a k element set S_k; $\sigma_1, \ldots, \sigma_k$ are all singleton subsets of S_k. For each matrix $A \in \mathcal{M}_n(\mathbb{B})$, the p^{th} *constituent* [11] of A, A_p, is the $n \times n$ binary Boolean matrix whose $(i,j)^{\text{th}}$ entry is 1 if and only if $a_{i,j} \supseteq \sigma_p$. Via the constituents, A can be written uniquely as $\sum_{p=1}^{k} \sigma_p A_p$, which is called the *canonical form* of A. It follows from the uniqueness of the canonical form that for all $1 \leq p \leq k$

(i) $(A+B)_p = A_p + B_p$,

(ii) $(AB)_p = A_p B_p$,

(iii) $\alpha A)_p = \alpha_p A_p$

for all matrices A, B in $\mathcal{M}_n(\mathbb{B})$ and for all α in $\mathbb{B}$.

Let A be a matrix in $\mathcal{M}_n(\mathbb{B})$. Then a p^{th} constituent of A may be a key concluding whether A is idempotent or not. For example, consider

$$A = \begin{bmatrix} \sigma_1 & \sigma_2 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_2(\mathbb{B}_2).$$

Then the 2^{th} constituent of A is

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_2(\mathbb{B}_1),$$

and A_2 is non-idempotent in $\mathcal{M}_2(\mathbb{B}_1)$ by Lemma 3.2. Theorem 4.2 (below) shows that A is non-idempotent in $\mathcal{M}_2(\mathbb{B}_2)$.

Theorem 4.2. *Let A be a matrix in $\mathcal{M}_n(\mathbb{B})$. Then A is idempotent if and only if all p^{th} constituents of A are idempotent in $\mathcal{M}_n(\mathbb{B}_1)$.*

Proof. Let $A = \sum_{p=1}^{k} \sigma_p A_p$ be the canonical form of A. If A is idempotent in $\mathcal{M}_n(\mathbb{B})$, then we have

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

$$(A^2 =)\sigma_1 A_1^2 + \cdots + \sigma_k A_k^2 = \sigma_1 A_1 + \cdots + \sigma_k A_k (= A). \tag{4.1}$$

If we multiply σ_p on both sides in (4.1), then we have $\sigma_p A_p^2 = \sigma_p A_p$ for all $p = 1, \ldots, k$. Suppose that some p^{th} constituent of A is not idempotent in $\mathcal{M}_n(\mathbb{B}_1)$ so that $A_p^2 \neq A_p$. Then there exist indices i and j such that $(i,j)^{\text{th}}$ entries of A_p and A_p^2 are different in $\mathbb{B}_1 = \{0, 1\}$. If the $(i,j)^{\text{th}}$ entry of A_p is 1, then that of A_p^2 is 0. Thus the $(i,j)^{\text{th}}$ entry of A contains σ_p, while A^2 does not. Thus we have $A^2 \neq A$, a contradiction. Similarly, if the $(i,j)^{\text{th}}$ entries of A_p and A_p^2 are 0 and 1, respectively, then we have $A^2 \neq A$, a contradiction. Therefore all p^{th} constituents of A are idempotent in $\mathcal{M}_n(\mathbb{B}_1)$.

The converse follows from the definition of the canonical form of A.
■

Thus we obtain the characterizations of all idempotent matrices over the general Boolean algebra $\mathbb{B}$.

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

5 The case of fuzzy semiring

We recall that for the fuzzy semiring $\mathbb{F}$, two operations are defined as

$$x + y = \max(x, y) \quad \text{and} \quad xy = \min(x, y)$$

for all $x, y \in \mathbb{F}$.

Let α be a fixed member of $\mathbb{F}$, other than 1. For each $x \in \mathbb{F}$, define $x^\alpha = 0$ if $x \leq \alpha$, and $x^\alpha = 1$ otherwise. Then the mapping $x \to x^\alpha$ is a homomorphism of $\mathbb{F}$ onto $\mathbb{B}_1$. Its entrywise extension to a mapping $A \to A^\alpha$ of $\mathcal{M}_n(\mathbb{F})$ onto $\mathcal{M}_n(\mathbb{B}_1)$ preserves matrix sums and products and multiplication by scalars. We call A^α the α-*pattern* of A.

Let $A = [a_{i,j}]$ be a matrix in $\mathcal{M}_n(\mathbb{F})$. Then an $a_{i,j}$-pattern of A may be a key concluding whether A is idempotent or not. For example, let

$$A = [a_{i,j}] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} \in \mathcal{M}_2(\mathbb{F}).$$

Then the $a_{2,2}(= \frac{1}{4})$-pattern of A is

$$A^{\frac{1}{4}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{M}_2(\mathbb{B}_1),$$

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

and $A^{\frac{1}{4}}$ is not idempotent in $\mathcal{M}_2(\mathbb{B}_1)$ by Lemma 3.9. Theorem 5.1 (below) shows that A is not idempotent in $\mathcal{M}_2(\mathbb{F})$.

Theorem 5.1. *Let $A = [a_{i,j}]$ be a matrix in $\mathcal{M}_n(\mathbb{F})$. Then A is idempotent if and only if all $a_{i,j}$-patterns of A are idempotent in $\mathcal{M}_n(\mathbb{B}_1)$.*

Proof. Let A be idempotent in $\mathcal{M}_n(\mathbb{F})$. Then all $a_{i,j}$-patterns of A are idempotent in $\mathcal{M}_n(\mathbb{B}_1)$ because each $a_{i,j}$-pattern of A is a homomorphism of $\mathcal{M}_n(\mathbb{F})$ onto $\mathcal{M}_n(\mathbb{B}_1)$.

Conversely, assume that each $a_{i,j}$-pattern $A^{a_{i,j}}$ of A is idempotent in $\mathcal{M}_n(\mathbb{B}_1)$. Suppose that $A^2 \neq A$. Then for some $(i,j)^{\text{th}}$ entries of A and A^2, we have

$$a_{i,j} \neq \sum_{k=1}^{n} a_{i,k}a_{k,j}. \tag{5.1}$$

If $a_{i,j} < \sum_{k=1}^{n} a_{i,k}a_{k,j}$, then the $(i,j)^{\text{th}}$ entry of $A^{a_{i,j}}$ is 0, but that of $(A^{a_{i,j}})^2$ is 1, a contradiction to the fact that $a_{i,j}$-pattern of A is idempotent in $\mathcal{M}_n(\mathbb{B}_1)$. Hence we have $a_{i,j} > \sum_{k=1}^{n} a_{i,k}a_{k,j}$. We notice that the right side of (5.1) is just $a_{i,k}a_{k,j}$ for some $k \in \{1, \ldots, n\}$. Furthermore we have $a_{i,k}a_{k,j} = a_{i,k}$ or $a_{k,j}$. If $a_{i,k}a_{k,j} = a_{i,k}$, then $a_{i,j} > \sum_{k=1}^{n} a_{i,k}a_{k,j} = a_{i,k}$, and hence the $(i,j)^{\text{th}}$ entry of $A^{a_{i,k}}$ is 1, but that of $(A^{a_{i,k}})^2$ is 0, a contradiction. Similarly if $a_{i,k}a_{k,j} = a_{k,j}$, then we have $(A^{a_{k,j}})^2 \neq A^{a_{k,j}}$, a contradiction. Therefore A is idempotent in $\mathcal{M}_n(\mathbb{F})$. ■

Thus we characterize all idempotent matrices over the fuzzy semiring $\mathbb{F}$.

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

6 The case of nonnegative integers

In this section, $\mathbb{Z}_+$ denote the semiring of all nonnegative integers.

Let A be an idempotent matrix in $\mathcal{M}_n(\mathbb{Z}_+)$. Then Corollary 3.5 tell us that all diagonal entries of A are either 0 or 1.

Lemma 6.1. *Let A be a matrix in $\mathcal{M}_n(\mathbb{Z}_+)$. Assume that $E_{i,j} \sqsubseteq A$ for some i, j with $i \neq j$. If $E_{i,i}, E_{j,j} \sqsubseteq A$, then A is non-idempotent.*

Proof. Since $E_{i,j} \sqsubseteq A$ and $E_{i,i}, E_{j,j} \sqsubseteq A$, we have that $a_{i,j}, a_{i,i}$ and $a_{j,j}$ are all nonzero. Assume that A is idempotent. Then we have $a_{i,i} = a_{j,j} = 1$. Thus we have

$$\begin{aligned} A^2 &= (E_{i,i} + a_{i,j}E_{i,j} + E_{j,j} + \cdots)^2 \\ &= E_{i,i} + 2a_{i,j}E_{i,j} + E_{j,j} + \cdots . \end{aligned}$$

So the $(i,j)^{\text{th}}$ entry of A^2 is strictly greater than that of A, a contradiction. Hence A is non-idempotent. ■

Let $RP(i)$ be an i^{th} rectangle part of $A \in \mathcal{M}_n(\mathbb{Z}_+)$. Then $RP(i)$ is called *pure* if it has only nonzero diagonal entry.

Proposition 6.2. *If $RP(i)$ is an i^{th} rectangle part of an idempotent matrix $A \in \mathcal{M}_n(\mathbb{Z}_+)$, then it is pure.*

Proof. It follows from Lemma 6.1. ■

Lemma 6.3. *Let $A = [a_{i,j}]$ be a matrix in $\mathcal{M}_n(\mathbb{Z}_+)$ with $a_{i,i}a_{j,j} \neq 0$ for some indices i and j. If $\boldsymbol{R_i}$ and $\boldsymbol{C_j}$ are not (i,j)-disjoint, then A is non-idempotent.*

Proof. If $i \neq j$, the result follows from Proposition 3.8 and and Lemma 6.1. So we may assume that $i = j$. Suppose that A is idempotent and, $\boldsymbol{R_i}$ and $\boldsymbol{C_i}$ are not (i,j)-disjoint. Then there exist at least two off-diagonal weighted cells $a_{i,x}E_{i,x} \sqsubseteq \boldsymbol{R_i}$ and $a_{y,i}E_{y,i} \sqsubseteq \boldsymbol{C_i}$ such that $(a_{i,x}E_{i,x})(a_{y,i}E_{y,i}) \neq O_n$. By Proposition 2.6, we have $x = y$. Since $E_{x,i} \sqsubseteq A$ and $E_{i,x} \sqsubseteq A$, their product $E_{x,i}E_{i,x} = E_{x,x}$ is in A by Lemma 2.11 because A is idempotent. Thus we have $E_{i,x} \sqsubseteq A$ and $E_{i,i}, E_{x,x} \sqsubseteq A$. By Lemma 6.1, A is non-idempotent, a contradiction. ■

Example 6.4. Consider a matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 0 & 6 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{M}_4(\mathbb{Z}_+).$$

Then A is the sum of one 1^{st} pure rectangle part and one 4^{th} line part. But $\boldsymbol{R_1}$ and $\boldsymbol{C_4}$ are not $(1,4)$-disjoint. By Lemma 6.3, A is non-idempotent. ■

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

Theorem 6.5. *Let*

$$A = \sum_{i=1}^{m} \alpha_i E_i + \sum_{j=1}^{k} \beta_j F_j$$

be a matrix in $\mathcal{M}_n(\mathbb{Z}_+)$, where E_i are diagonal cells, and F_j off-diagonal cells with nonzero scalars α_i and β_j in $\mathbb{Z}_+$. Then A is idempotent if and only if it is the sum of s disjoint pure rectangle parts and t disjoint line parts of A, and the followings are satisfied:

(1) *each rectangle part is idempotent,*

(2) *each line part is idempotent,*

(3) $m = s + t$ *and* $\alpha_i = 1$ *for all* $i = 1, \ldots, m$,

(4) *for any* $a, b \in \{1, \ldots, n\}$, $\boldsymbol{R_a}$ *and* $\boldsymbol{C_b}$ *are* (i, j)*-disjoint.*

Proof. The necessity is trivial. We now prove the sufficiency. Suppose that $A = [a_{i,j}]$ is idempotent in $\mathcal{M}_n(\mathbb{Z}_+)$. following Corollary 3.4-(2), we may assume that $m \geq 1$. Let $E_{i,j} \sqsubseteq A$. It follows from Theorem 3.11 and Proposition 6.2 that $E_{i,j}$ is in either a pure rectangle part or a line part of A. So we may assume that A has s disjoint pure rectangle parts and t disjoint line parts, where $s, t \geq 0$. Then we have $m = s + t$ and $\alpha_i = 1$ for all $i = 1, \ldots, m$ by Lemma 3.1. Thus (3) is satisfied. (4) follows from Lemma 6.3. (1) and (2) are obvious by (4). ■

Thus we obtain characterizations of all idempotent matrices over the semiring of all nonnegative integers.

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

References

[1] J. K. Baksalary, F. Pukelsheim and G. P. H. Styan, *Some properties of matrix partial orderings*, Linear Algebra Appl. **119**(1989), 57-85.

[2] R. B. Bapat, S. K. Jain and L. E. Snyder, *Nonnegative idempotent matrices and minus partial order*, Linear Algebra Appl. **261**(1997), 143-154.

[3] L. B. Beasley and N. J. Pullman, *Linear operators strongly preserving idempotent matrices over semirings*, Linear Algebra Appl. **160**(1992), 217-229.

[4] L. B. Beasley, K. T. Kang and S. Z. Song, *Idempotent matrix preservers over Boolean algebras*, J. Korean Math. Soc., To appear.

[5] L. B. Beasley, A. E. Guterman, K. T. Kang and S. Z. Song, *Idempotent Boolean matrices and minus order majorization*, Linear Algebra Appl., To submit.

[6] R. Chaudhuri and A. Mukherjea, *Idempotent Boolean matrices*, Semigroup Forum, **21**(1980), 273-282.

[7] D. A. Gregory and N. J. Pullman, *Semiring rank* : *Boolean rank and nonnegative rank factorizations*, J. Combin. Inform. System Sci., **8**(1983), 223-233.

[8] R. E. Hartwig, *How to partially order regular elements*, Math. Japonica, **25**(1980), no. 1, 1-13.

[9] R. E. Hartwig and G. P. H. Styan, *On some characterizations of the "star" partial ordering for matrices and rank subtractivity*, Linear Algebra Appl., **82**(1986), 145-161.

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

[10] K. H. Kim, *Boolean Matrix Theory and Applications*, Pure and Applied Mathematics, V.70, Marcel Dekker, New York, 1982.

[11] S. Kirkland and N. J. Pullman, *Linear operators preserving invariants of non-binary matrices*, Linear and Multilinear Algebra, **33**(1992), 295-300.

[12] V. N. Kolokoltsov and V. P. Maslov, *Idempotent Analysis and its Applications*, Mathematics and its Applications, V. 401, Dordrecht: Kluwer Academic Publishers, 1997.

[13] J. Lambek, *Lectures on Rings and Modules*, Second edition. Chelsea Publishing Co., New York, 1976.

[14] D. Rosenblatt, *On the graphs of finite idempotent Boolean relation matrices*, J. Res. Nat. Bureau of Standards, **67B**(1963), 249-256.

[15] B. Schein, *A construction for idempotent binary relations*, Proceedings of the Japan Academy, **46**(1970), 246-247.

[16] S. Z. Song and K. T. Kang, *Types and enumeration of idempotent matrices*, Far East J. Math. Sci. **3**(2001), 1029-1042.

[17] S. Z. Song, K. T. Kang and Y. B. Jun, *Linear preservers of Boolean nilpotent matrices*, J. Korean Math. Soc., To appear.

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

<국문 초록>

반환상의 재귀행렬의 구조 분석

본 논문에서는 이항 부울 대수상의 행렬이 재귀행렬(idempotent matrix)이 되기 위한 필요충분조건에 대한 연구결과를 다양한 반환상의 행렬들로 확장하여 연구한다.

제 2절에서는 본 연구에 필요한 정의와 그들의 표시방법을 제시한다.

제 3절에서는 영인자(zero-divisor)가 없는 반환 상에서, 한 행렬이 재귀행렬이 되기 위한 충분조건을 제시하고, 이를 증명한다.

제 4절에서는 일반적인 부울 대수 위에서, 한 행렬이 재귀행렬이 되기 위한 필요충분조건은 그 행렬의 각 성분별 행렬(constituent matrix)들이 이항 부울 대수 위에서 재귀행렬이 되는 것임을 밝힌다.

제 5절에서는 퍼지 반환 상에서, 한 행렬이 재귀행렬이 되기 위한 필요충분조건은 그 행렬의 각 원소별 형식행렬(pattern matrix)들이 이항 부울 대수 위에서 재귀행렬이 되는 것임을 밝힌다.

끝으로, 제 6절에서는 비음의 정수반환상상에서, 한 행렬이 재귀행렬이 되기 위한 필요충분조건은 그 행렬이 직사각형 행렬부분과 선분행렬부분의 합으로 나타나면서 4가지 조건을 만족하는 것임을 밝힌다.

제주대학교 중앙도서관
JEJU NATIONAL UNIVERSITY LIBRARY

감사의 글

5학기 동안 여러 가지로 많은 어려움이 있었지만 많은 분들의 도움으로 무사히 마칠 수 있었습니다.

우선 본 논문이 완성되기까지 연구에 바쁘신 가운데도 부족한 저에게 항상 깊은 관심과 배려로 많은 가르침을 주시고 지도해주신 송석준 교수님께 깊은 감사를 드립니다. 그리고 이 논문이 완성되기까지 많은 관심과 충고로 저를 이끌어 주신 강경태 선생님의 고마움도 잊지 않겠습니다. 그리고 대학원 5학기 동안 훌륭한 강의를 해 주시고 여러 도움을 주신 교수님들께도 감사드리며 5학기 동안 기쁠 때 같이 웃고 힘들 때 서로 의지하며 함께 생활해 온 선생님들께도 감사드립니다. 그리고 무엇보다도 항상 든든한 힘이 되어준 저의 소중한 가족과 부모님들께도 깊은 고마움과 사랑의 마음을 전합니다. 이 모든 분들과 주위에서 격려하고 용기를 주신 모든 분들의 사랑과 기대에 어긋남이 없도록 앞으로도 열심히 생활하겠습니다.

2006년 6월

제주대학교 중앙도서관