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COMPARISONS OF RANK, COLUMN RANK AND MAXIMAL COLUMN RANK OF MATRICES OVER MAX ALGEBRA AND THEIR LINEAR PRESERVERS

濟州大學校 大學院

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COMPARISONS OF RANK, COLUMN RANK AND MAXIMAL COLUMN RANK OF MATRICES OVER MAX ALGEBRA AND THEIR LINEAR PRESERVERS

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막스행렬 계수들의 비교와
그들의 선형보존자

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ABSTRACT

COMPARISONS OF RANK, COLUMN RANK AND MAXIMAL COLUMN RANK OF MATRICES OVER MAX ALGEBRA AND THEIR LINEAR PRESERVERS

Boolean rank, column rank and maximal column rank over Boolean matrices have been studied and developed so far. And their linear preservers also have been characterized over Boolean matrices in the previous researches.

In this thesis, we compare the rank and column rank using a matrix function and we obtain the values of this function on the matrices over max algebra. We also characterize the linear operators that preserve column rank over max algebra. We show that a linear operator on the set $\mathcal{M}$ of $m \times n$ matrices over max algebra preserves column rank of each matrix in $\mathcal{M}$ if and only if it is a congruence operator, which has the form of multiplications by monomial matrices on both sides of the given matrix.

Moreover, we investigate the relationships between rank and maximal column rank using a matrix function and we determine the values of this function on the matrices over max algebra. We also obtain some characterizations of the linear operators preserving maximal column rank over max algebra. One of them is that a linear operator on $\mathcal{M}$ preserves maximal column rank of each matrix if and only if it preserves maximal column ranks 1, 2 and 3.

1 Introduction

One of the most active and important subjects in matrix theory during the past century is the study of those linear operators on matrices that leave certain properties or relations of matrices invariant. Such topics are usually called *Linear Preserver Problems*(LPP). The earliest papers in our reference list on LPP are [Frobenius, 1897] and [Kantor, 1897]. Since much effort has been devoted to this type of problem, there have been several excellent survey papers such that [Marcus, 1962], [Marcus, 1971], [Grone, 1976], and so on. Since Grone's thesis was published, this topic have been studied by many authors. And various techniques have been introduced for the researches of this topic.

Let $\mathcal{M}$ be a space of matrices. Linear preserver problems usually fall into one of four categories.

(i) Suppose that P is a certain property of matrices. Characterize those linear operators T on $\mathcal{M}$ that preserve property P in the sense that

$$T(X) \text{ satisfies } P \text{ whenever } X \text{ satisfies } P.$$

For example, let $\mathcal{M}$ be the set of all $n \times n$ complex matrices, and let P be nonsingularity. Then the problem is to classify all linear maps T satisfy

$$T(X) \text{ is nonsigular whenever } X \text{ is nonsingular.}$$

The answer is given in [Marcus, Purves, 1959]. The map T will preserve nonsingularity if and only if there exist invertible $n \times n$ complex matrices M and N such that

$$T(X) = MXN \quad \text{for all } X \in \mathcal{M}, \text{ or} \tag{1.1}$$

$$T(X) = MX^tN \quad \text{for all } X \in \mathcal{M}. \tag{1.2}$$

where X^t denotes the transpose of X. It should be noted that many linear preserver problems have answers like that of (1.1) and (1.2), where M and N have to meet certain specifications.

(ii) Suppose that S is a subset of $\mathcal{M}$. Characterize those linear operators T on $\mathcal{M}$ that map S (onto) itself.

For example, let $\mathcal{M}$ be the $n \times n$ complex matrices and let S be the group of unitary matrices. Then (see [Marcus, 1959]) a linear transformation T preserves the unitary group if and only if T has the form (1.1) or (1.2) with M and N unitary.

We remark that problems of type (i) and (ii) could overlap.

(iii) Let F be a scalar-valued, vector-valued, or set-valued function on $\mathcal{M}$. Characterize those linear operators T on $\mathcal{M}$ that preserve F in the sense that $F(T(X)) = F(X)$ for all $X \in \mathcal{M}$.

For example, let $\mathcal{M}$ be the $n \times n$ complex matrices and let $F(X)$ be the spectrum of X (including multiplicities). Then a linear operator T on $\mathcal{M}$ preserves spectrum if and only if T has the form (1.1) or (1.2) with $N = M^{-1}$.

(iv) Let $\approx$ be a certain relation on $\mathcal{M}$. Characterize those linear operators on $\mathcal{M}$ that preserve $\approx$ in the sense that

$$T(X) \approx T(Y) \quad \text{whenever} \quad X \approx Y.$$

For example, let $\mathcal{M}$ be the set of all $n \times n$ complex symmetric matrices, and let $\approx$ be the unitary congruence relation, that is, $X \approx Y$ if there exists a unitary U such that $Y = UXU^t$. Then a linear operator T on $\mathcal{M}$ preserves unitary congruence if and only if there exists a unitary matrix N and $M = N^t$ such that (1.1) or (1.2) holds.

On practical grounds, for many physical problems one might want to apply certain transformations to a system so that the transformed system has a simpler structure. On the other hand, one might want to leave certain important properties, subsets or relations in the original system invariant. Since many physical systems may be described in terms of matrices, and linear transformations are the easiest and most common operations used to transform a system, results on linear preserver problems are applicable.

Besides the practical motivations mentioned above, one might study this subject pure for intellectual curiosity. For example, if one defines a linear operator on the $n \times n$ complex matrices by $T(X) = UXU^*$ where U is unitary, we observe that T preserves many important properties of matrices such as spectrum, positive definiteness, normality, rank, etc. One might ask which of the foregoing properties would be sufficient to force T to be a unitary congruence. In some cases, very mild assumptions may be enough.

Among LPP, rank-preserver problems have been the subject of research by many authors. Let $\mathbb{F}$ be an algebraically closed field. Marcus and Moyls [1] and Westwick [2] have shown that

Over $\mathbb{F}$, T preserves rank 1 if and only if T is a $(U, V)-$operator. (1.3)

Also Lautemann [3] have shown that

Over $\mathbb{F}$, T is a rank preserver if and only if T is a $(U, V)-$operator. (1.4)

But the above results are different in semirings as Boolean semiring, fuzzy semiring, etc.

In Boolean matrices, Beasley and Pullman [5] characterized those linear operators that preserve binary Boolean ranks. Kirkland and Pullman [10] had characterizations of linear operators preserving ranks of non-binary Boolean matrices. Song [11] characterized those linear operators preserving Boolean column ranks, and Song and Lee [14] had characterizations of linear operators preserving column ranks of non-binary Boolean matrices. Hwang, Kim and Song [12] characterized those linear operators that preserve maximal column ranks of non-binary Boolean matrices, and Song and Yang [20] had characterizations of linear operators preserving maximal column ranks of non-binary Boolean matrices.

In fuzzy matrices, Beasley and Pullman [7] obtained characterizations of linear operators that preserve fuzzy ranks. Song [13] characterized the column rank preserver case. Song and Park [21] had characterizations of the maximal column rank preserver.

Beasley, Gregory and Pullman [6] considered the nonnegative part $\mathbb{R}^+$ of reals $\mathbb{R}$. They obtained characterizations of linear operators that preserve ranks of matrices over $\mathbb{U}^+$, the nonnegative part of unique factorization domain $\mathbb{U}$ in $\mathbb{R}$. But the characterizations preserving column ranks and maximal column ranks are hard subjects, so they are open until now. Here are some partial responds. Beasley and Song [9] obtained characterizations of linear operators that preserve column ranks of matrices over $\mathbb{Z}^+$, the nonnegative part of integers. Song and Hwang [16] characterized those linear operators that preserve spanning column ranks of matrices over $\mathbb{S}^+$, the nonnegative part of unique factorization domain $\mathbb{S}$(a subset of $\mathbb{R}$) which contains only one unit. Also Song [17] had characterizations of linear operators preserving maximal column ranks of matrices over $\mathbb{Z}^+$.

Max algebra has been a great deal of interest by many authors since this system allows one to express in a linear fashion, phenomena that are nonlinear in the conventional algebra. It has applications in many diverse areas such as parallel computation, transportation networks and scheduling. We refer to [15 and 18] for a description of such systems and their applications.

Bapat, Pati and Song [19] exhibited characterizations of the linear operators that preserve several invariants of matrices over max algebra. But they did not deal with column rank and maximal column rank preserver over max algebra.

In this dissertation we study on the column rank and maximal column rank of matrices over max algebra. Consequently, in section 5, we analyze the relationships between rank and column rank and also obtain characterizations of the linear operators that preserve the column ranks of matrices over max algebra. In section 6, we investigate the relationships between rank and maximal column rank of matrices over max algebra and also extend the study on known properties of linear operators preserving the rank of matrices over max algebra carry over to linear operators preserving maximal column ranks.

2 Preliminaries and basic results

2.1 Boolean algebra and definitions of ranks

Let $\mathbb{B} = \{0, 1\}$ be the (binary) Boolean algebra equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$. Let $\mathcal{M}_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ matrices with entries in $\mathbb{B}$. Then the usual definition for adding and multiplying matrices over fields are applied to Boolean matrices as well.

An $n \times n$ Boolean matrix A is said to be *invertible* if there exists some $X \in \mathcal{M}_{n,n}(\mathbb{B})$ such that $AX = XA = I_n$, where I_n is the $n \times n$ identity matrix. It is well-known ([5]) that the permutation matrices are the only invertible matrices in $\mathcal{M}_{n,n}(\mathbb{B})$ and $A^{-1} = A^t$ when A is invertible.

Definition 2.1. ([5]) Let A be a nonzero $m \times n$ Boolean matrix. If there is the least integer k for which there exist $m \times k$ and $k \times n$ Boolean matrices B and C with $A = BC$, then we call that A has *Boolean rank*(or *rank*) k and denote $b(A) = k$. The Boolean rank of the zero matrix is zero.

A *Boolean vector space* $\mathcal{V}$ is any subset of $\mathbb{B}^m[= \mathcal{M}_{m,1}(\mathbb{B})]$ containing 0 which is closed under addition. If $\mathcal{V}$ and $\mathcal{W}$ are vector space with $\mathcal{V} \subseteq \mathcal{W}$, then $\mathcal{V}$ is called a *subspace* of $\mathcal{W}$. We identity $\mathcal{M}_{m,n}(\mathbb{B})$ with $\mathbb{B}^{mn}$ in the usual way when we discuss it as a Boolean vector space and consider its subspaces.

Let $\mathcal{V}$ be a Boolean vector space. If S is a subset of $\mathcal{V}$, then $< S >$ denotes the intersection of all subspaces of $\mathcal{V}$ containing S, which is a subspace of $\mathcal{V}$ too, called the *subspace generated by* S. If $S = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_p\}$, then $< S >= \{\sum_{i=1}^{p} x_i \boldsymbol{s_i} : x_i \in \mathbb{B}\}$ is the set of all linear combinations of the elements in S. In particular, $\langle \phi \rangle = \{0\}$. Define the *dimension* of $\mathcal{V}$, written as $\dim(\mathcal{V})$, to be the minimum of the cardinalities of all subsets S of $\mathcal{V}$ generating $\mathcal{V}$. We call a generating set of cardinality equal to $\dim(\mathcal{V})$ a *basis* of $\mathcal{V}$. It is well known that every Boolean vector space $\mathcal{V}$ has only one basis (see [5]).

A subset of $\mathcal{V}$ is called *linearly independent* if none of its members is a linear combination of the others. Evidently every basis is independent. The subspace of $\mathbb{B}^m$ generated by the columns of an $m \times n$ Boolean matrix A is called the *column space* of A, and denoted $< A >$.

Definition 2.2. ([8, 11]) Let A be any $m \times n$ Boolean matrix. Then the *column rank* of A is defined by the dimension of the column space of A, and denoted $c(A)$.

It follows that $0 \leq b(A) \leq c(A) \leq \min(m, n)$ for any $A \in \mathcal{M}_{m,n}(\mathbb{B})$.

Beasley and Pullman showed the following:

Lemma 2.3. ([11]) *Let $\mu(\mathbb{B}, m, n)$ be the largest integer k such that for all $m \times n$ Boolean matrices, $r(A) = c(A)$ if $r(A) \leq k$. Then*

$$\mu(\mathbb{B}, m, n) = \begin{cases} 1 & \textit{if } \min(m, n) = 1; \\ 3 & \textit{if } m \geq 3, \textit{ and } n = 3; \\ 2 & \textit{otherwise.} \end{cases}$$

Definition 2.4. ([12]) Let A be any $m \times n$ Boolean matrix. Then the *maximal column rank* of A is defined by the maximal number of the columns of A which are linearly independent, and denoted $mc(A)$.

It follows that

$$0 \leq b(A) \leq c(A) \leq mc(A) \leq \min(m,n)$$

for any $A \in \mathcal{M}_{m,n}(\mathbb{B})$.

Let $\alpha(\mathbb{B},m,n)$ be the largest integer k such that for all $m \times n$ Boolean matrices, $c(A) = mc(A)$ if $c(A) \leq k$ and there is at least one $m \times n$ Boolean matrix A with $c(A) = k$. Hwang and Song showed the following:

Lemma 2.5. ([12]) *For $m \times n$ Boolean matrices, we have the values of α as follows;*

$$\alpha(\mathbb{B},m,n) = \begin{cases} 1 & \text{if } \min(m,n) = 1; \\ 3 & \text{if } m \geq 3, \text{ and } n = 3; \\ 4 & \text{if } m \geq 3, \text{ and } n = 4; \\ 2 & \text{otherwise.} \end{cases}$$

2.2 Max algebra and definition of rank

The *max algebra* consists of the set $\mathbb{R}_{\max}$, where $\mathbb{R}_{\max}$ is the set of non-negative real numbers, equipped with two binary operations, denoted by $\oplus$ and $\cdot$ (and to be referred to as addition and multiplication over the max algebra), respectively. The operations are defined as follows:

$$a \oplus b = max(a,b) \quad \text{and} \quad a \cdot b = ab.$$

That is, their sum is the maximum of a and b and their product is the usual product. We denote $a_1 \oplus \cdots \oplus a_n$ by $\oplus_{i=1}^{n} a_i$.

The (i,j)th entry of a matrix A is denoted by a_{ij} or $A(i,j)$. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices over $\mathbb{R}_{\max}$, then the sum of A and B is denoted by $A \oplus B$, which is the $m \times n$ matrix with $a_{ij} \oplus b_{ij}$ as its (i,j)th entry. If $c \in \mathbb{R}_{\max}$, then cA is the matrix $[ca_{ij}]$. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then their product is denoted by $A \otimes B$, which is the $m \times p$ matrix with $max_r\{a_{ir}b_{rj}\}$ as its (i,j)th entry. For $m \times n$ matrices A and B, $A \geq B$ means $a_{ij} \geq b_{ij}$ for all i, j. The identity matrix of an appropriate order is denoted by I. And the transpose of A, denoted by A^t, is defined in the usual way, i.e. $A^t(i,j) = A(j,i)$ for all i, j.

For example, if

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix},$$

then we have

$$A \oplus B = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 4 \\ 3 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A \otimes B = \begin{bmatrix} 4 & 2 & 3 \\ 12 & 4 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

It can be easily proved that the product $\otimes$ is associative and that it distributes over the sum $\oplus$.

Let S be a subset of $(\mathbb{R}_{\max})^n$, where n is a positive integer. Then $span(S)$ is the set defined as follows:

$$span(S) = \{\boldsymbol{x} | \boldsymbol{x} = \oplus_{i=1}^{n} \alpha_i \boldsymbol{x}_i \quad \text{with} \quad \boldsymbol{x}_i \in S \quad \text{and} \quad \alpha_i \in \mathbb{R}_{\max}\}.$$

A *semimodule* over $\mathbb{R}_{\max}$ generated by S is the $span(S)$. The elements of a semimodule are called *vectors*. Given a semimodule $\mathcal{V}$, if there exists a finite subset S of $\mathcal{V}$ such that $\mathcal{V} = span(S)$, then $\mathcal{V}$ is called a *finitely generated semimodule*.

Let $\mathcal{V}$ be a semimodule over $\mathbb{R}_{\max}$. The set of vectors $\{\boldsymbol{v}_i | \, i \in I\}$ is called a *weak basis* of $\mathcal{V}$ if $span(\{\boldsymbol{v}_i | \, i \in I\}) = \mathcal{V}$ and no proper subset of $\{\boldsymbol{v}_i | \, i \in I\}$ span $\mathcal{V}$.

Definition 2.6. A set S of vectors in a semimodule $\mathcal{V}$ is called *linearly dependent* if there exists $\boldsymbol{x} \in S$ such that $\boldsymbol{x} \in span(S - \{\boldsymbol{x}\})$. A set S of vectors in $\mathcal{V}$ is called *linearly independent* if it is not linearly dependent.

Thus an independent set cannot contain a zero vector. Also a weak basis of a semimodule is linearly independent.

In the followings, all matrices and semimodules are assumed to be defined over max algebra.

Theorem 2.7. *Let $\mathcal{V}$ be a semimodule. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two weak bases of $\mathcal{V}$. Then for $\boldsymbol{x} \in \mathcal{B}_1$ there exists a unique $\boldsymbol{y}_x \in \mathcal{B}_2$ such that $\boldsymbol{y}_x = \alpha \boldsymbol{x}$ for some nonzero $\alpha \in \mathbb{R}_{\max}$, and for $\boldsymbol{y} \in \mathcal{B}_2$ there exists a unique $\boldsymbol{x}_y \in \mathcal{B}_1$ such that $\boldsymbol{x}_y = \beta \boldsymbol{y}$ for some nonzero $\beta \in \mathbb{R}_{\max}$. In particular $|\mathcal{B}_1| = |\mathcal{B}_2|$.*

Proof. Let $\boldsymbol{x}$ be any vector in $\mathcal{B}_1$. Since $\mathcal{B}_2$ is a weak basis of $\mathcal{V}$, there exist $c_1, c_2, \cdots, c_n \in \mathbb{R}_{\max}$ and vectors $\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_n \in \mathcal{B}_2$ such that

$$\boldsymbol{x} = \oplus_{i=1}^{n} c_i \, \boldsymbol{y}_i. \tag{2.1}$$

Since each $\boldsymbol{y}_i$ is a linear combination of a finite number of elements of $\mathcal{B}_1$, we can assume that there exists a set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_m\} \subseteq \mathcal{B}_1$ such that

$$\boldsymbol{y}_i = \oplus_{j=1}^{m} a_i{}^j\, \boldsymbol{x}_j, \tag{2.2}$$

where some of a_i^j might be zero. From (2.1) and (2.2), we have

$$\boldsymbol{x} = \max\{c_i\, a_i^j\, \boldsymbol{x}_j \;:\; i = 1, \cdots, n;\; j = 1, \cdots, m;\; \boldsymbol{x}_j \in \mathcal{B}_1.\} \tag{2.3}$$

Since $\mathcal{B}_1$ is linearly independent it follows that $\boldsymbol{x} = \boldsymbol{x}_{j_0}$, for some j_0. (We note here that there exists exactly one such j_0 for otherwise $\mathcal{B}_1$ will be linearly dependent.) Now we claim that

$$\text{at least one element of } \{c_i\, a_i^{j_0} \;:\; i = 1, \cdots, n\} \text{ is } 1. \tag{2.4}$$

For, suppose that (2.4) does not hold. Then we have that no $c_i\, a_i^{j_0}$ can be more than 1, because the right hand side of (2.3) evaluates to a vector strictly greater than $\boldsymbol{x}_{j_0} = \boldsymbol{x}$, which is the left hand side of (2.3), a contradiction. So, let $c_i\, a_i^{j_0} < 1$, for $i = 1, \cdots, n$. Thus (2.3) reduces to

$$\boldsymbol{x}_{j_0} = \boldsymbol{x} = \max\{c_i\, a_i^j\, \boldsymbol{x}_j \;:\; i = 1, \cdots, n;\; j = 1, \cdots, m;\; j \neq j_0\}. \tag{2.5}$$

The above equation implies that $\boldsymbol{x}_{j_0}$ is a linear combination of $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m\} - \{\boldsymbol{x}_{j_0}\}$, which is a contradiction to the fact that $\mathcal{B}_1$ is a linearly independent set. Thus we have $c_i\, a_i^{j_0} = 1$, for some i and j_0. Hence (2.4) holds.

From (2.4), we may assume that $c_{i_0}\, a_{i_0}^{j_0} = 1$, and hence $c_{i_0} \neq 0$. Thus, using (2.3), we obtain that

$$c_{i_0}\, a_{i_0}^{j_0}\, \boldsymbol{x}_{j_0} = \boldsymbol{x} \geq c_i\, a_i^j\, \boldsymbol{x}_j$$

for $i = 1, \cdots, n$ and $j = 1, 2, \cdots, m$. In particular,

$$c_{i_0}\, a_{i_0}^{j_0}\, \boldsymbol{x}_{j_0} \geq c_{i_0}\, a_{i_0}^{j}\, \boldsymbol{x}_j \tag{2.6}$$

for $j = 1, \cdots, m$. Multiplying $1/c_{i_0}$ in both sides of (2.6), we have that

$$a_{i_0}^{j_0}\, \boldsymbol{x}_{j_0} \geq a_{i_0}^{j}\, \boldsymbol{x}_j$$

for $j = 1, \cdots, m$, and hence

$$\boldsymbol{y}_{i_0} = \oplus_j a_{i_0}^{j}\, \boldsymbol{x}_j = a_{i_0}^{j_0}\, \boldsymbol{x}_{j_0} = a_{i_0}^{j_0}\, \boldsymbol{x}.$$

Thus for $\boldsymbol{x} \in \mathcal{B}_1$ there exists $\boldsymbol{y}_x \in \mathcal{B}_2$, that is $\boldsymbol{y}_{i_0}$ and $\alpha = a_{i_0}^{j_0}$ such that $\boldsymbol{y}_x = \alpha\, \boldsymbol{x}$. The vector $\boldsymbol{y}_x$ is unique, because if there exist two vectors in $\mathcal{B}_2$ such that $\boldsymbol{y}_1 = \alpha_1\, \boldsymbol{x}$ and $\boldsymbol{y}_2 = \alpha_2\, \boldsymbol{x}$, then $\boldsymbol{y}_1$ is a scalar multiple of $\boldsymbol{y}_2$ and this is a contradiction to the fact that $\mathcal{B}_2$ is linearly independent.

Similarly it can be shown that for each $\boldsymbol{y} \in \mathcal{B}_2$ there exists a unique vector $\boldsymbol{x}_y$ in $\mathcal{B}_1$ such that $\boldsymbol{x}_y = \beta \boldsymbol{y}$, for some $\beta \neq 0$.

It is evident from the above discussion that the function $f : \mathcal{B}_1 \to \mathcal{B}_2$ defined by $f(\boldsymbol{x}) = \boldsymbol{y}_x$ is a bijection. Thus the proof is complete. □

This theorem shows that all weak basis for a semimodule $\mathcal{V}$ have the same number of vectors, even if there are many different weak basis for a semimodule. Thus the cardinality of a weak basis is called the *weak dimension* of the semimodule $\mathcal{V}$, denoted by $\dim_w(\mathcal{V})$.

Let $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ denote the set of all $m \times n$ matrices with entries from $\mathbb{R}_{\max}$. We identity $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ with $(\mathbb{R}_{\max})^{mn}$ in the usual way when we discuss it as a semimodule.

Let $\Delta = \{(i, j) |\, 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and by $\mathbb{E}$ we denote the standard weak basis of $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$, that is

$$\mathbb{E} = \{E_{ij} : i = 1, \cdots, m,\ j = 1, \cdots, n\},$$

where the (i, j)th entry of the $m \times n$ matrix E_{ij} is 1 and all other entries are zero. Then $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ is a finitely generated simimodule with mn as the weak dimension.

Definition 2.8. ([19]) Let A be an $m \times n$ matrix over max algebra. Then the *rank* or *factor rank* of A, denoted by $r(A)$, is the number defined as the least integer k for which there exist $m \times k$ and $k \times n$ matrices B and C with $A = B \otimes C$. The rank of the zero matrix is zero.

We can easily obtain that $0 \le r(A) \le \min(m,n)$.

A square matrix A is called *invertible* if there exists a matrix B such that $A \otimes B = B \otimes A = I$. A square matrix A is called *monomial* if it has exactly one nonzero element in each row and column.

Lemma 2.9. *Let A be a square matrix over $\mathbb{R}_{\max}$. Then A is invertible if and only if A is monomial.*

Proof. Let $A = [a_{ij}]$ be an invertible matrix over $\mathbb{R}_{\max}$. Define a square matrix $B = [b_{ij}]$ by

$$b_{ij} = \begin{cases} 0 & \text{if } a_{ij} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Then B is an invertible Boolean matrix. It is well known [5] that the matrix B is a permutation matrix. Thus A is monomial.

The converse follows from the definition of monomial. □

3 Linear operator preserving Boolean ranks over Boolean algebra

3.1 Rank-preserving operator

Let $\mathcal{V}$ and $\mathcal{W}$ be two Boolean vector spaces. Then a mapping $T : \mathcal{V} \to \mathcal{W}$ which preserves sums and 0 is said to be a (Boolean) *linear transformation*. If $\mathcal{V} = \mathcal{W}$, the *operator* is used instead of transformation.

Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$. Then we say that T is a

(1) (U, V)-*operator* if there exist invertible Boolean matrices U and V such that $T(A) = UAV$ for all A in $\mathcal{M}_{m,n}(\mathbb{B})$, or $m = n$ and $T(A) = U\,A^t\,V$ for all A in $\mathcal{M}_{m,n}(\mathbb{B})$;

(2) *rank preserver* if $b((T(A)) = b(A)$ for all A in $\mathcal{M}_{m,n}(\mathbb{B})$;

(3) *rank-1 preserver* if $b((T(A)) = 1$ whenever $b(A) = 1$ for A in $\mathcal{M}_{m,n}(\mathbb{B})$.

Let $\mathbb{F}$ be an algebraically closed field. Then Marcus, Moyls [1] and Westwick [2] showed that if T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{F})$ and T maps rank-1 matrices to rank-1 matrices(*i.e.* T preserves rank-1 matrices), then (and only then) T is a (U, V)-operator. This result does not hold for the (binary) Boolean case.

The following example shows that not all rank-1-preserving operators T are of the form $T(X) = UXV$ or $T(X) = UX^tV$ for some invertible Boolean matrices U and V, contrary to the situation for algebraically closed fields.

Example 3.1. Let

$$T\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = (b+e+c+f)\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 & d \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, T is a linear operator and $b(T(X)) = 1$ whenever $b(X) = 1$ (in fact whenever $X \neq 0$). If there existed invertible Boolean matrices U and V such that $T(X) = UXV$ for all $X \in \mathcal{M}_{2,3}(\mathbb{B})$, then for $j = 1, 2, 3$, we have $T(E_{1j}) = \mathbf{u}\mathbf{v}_j^t$, where $\mathbf{u}$ is the first column of U and $\mathbf{v}_j$ is the jth column of V. But

$$T(E_{11}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1, 0, 0] \text{ and } T(E_{12}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1, 1, 1],$$

and hence

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which is a contradiction. □

Suppose that U and V are invertible members of $\mathcal{M}_{m,m}(\mathbb{B})$ and $\mathcal{M}_{n,n}(\mathbb{B})$ respectively, and T is the operator on $\mathcal{M}_{m,n}(\mathbb{B})$ defined by $T(X) = UXV$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B})$. Clearly T is linear. Moreover $T(X)$ has rank 1 whenever X has rank 1. For, suppose that X has rank 1, so that $X = \mathbf{a}\mathbf{b}^t$ where $\mathbf{a} \neq 0$ and $\mathbf{b} \neq 0$. Then $T(X) = U\mathbf{a}\mathbf{b}^tV = (U\mathbf{a})(V^t\mathbf{b})^t$, and since U and V^t are invertible, neither $U\mathbf{a}$ nor $V^t\mathbf{b}$ is 0, so $T(X)$ has rank 1. It follows that all Boolean (U, V)-operators are rank-1 preservers.

Theorem 3.2. ([5]) *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$, then the following statements are equivalent;*

(1) *T is invertible and preserves rank* 1;

(2) *T preserves the ranks* 1 *and* 2 *and preserves the dimension of all rank*-1 *spaces;*

(3) *T is a (U, V)-operator.*

Theorem 3.3. ([5]) *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$, then the following statements are equivalent;*

(1) *T is a rank preserver;*

(2) *T preserves the ranks* 1 *and* 2*;*

(3) *T is a (U, V)-operator.*

3.2 Column rank-preserving operator

Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$. Then we say that T is a

(1) *congruence operator* if there exist invertible Boolean matrices U and V such that $T(A) = UAV$ for all A in $\mathcal{M}_{m,n}(\mathbb{B})$;

(2) *transposition operator* if $m = n$ and $T(A) = A^t$, the transpose matrix of A, for all A in $\mathcal{M}_{m,n}(\mathbb{B})$;

(3) *column rank preserver* if $c(T(A)) = c(A)$ for all A in $\mathcal{M}_{m,n}(\mathbb{B})$.

(4) *preserves column rank k* if $c(T(A)) = k$ whenever $c(A) = k$ for all A in $\mathcal{M}_{m,n}(\mathbb{B})$.

Using the function μ in Lemma 2.3, we can apply the results for ranks 1 and 2 in Theorem 3.2 to those for column ranks 1 and 2. Thus we obtain the following Theorem 3.4.

Theorem 3.4. ([11]) *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$, then the following statements are equivalent;*

(1) *T is invertible and preserves column rank 1;*

(2) *T preserves the column ranks 1 and 2 and preserves the dimension of all column-rank-1 spaces;*

(3) *T is in the group of operators generated by congruence and transposition operators.*

But some rank preservers do not preserve any column ranks as shown in Lemma 3.6 below.

Example 3.5. Consider a Boolean matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then none of the columns of A are linear combinations of the others. So $c(A) = 4$. But $c(A^t) = 3$ because the fourth column of A^t is the sum of the first and second columns of A^t, and $b(A) = b(A^t) \leq c(A^t) = 3$. By Lemma 2.3, $b(A)$ is greater than 2. So $b(A) = 3$. This shows that, for $A \in \mathcal{M}_{m,m}(\mathbb{B})$ with $m \geq 4$, $b(A) < c(A)$ is possible. □

Lemma 3.6. ([11]) *If T is a transposition operator on $\mathcal{M}_{m,m}(\mathbb{B})$ with $m \geq 4$, then T does not preserve column rank c for $c \geq 3$ but preserves all Boolean ranks.*

Theorem 3.7. ([11]) *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ with $n \geq m \geq 4$, then the following statements are equivalent;*

(1) *T is a column rank preserver;*

(2) *T preserves the column ranks $1, 2$ and 3;*

(3) *T is a congruence operator;*

(4) *T is bijective and preserves column ranks 1 and 3.*

3.3 Maximal column rank-preserving operator

If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$, then we say that T is a *maximal column rank preserver* if $mc(T(A)) = mc(A)$ for all A in $\mathcal{M}_{m,n}(\mathbb{B})$. T *preserves maximal column rank k* if $mc(T(A)) = k$ whenever $mc(A) = k$ for all A in $\mathcal{M}_{m,n}(\mathbb{B})$.

Using the function α in Lemma 2.5, we can apply the results for ranks 1 and 2 in Theorem 3.2 to those for maximal column ranks 1 and 2. Thus we obtain the following Theorem 3.8.

Theorem 3.8. ([12]) *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$, then the following statements are equivalent;*

(1) *T is invertible and preserves maximal column rank 1;*

(2) *T preserves maximal column ranks 1 and 2 and preserves the dimension of all maximal-column-rank-1 spaces;*

(3) *T is in the group of operators generated by congruence and transposition operators.*

But some rank preservers do not preserve any maximal column ranks as shown in Lemma 3.9 below.

Consider the Boolean matrix A in Example 3.5. Since all columns of A are linearly independent, we have $mc(A) = 4$. But $mc(A^t) = 4$ because the fourth column of A^t is the sum of the first and second columns of A^t and the first three columns of A^t are linearly independent. And $b(A) = b(A^t) \leq mc(A^t) = 3$. By Lemma 2.5, $b(A)$ is greater than 2. So $b(A) = 3$. This shows that, for $A \in \mathcal{M}_{m,m}(\mathbb{B})$ with $m \geq 4$, $b(A) < mc(A)$ is possible.

Lemma 3.9. ([12]) *If T is a transposition operator on $\mathcal{M}_{m,m}(\mathbb{B})$ with $m \geq 4$, then T does not preserve maximal column rank r for $r \geq 3$ but preserves all Boolean ranks.*

Theorem 3.10. ([12]) *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ with $n \geq m \geq 4$, then the following statements are equivalent;*

(1) *T is a maximal column rank preserver;*

(2) *T preserves the maximal column ranks 1, 2 and 3;*

(3) *T is a congruence operator;*

(4) *T is bijective and preserves maximal column ranks 1 and 3.*

4 Linear operator preserving rank over max algebra

4.1 Rank-1 preserving operator

If $\mathcal{V}$ is a semimodule over $\mathbb{R}_{\max}$, a mapping $T : \mathcal{V} \to \mathcal{W}$ is called a *linear transformation* if T has the following two properties:

(1) $T(0) = 0$ and

(2) $T(\alpha\boldsymbol{x} \oplus \beta\boldsymbol{y}) = \alpha T(\boldsymbol{x}) \oplus \beta T(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$, $\alpha, \beta \in \mathbb{R}_{\max}$.

If $\mathcal{V} = \mathcal{W}$, the *operator* is used instead of transformation. By $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ we denote the semimodule of all $m \times n$ matrices with entries from $\mathbb{R}_{\max}$.

When T is a linear transformation on a semimodule $\mathcal{V}$, its behavior on the weak basis of $\mathcal{V}$ determines the behavior of T completely. This proves:

Lemma 4.1. *Let $\mathcal{V}$ be a finitely generated semimodule. For every linear transformation T on $\mathcal{V}$*

$$dim_w(T(\mathcal{V})) \leq dim_w(\mathcal{V}).$$

Proof. Let $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n\}$ be a weak basis of $\mathcal{V}$. Then $\{T(\boldsymbol{x}_1), \cdots, T(\boldsymbol{x}_n)\}$ span $T(\mathcal{V})$. Thus by Theorem 2.7, the proof is complete. □

Let $\mathcal{V}$ and $\mathcal{W}$ be two semimodules over $\mathbb{R}_{\max}$. A linear transformation $T : \mathcal{V} \to \mathcal{W}$ is called *injective* if $T(\boldsymbol{x}) = T(\boldsymbol{y})$ implies $\boldsymbol{x} = \boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$. The map T is called *surjective* if $T(\mathcal{V}) = \mathcal{W}$. The map T is called *invertible* if it is both injective and surjective.

Lemma 4.2. *Let $\mathcal{V}$ and $\mathcal{W}$ be finitely generated semimodules. If $T : \mathcal{V} \to \mathcal{W}$ is injective, then*

$$\dim_w(T(\mathcal{V})) = \dim_w(\mathcal{V}).$$

Proof. Let $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n\}$ be a weak basis of $\mathcal{V}$. Then $S = \{T(\boldsymbol{x}_1), \cdots, T(\boldsymbol{x}_n)\}$ spans $T(\mathcal{V})$. Suppose that S is linearly dependent and let $T(\boldsymbol{x}_n) = \oplus_{i=1}^{n-1}\alpha_i T(\boldsymbol{x}_i)$. Then T being injective we get $\boldsymbol{x}_n = \oplus_{i=1}^{n-1}\alpha_i \boldsymbol{x}_i$, a contradiction to the fact that $\{\boldsymbol{x}_i : i = 1, \cdots, n\}$ is linearly independent. Thus S is a weak basis of $T(\mathcal{V})$. □

Henceforth we will be talking about finitely generated semimodules only.

Lemma 4.3. *Let $T : \mathcal{V} \to \mathcal{W}$ be a surjective linear transformation. Then T is invertible if and only if T preserves the weak dimension of every sub-semimodule of $\mathcal{V}$.*

Proof. If T is invertible, then T preserves the weak dimension of every sub-semimodule of $\mathcal{V}$ by Lemma 4.2. Conversely, if T preserves the weak dimension of every sub-semimodule of $\mathcal{V}$, then T must be injective, for otherwise, there exist some $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathcal{V}$ such that $\boldsymbol{x} \neq \boldsymbol{y}, T(\boldsymbol{x}) = T(\boldsymbol{y})$, which would imply that the weak dimension of the sub-semimodule generated by $\boldsymbol{x}, \boldsymbol{y}$ is not preserved under T. □

Corollary 4.4. *If $T : \mathcal{V} \to \mathcal{V}$ is a linear operator, then the following statements are equivalent;*

(1) *T is invertible;*

(2) *T preserves the weak dimension of every sub-semimodules of $\mathcal{V}$;*

(3) *T permutes the weak basis of $\mathcal{V}$, with some nonzero scalar multiplication.*

Proof. The equivalence of (1) and (2) follows from Lemma 4.3. To show that (1) and (3) are equivalent, note that, if T is invertible and $\{\boldsymbol{x}_i : i = 1, \cdots, n\}$ is a weak basis of $\mathcal{V}$, then $\{T(\boldsymbol{x}_i) : i = 1, \cdots, n\}$ is a weak basis of $T(\mathcal{V}) = \mathcal{V}$. Now applying Theorem 2.7, we see that (3) is true for T. On the other hand, if T is a linear operator satisfying (3), then naturally $T(\boldsymbol{x}) = A\boldsymbol{x}$, where A is a monomial. Clearly T is invertible. □

We say that a linear operator T on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ *preserves rank* k, if for all $m \times n$ matrix A over max algebra, $r(A) = k$ implies $r(T(A)) = k$. Suppose T is an invertible operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$. We know, by Corollary 4.4, that $T(E_{ij}) = \alpha_{ij}\, E_{pq}$, for some nonzero $\alpha_{ij} \in \mathbb{R}_{\max}$, where p and q are the first and second coordinate of $T(E_{ij})$, respectively, which are depended on the coordinates i and j. For this invertible linear operator T, define the $m \times n$ array τ whose (i, j)th entry is $\tau(i, j) = \alpha_{ij}\,(p, q)$ for all i, j. The array τ is called the *representation* of T.

Lemma 4.5. *If T is an invertible linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ that preserves the rank of every rank-1 matrix and τ is the representation of T, then there exist permutations (matrices corresponding to these permutations) U and V of $1, \cdots, m$ and $1, \cdots, n$, respectively and there exist invertible diagonal matrices C and D such that*

(a) $\tau(i, j) = C(i, i)\, D(j, j)\ (\,U(i),\, V(j)\,)$ *for all $(i, j) \in \Delta$ and in this case*

$$T(A) = C \otimes U \otimes A \otimes V \otimes D$$

holds for all $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$ or

(b) $m = n$ *and* $\tau(i, j) = C(i, i)\, D(j, j)\ (\,V(j),\, U(i)\,)$ *for all $(i, j) \in \Delta$ and* *in this case*

$$T(A) = C \otimes U \otimes A^t \otimes V \otimes D$$

holds for all $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$.

Proof. We follow 8 steps for this proof.

Step 1. Any two entries in the same row (or column) of τ satisfy exactly one of the followings:

(i) The entries have a common first coordinate.

(ii) The entries have a common second coordinate.

Proof of Step 1. Suppose $\tau(i,j) = \alpha_{ij}(p,q)$ and $\tau(i,k) = \alpha_{ik}(r,s)$ for $j \neq k$. Then $T(E_{ij}) = \alpha_{ij}E_{pq}$, and $T(E_{ik}) = \alpha_{ik}E_{rs}$. Since $E_{ij} \oplus E_{ik}$ is a rank one matrix and T preserves the rank of rank one matrices, $\alpha_{ij}E_{pq} \oplus \alpha_{ik}E_{rs}$ must be a rank one matrix. But this is possible only if either $p = r$ or $q = s$. Suppose both $p = r$ and $q = s$ occurs simultaneously. Then

$$T(E_{ij}) = \alpha_{ij}E_{pq} = T\left(\frac{\alpha_{ij}}{\alpha_{ik}} E_{ik}\right)$$

and thus T is not injective, a contradiction. Hence we have $p = r$ or $q = s$, but not both.

Step 2. If two entries in the ith row of τ have a common first coordinate, then each entry in the ith row has the same first coordinate for all i.

Proof of Step 2. Let $T(E_{ij}) = \alpha_{ij}E_{pq}$, $T(E_{ik}) = \alpha_{ik}E_{pr}$ and $T(E_{il}) = \alpha_{il}E_{st}$ for distinct i, k and l. Suppose $s \neq p$. Comparing $T(E_{ik})$ and $T(E_{il})$ we get $t = r$. Comparing $T(E_{ij})$ and $T(E_{il})$ we get $t = q$. Thus we have

$r = q$, and hence both the first coordinate and the second coordinate of $T(E_{ij})$ and $T(E_{ik})$ are the same and this is a contradiction to Step 1. Thus we have $s = p$.

Step 3. If the entries in the ith row of τ have a common first coordinate, then the second coordinates constitute a permutation V of $\{1, \cdots, n\}$.

Proof of Step 3. It follows from Step 1.

Step 4. If the entries in the ith row of τ have a common first coordinate, then the entries in the lth row of τ also have a common first coordinate, for any $l = 1, \cdots, m$.

Proof of Step 4. With reference to Step 1, assume that $T(E_{lj}) = \alpha_{lj}E_{uv}$ and $T(E_{lk}) = \alpha_{lk}E_{wv}$ with $u \neq w$. Also assume that $T(E_{ij}) = \alpha_{ij}E_{pq}$ and $T(E_{ik}) = \alpha_{ik}E_{pr}$ with $q \neq r$. Since $E_{ij} \oplus E_{ik} \oplus E_{lj} \oplus E_{lk}$ is a rank one matrix, $T(E_{ij} \oplus E_{ik} \oplus E_{lj} \oplus E_{lk})$ should be a rank one matrix. But

$$T(E_{ij} \oplus E_{ik} \oplus E_{lj} \oplus E_{lk}) = \alpha_{ij}E_{pq} \oplus \alpha_{ik}E_{pr} \oplus \alpha_{lj}E_{uv} \oplus \alpha_{lk}E_{wv},$$

can never be a rank one matrix, which is a contradiction to the fact that T preserves rank one.

Step 5. If the entries in the ith row of τ have a common first coordinate, say, p, then the entries in the lth row $(i \neq l)$ have a common first coordinate, say q, then $p \neq q$.

Proof of Step 5. Assume that $p = q$. We know by Step 3 that $T(E_{ij}) = \alpha_{ij}E_{p,V(j)}$, $j = 1, \cdots, n$. Suppose that $T(E_{l1}) = \alpha_{l1}E_{qh} = \alpha_{l1}E_{ph}$, for some $h \in \{1, \cdots, n\}$. Since V is a permutation of $\{1, \cdots, n\}$, there exists r such that $V(r) = h$. But $T(E_{ir}) = \alpha_{ir} E_{p,V(r)} = \alpha_{ir} E_{ph}$ by Step 3. Thus we get

$$T(E_{l1}) = \alpha_{l1} E_{p,h} = T\left(\frac{\alpha_{l1}}{\alpha_{ir}} E_{ir}\right)$$

and this is a contradiction to the fact that T is injective.

Step 6. Suppose that the entries of the ith row of τ have a common first coordinate. Let V and V^* be the permutation of $\{1, \cdots, n\}$ corresponding the second coordinates of the ith and the lth row of τ, respectively (guaranteed by Step 3 and Step 4). Then $V = V^*$.

Proof of Step 6. Follows directly from Step 1. In fact, by Step 1, $T(E_{i1})$ and $T(E_{l1})$ should have either the same first coordinate or the same second coordinate. But the first coordinates are different by Step 5. Thus the second coordinates must agree. The situations are the same for $T(E_{is})$ and $T(E_{ls})$, $s = 2, \cdots, n$. Thus the permutations V and V^* corresponding to the second coordinates of the ith and the lth low must be the same.

Step 7. Suppose that the entries of the ith row have a common first coordinate. Then the entries of the jth column have a common second coordinate and the first coordinates of the entries of the jth column constitute a permutation U of $\{1, \cdots, m\}$.

Proof of Step 7. Similar to Step 3 in view of Step 5.

Step 8. Suppose that the entries of the ith row have a common first coordinate. Then we have followings:

(a) There exist permutations U of $\{1, 2, \cdots, m\}$ and V of $\{1, 2, \cdots, n\}$

such that

$$T(E_{ij}) = \alpha_{ij} E_{U(i),V(j)},\ i = 1, \cdots, m,\ j = 1, \cdots, n.$$

(b) For all $i, l \in \{1, 2, \cdots, m\}$ and all $j, k \in \{1, 2, \cdots, n\}$,

$$\frac{\alpha_{ij}}{\alpha_{ik}} = \frac{\alpha_{lj}}{\alpha_{lk}}.$$

Thus there exist diagonal matrices C and D such that $\alpha_{ij} = C(i,i)\, D(j,j)$ for all $i \in \{1, \cdots, m\}$ and $j \in \{1, \cdots, n\}$.

(c) For any $m \times n$ matrix A,

$$T(A) = C \otimes U \otimes A \otimes V \otimes D,$$

where U and V are the matrices corresponding to the permutations of $\{1, \cdots, m\}$ and $\{1, \cdots, n\}$, respectively.

Proof of Step 8.

(a). It follows from Steps 1 to 7.

(b). Since $E_{ij} \oplus E_{ik} \oplus E_{lj} \oplus E_{lk}$ is a rank one matrix, $T(E_{ij} \oplus E_{ik} \oplus E_{lj} \oplus E_{lk})$ is also a rank one matrix. Using (a), we have

$$T(E_{ij} \oplus E_{ik} \oplus E_{lj} \oplus E_{lk}) = \alpha_{ij} E_{U(i),V(j)} \oplus \alpha_{ik} E_{U(i),V(k)} \\ \oplus \alpha_{lj} E_{U(l),V(j)} \oplus \alpha_{lk} E_{U(l),V(k)}.$$

Since the matrix in the right hand side of the above equation has rank one, it follows that $\frac{\alpha_{ij}}{\alpha_{ik}} = \frac{\alpha_{lj}}{\alpha_{lk}}$. Thus if we take $C(1,1) = 1$, $D(1,1) = \alpha_{11}$, $C(i,i) = \alpha_{i1}/\alpha_{11}$ and $D(j,j) = \alpha_{1j}$ for all $i \in \{2, \cdots, m\}$ and $j = \{2, \cdots, n\}$, then we have the invertible diagonal matrices C and D.

(c). This is easy in view of (a) and (b). In fact

$$\begin{aligned} T(A) &= T\left(\oplus_{i,j} A(i,j)\, E_{ij}\right) = \oplus_{i,j} A(i,j)\, T\left(E_{ij}\right) \\ &= \oplus_{i,j} C(i,i)\, A(i,j)\, E_{U(i),V(j)}\, D(j,j) = C \otimes U \otimes A \otimes V \otimes D. \end{aligned}$$

A similar argument shows that if the entries in the first row of τ have a common second coordinate, then $m = n$ and there exist permutations (matrices) U and V of $\{1, \cdots, n\}$ and there exist invertible diagonal matrices C and D such that $T(A) = C \otimes U \otimes A^t \otimes V \otimes D$, for all $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$.

Thus the proof of the lemma 4.5 is complete. □

Now, we define a sub-semimodule of $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ whose nonzero members have rank 1 as a *rank-1-sub-semimodule.*

Lemma 4.6. *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ that preserves the weak dimension of all rank-1-sub-semimodules, then the restriction of T to the rank one matrices is injective or T reduces the rank of some rank two matrix to one.*

Proof. Let $\mathcal{M}^1 = \{A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max}) : r(A) = 1\}$. For each $B \in \mathcal{M}^1$, define $\mathcal{W}_B = \text{span}\{X \in \mathcal{M}^1 : T(X) = T(B)\}$. Note that $B \in \mathcal{W}_B$ and $dim_w(T(\mathcal{W}_B)) = 1$. Then we have two cases.

Case 1. For all $B \in \mathcal{M}^1$, $\mathcal{W}_B$ is a semimodule containing the nonzero matrices of rank one and zero matrix only. Then by the hypothesis that T preserves the weak dimension of all rank one sub-semimodules, we have $dim_w(\mathcal{W}_B) = dim_w(T(\mathcal{W}_B)) = 1$. Hence $\mathcal{W}_B =$ span $(\{B\})$. Thus T is injective.

Case 2. There exists a $B \in \mathcal{M}^1$ such that $dim_w(\mathcal{W}_B) > 1$. Then there exist X_0, $Y_0 \in \{X \in \mathcal{M}^1 : T(X) = T(B)\}$ such that $r(X_0 \oplus Y_0) = 2$. Also $T(X_0 \oplus Y_0) = T(B)$ has rank 1. Hence T reduces the rank of some rank two matrix to one. □

Corollary 4.7. *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ that*

(i) *preserves the ranks of all rank one and rank two matrices, and*

(ii) *preserves the weak dimension of all rank one sub-semimodules,*

then

(a) *T is invertible and*

(b) *T^{-1} satisfies* (i) *and* (ii).

Proof. It follows from Lemma 4.6 that T is injective on the set of all rank one matrices. Note that the weak basis $\mathbb{E}$ of $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ is a set of rank one matrices. Let $E_{ij} \in \mathbb{E}$ and C be a rank one matrix such that $T(C) = E_{ij}$. Since $C \neq 0$, we can choose a nonzero $\alpha \in \mathbb{R}_{\max}$ and $F \in \mathbb{E}$ such that $\alpha C \geq F$. Since T is a linear operator and we are dealing with nonnegative real numbers, we have $T(\alpha C) \geq T(F)$, that is $\alpha E_{ij} \geq T(F)$. Thus it follows that

$$T(F) = \beta E_{ij}$$

for some nonzero $\beta \in \mathbb{R}_{\max}$.

It is easy to see that C has exactly one nonzero entry. In fact, if C has more than one nonzero entry, then we can get a nonzero $\gamma \in \mathbb{R}_{\max}$ and an $F' \in \mathbb{E}$ with $F' \neq F$ such that $\gamma C \geq F'$. In a similar method as the above, we can get that

$$T(F') = \delta E_{ij}$$

for some nonzero δ. Thus $T\left(\frac{1}{\beta}F\right) = E_{ij} = T\left(\frac{1}{\delta}F'\right)$ whereas $\frac{1}{\beta}F \neq \frac{1}{\delta}F'$. This contradicts the fact that T is injective on the set of rank one matrices.

Thus C has to be of the form $\alpha_{ij}E_{lk}$. We remind that C is the pre-image of E_{ij}. Since T is injective, E_{ij} can not have more than one pre-images. Since $\mathbb{E}$ is a finite set we conclude that T permutes $\mathbb{E}$ with some nonzero scalar multiplication. Hence by Corollary 4.4-(3), T is invertible. The rest of the proof is trivial. □

We say that a linear operator T on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ is a *(U,V)-operator* if there exist monomials $U \in \mathcal{M}_{m,m}(\mathbb{R}_{\max})$ and $V \in \mathcal{M}_{n,n}(\mathbb{R}_{\max})$ such that either $T(A) = U \otimes A \otimes V$ for all A in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ or $m = n$, $T(A) = U \otimes A^t \otimes V$ for all A in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$.

Theorem 4.8. *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ then the followings are equivalent;*

(1) *T is invertible and preserves rank* 1*;*

(2) *T preserves ranks* 1 *and* 2 *and preserves the weak dimension of all rank-1-sub-semimodules;*

(3) *T is a (U,V)-operator.*

Proof. Lemma 4.5 shows that (1) implies (3). Corollary 4.7 shows that (2) implies (1). To show that (3) implies (2), note that (U,V)-operators are preservers of all rank (it can be seen easily, in view of the fact that $r(A)$ is the smallest integer k such that there exist k rank one matrices whose sum is A) and now apply Lemma 4.2. □

4.2 Rank-preserving operator

In this section, we obtain characterizations of the linear operators which preserve rank of matrices over max algebra.

We say that a linear operator T on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ is a *rank preserver* if T preserves all ranks.

Lemma 4.9. *Let $A, B \in \mathcal{M}_{m,n}(\mathbb{R}_{\max}), A \neq B, m > 1, n > 1$ with $r(A) = r(B) = 1$. Then*

(i) *if the number of nonzero entries in A is more than that of B, then there exists $C \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$ such that $r(A \oplus C) = 1$ and $r(B \oplus C) = 2$.*

(ii) *if the number of nonzero entries in A is equal to that of B, then there exists $C \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$ such that either $r(A \oplus C) = 1$ and $r(B \oplus C) = 2$ or $r(A \oplus C) = 2$ and $r(B \oplus C) = 1$.*

Proof. (i). If $r(A \oplus B) = 2$, then it holds with $C = A$. So assume that $r(A \oplus B) = 1$. Then there exist i_0, j_0 such that $A(i_0, j_0) \neq 0$ but $B(i_0, j_0) = 0$. Consider the following three cases.

Case 1. $A \oplus B$ has at least two nonzero rows and two nonzero columns. Define an $m \times n$ matrix C as the following:

$$C(i,j) = \begin{cases} \frac{1}{2}A(i_0, j_0), & \text{if } (i,j) = (i_0, j_0), \\ A(i,j), & \text{otherwise.} \end{cases}$$

Then $A \oplus C = A$ and hence $r(A \oplus C) = 1$. But $B \oplus C$ is the same as $B \oplus A$, except the (i_0, j_0)th entry of them. That is, $(B \oplus C)(i_0, j_0) = \frac{1}{2}A(i_0, j_0)$

and $(B \oplus A)(i_0, j_0) = A(i_0, j_0)$ are different. Therefore $r(B \oplus C) = 2$ since $r(B \oplus A) = 1$.

Case 2. $A \oplus B$ has exactly one nonzero row, that is the i_0th row. Then the i_0th rows A and B are nonzero, respectively, and all the other rows are zero. Let i_1th row of A be zero.

Define an $m \times n$ matrix C as the following:

$$C(i,j) = \begin{cases} \frac{1}{2}A(i_0, j_0), & \text{if} \quad (i,j) = (i_0, j_0), \\ A(i_0, j), & \text{if} \quad i = i_0, \ j \neq j_0, \\ A(i_0, j), & \text{if} \quad i = i_1, \\ A(i,j), & \text{otherwise.} \end{cases}$$

Then $A \oplus C$ is of rank one. But $B \oplus C$ is the same as $C \oplus A$, except that the (i_0, j_0)th entry of them. That is, $(C \oplus A)(i_0, j_0) = A(i_0, j_0)$ and $(B \oplus C)(i_0, j_0) = \frac{1}{2}A(i_0, j_0)$ are different. Therefore $r(B \oplus C) = 2$ since $r(C \oplus A) = 1$.

Case 3. $A \oplus B$ has exactly one nonzero column. The proof of this case is similar to Case 2.

(ii). The proof is similar to that of (i) if we keep in mind that in the proof of (i), $\frac{1}{2}A(i_0, j_0)$ is a positive real number between $A(i_0, j_0)$ and $B(i_0, j_0)$ when $A(i_0, j_0)$ is strictly larger than $B(i_0, j_0)$. □

Lemma 4.10. *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ with $m > 1$, $n > 1$, and T is not invertible but preserves rank 1, then T decreases the rank of some rank two matrix.*

Proof. By the proof of Corollary 4.7, T is not injective on the set of all rank one matrices. So there exist distinct rank one matrices X and Y such that $T(X) = T(Y)$. Without loss of generality, we may assume that the number of nonzero entries in X is more than or equal that of Y. By Lemma 4.9, there is some matrix C such that

$$\text{either} \quad r(X \oplus C) = 2,\ r(Y \oplus C) = 1 \quad \text{or} \quad r(X \oplus C) = 1,\ r(Y \oplus C) = 2.$$

Then, in the former case, $T(X \oplus C) = T(X) \oplus T(C) = T(Y) \oplus T(C) = T(Y \oplus C)$ is a rank one matrix. Thus T decreases the rank of rank two matrix $X \oplus C$. Similarly, in the latter case, T also decreases the rank of $Y \oplus C$. □

Theorem 4.11. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ with $m > 1$ and $n > 1$. Then T is a rank preserver if and only if T is a (U, V)-operator.*

Proof. By Theorem 4.8 and Lemma 4.10, we see that the necessity of the condition is satisfied. The sufficiency is trivial in view of Theorem 4.8 and the fact that $r(A)$ is the smallest integer k such that there exist k rank one matrices whose sum is A. □

Theorem 4.12. *Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$. Then T is a rank preserver if and only if T preserves ranks 1 and 2.*

Proof. We may assume $m > 1$ and $n > 1$. If T preserves ranks 1 and 2, then T is invertible by Lemma 4.10. Thus T is a rank preserver by Theorem 4.8 and Theorem 4.11. The converse is trivial. □

Thus we have characterized the linear operators that preserve rank of matrices over max algebra.

5 Linear operator preserving column rank over max algebra

5.1 Comparison of rank and column rank

The sub-semimodule of $(\mathbb{R}_{\max})^m$ generated by the columns of an $m \times n$ matrix A is called the *column space* of A, and denoted $< A >$. The *column rank*, $c(A)$, of $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$ is the weak dimension of the column space of A. The column rank of the zero matrix is zero.

It follows that

$$0 \leq r(A) \leq c(A) \leq n \tag{5.1}$$

for all matrices $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$.

The column rank of a matrix may strictly exceed its rank over $\mathbb{R}_{\max}$. For example, we consider a matrix with nonzero elements $a, b, c, d, e, f \in \mathbb{R}_{\max}$

$$A = \begin{bmatrix} 0 & 0 & a & b \\ c & 0 & 0 & d \\ 0 & e & f & 0 \end{bmatrix} \in \mathcal{M}_{3,4}(\mathbb{R}_{\max}). \tag{5.2}$$

Then Example 5.6 (below) implies that $r(A) = 3$, but $c(A) = 4$.

Lemma 5.1. *For any $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$, $r(A) = 1$ if and only if $c(A) = 1$.*

Proof. If $r(A) = 1$, then A can be factored as

$$A = \boldsymbol{b} \otimes [\, c_1 \ \cdots \ c_n \,],$$

where $\boldsymbol{b}$ is an $m \times 1$ matrix and $[\, c_1 \ \cdots \ c_n \,]$ is an $1 \times n$ matrix. Since $r(A) = 1$, $\boldsymbol{b}$ is not a zero column vector. Then it is obvious that $\mathrm{span}(\{\boldsymbol{b}\}) =< A >$. Therefore $c(A) = 1$. The converse follows from (5.1). □

Let $\mu(\mathbb{R}_{\max}, m, n)$ be the largest integer k such that for all $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$, $r(A) = c(A)$ if $r(A) \leq k$. The matrix in (5.2) shows that $\mu(\mathbb{R}_{\max}, 3, 4) < 3$. In general $0 \leq \mu(\mathbb{R}_{\max}, m, n) \leq n$. We also obtain that

$$r\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = r(A) \quad \text{and} \quad c\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = c(A) \qquad (5.3)$$

for all $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$.

Lemma 5.2. *If $c(A) > r(A)$ for some $p \times q$ matrix A over $\mathbb{R}_{\max}$, then for all $m \geq p$ and $n \geq q$, $\mu(\mathbb{R}_{\max}, m, n) < r(A)$.*

Proof. Since $c(A) > r(A)$ for some $p \times q$ matrix A, we have $\mu(\mathbb{R}_{\max}, p, q) < r(A)$ from the definition of μ. Let $B = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ be an $m \times n$ matrix containing A as a submatrix. Then by (5.3),

$$r(B) = r(A) < c(A) = c(B).$$

So, $\mu(\mathbb{R}_{\max}, m, n) < r(A)$ for all $m \geq p$ and $n \geq q$. □

Lemma 5.3. *For any $A \in \mathcal{M}_{2,n}(\mathbb{R}_{\max})$ with $n \geq 2$, $r(A) = 2$ if and only if $c(A) = 2$.*

Proof. Let $r(A) = 2$. If $n = 2$, then (5.1) implies that $c(A) = 2$. So we can assume that $n \geq 3$. Since any zero column does not change the weak dimension of the column space of A, we may assume that there is no zero column in A. Then we may write

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_i & \cdots & \boldsymbol{a}_n \end{bmatrix} = \begin{bmatrix} \alpha_1 & \cdots & \alpha_i & \cdots & \alpha_n \\ \beta_1 & \cdots & \beta_i & \cdots & \beta_n \end{bmatrix}.$$

Since $c(A) \geq 2$, there exist at least two different columns in A such that they are linearly independent.

Let $\frac{\alpha_i}{\beta_i} = \min_{\beta_h \neq 0}\{\frac{\alpha_h}{\beta_h}\}$ and $\frac{\beta_j}{\alpha_j} = \min_{\alpha_h \neq 0}\{\frac{\beta_h}{\alpha_h}\}$. Then any column $\begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix}$ of A can be written as

$$\begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} = \begin{cases} \frac{\alpha_k}{\alpha_j} \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} & \text{if } \beta_k = 0, \\ \frac{\beta_k}{\beta_i} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} & \text{if } \alpha_k = 0, \\ \frac{\alpha_k}{\alpha_j} \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} \oplus \frac{\beta_k}{\beta_i} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} & \text{if } \alpha_k \beta_k \neq 0. \end{cases}$$

Thus $c(A) = 2$. The converse follows from (5.1) and Lemma 5.1. □

Theorem 5.4. *For any $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$ with $m \geq 2$ and $n \geq 2$, $r(A) = 2$ if and only if $c(A) = 2$.*

Proof. Let $r(A) = 2$. Then A can be factored as $A = B \otimes C$ for some $m \times 2$ matrix B and $2 \times n$ matrix C, which are expressed as

$$B = [\,\boldsymbol{x}\ \boldsymbol{y}\,] \quad \text{and} \quad C = \begin{bmatrix} \alpha_1 & \cdots & \alpha_i & \cdots & \alpha_n \\ \beta_1 & \cdots & \beta_i & \cdots & \beta_n \end{bmatrix}.$$

If $n = 2$, then (5.1) implies that $c(A) = 2$. So we may assume that $n \geq 3$. Then Lemma 5.3 implies that $c(C) = 2$. And we consider the three cases in the proof of Lemma 5.3. But it is sufficient to consider the last case. Notice that any column of A is of the form $\alpha_h \boldsymbol{x} \oplus \beta_h \boldsymbol{y}$. Using the method in the proof of Lemma 5.3, we have

$$\alpha_h = \tfrac{\beta_h}{\beta_i}\alpha_i \oplus \tfrac{\alpha_h}{\alpha_j}\alpha_j \quad \text{and} \quad \beta_h = \tfrac{\beta_h}{\beta_i}\beta_i \oplus \tfrac{\alpha_h}{\alpha_j}\beta_j,$$

where

$$\tfrac{\beta_i}{\alpha_i} = \max\left\{ \tfrac{\beta_h}{\alpha_h} \,\middle|\, h = 1, \cdots, n \right\} \text{ and } \tfrac{\beta_j}{\alpha_j} = \min\left\{ \tfrac{\beta_h}{\alpha_h} \,\middle|\, h = 1, \cdots, n \right\}.$$

Then we have

$$\begin{aligned} \alpha_h \boldsymbol{x} \oplus \beta_h \boldsymbol{y} &= \left(\tfrac{\beta_h}{\beta_i}\alpha_i \oplus \tfrac{\alpha_h}{\alpha_j}\alpha_j\right)\boldsymbol{x} \oplus \left(\tfrac{\beta_h}{\beta_i}\beta_i \oplus \tfrac{\alpha_h}{\alpha_j}\beta_j\right)\boldsymbol{y} \\ &= \tfrac{\beta_h}{\beta_i}(\alpha_i \boldsymbol{x} \oplus \beta_i \boldsymbol{y}) \oplus \tfrac{\alpha_h}{\alpha_j}(\alpha_j \boldsymbol{x} \oplus \beta_j \boldsymbol{y}) \\ &\in \operatorname{span}(\{\alpha_i \boldsymbol{x} \oplus \beta_i \boldsymbol{y}, \alpha_j \boldsymbol{x} \oplus \beta_j \boldsymbol{y}\}). \end{aligned}$$

This shows that $\{\alpha_i \boldsymbol{x} \oplus \beta_i \boldsymbol{y}, \alpha_j \boldsymbol{x} \oplus \beta_j \boldsymbol{y}\}$ is a weak basis of the column space of A, which implies that $c(A) = 2$. The converse follows from (5.1) and Lemma 5.1. □

Lemma 5.5. *If all columns of an $m \times n$ matrix A over $\mathbb{R}_{\max}$ are linearly independent, then $c(A) = n$.*

Proof. Let $A = [\boldsymbol{a}_1 \ \cdots \ \boldsymbol{a}_n] \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$, where each $\boldsymbol{a}_i \in \mathcal{M}_{m,1}(\mathbb{R}_{\max})$ is a column of A. Let Ω be any weak basis of the column space of A. Suppose that there exists a column $\boldsymbol{a}_j$ such that $\alpha_j \boldsymbol{a}_j \notin \Omega$ for all nonzero $\alpha_j \in \mathbb{R}_{\max}$. Then we have

$$\alpha_j \boldsymbol{a}_j = \oplus_{i \neq j}^{n} \alpha_i \boldsymbol{a}_i, \ \text{ equivalently } \ \boldsymbol{a}_j = \oplus_{i \neq j}^{n} \frac{\alpha_i}{\alpha_j} \boldsymbol{a}_i,$$

which contradicts the fact that all columns of A are linearly independent. Hence $\dim_w(< A >) \geq n$, which implies that $c(A) = n$. □

Example 5.6. Consider a matrix

$$A = \begin{bmatrix} 0 & 0 & a & b \\ c & 0 & 0 & d \\ 0 & e & f & 0 \end{bmatrix} \in \mathcal{M}_{3,4}(\mathbb{R}_{\max})$$

with nonzero elements $a, b, c, d, e, f \in \mathbb{R}_{\max}$. Since all columns of A are linearly independent over $\mathbb{R}_{\max}$, we have $c(A) = 4$ by Lemma 5.5. Also $2 \leq r(A) \leq 3 = \min(3, 4)$ by Lemma 5.1. It follows from Theorem 5.4 that $r(A) \neq 2$. Therefore $r(A) = 3$. □

Theorem 5.7. *For $m \times n$ matrices over max algebra, we have the values of μ as follows ;*

$$\mu(\mathbb{R}_{\max}, m, n) = \begin{cases} 1 & \textit{if } \min(m, n) = 1; \\ 3 & \textit{if } m \geq 3, \ \textit{and } n = 3; \\ 2 & \textit{otherwise.} \end{cases}$$

Proof. If $\min(m, n) = 1$, then we have $\mu(\mathbb{R}_{\max}, m, n) = 1$ from Lemma 5.1.

Consider the matrix $A \in \mathcal{M}_{3,4}(\mathbb{R}_{\max})$ in Example 5.6. Then $r(A) = 3$ and $c(A) = 4$. Thus we have $\mu(\mathbb{R}_{\max}, m, n) \le 2$ for all $m \ge 3$ and $n \ge 4$ by Lemma 5.2. Suppose $m \ge 2$ and $n \ge 2$. Then we have $\mu(\mathbb{R}_{\max}, m, n) \ge 2$ for all $m \ge 2$ and $n \ge 2$ by Theorem 5.4. Finally, we consider the case with $m \ge 3$ and $n = 3$. Then we have $\mu(\mathbb{R}_{\max}, m, n) = 3$ by Lemma 5.1 and Theorem 5.4. Therefore we have the values of μ as required. □

5.2 Column rank-preserving operator

In this section we obtain characterizations of the linear operators that preserve the column rank of matrices over max algebra.

A linear operator T on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ is said to *preserve column rank* if $c(T(A)) = c(A)$ for all $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$. It *preserves column rank* r if $c(T(A)) = r$ whenever $c(A) = r$.

Lemma 5.8. *The column rank of a matrix over* $\mathbb{R}_{\max}$ *is preserved under pre- or post-multiplication by an invertible matrix.*

Proof. For the case of pre-multiplication, let A be any $m \times n$ matrix and U be an $m \times m$ invertible matrix over $\mathbb{R}_{\max}$. By Lemma 2.9, U is monomial. If $c(A) = r$, then there exists a weak basis $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_r\}$ of the column space of A such that $\dim_w(< A >) = r$. Then $\{U \otimes \boldsymbol{x}_1, \cdots, U \otimes \boldsymbol{x}_r\}$ is clearly a weak basis of the column space of $U \otimes A$. Thus $c(U \otimes A) = r$. Conversely, if $c(U \otimes A) = r$, then there exists a weak basis $\{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_r\}$ of the column space of $U \otimes A$ such that $\dim_w(< U \otimes A >) = r$. Then $\{U^{-1} \otimes \boldsymbol{y}_1, \cdots, U^{-1} \otimes \boldsymbol{y}_r\}$ is clearly a weak basis of the column space of

$U^{-1} \otimes U \otimes A = A$. Hence $c(A) = r$.

For the case of post-multiplication, let V be an invertible matrix in $\mathcal{M}_{n,n}(\mathbb{R}_{\max})$. By Lemma 2.9, V is monomial. Let v_i be the nonzero entry of the ith column of V. Then we have

$$A \otimes V = [\, \boldsymbol{a}_1 \, \boldsymbol{a}_2 \, \cdots \, \boldsymbol{a}_n \,] \otimes V = [\, v_1 \boldsymbol{a}_{i(1)} \, v_2 \boldsymbol{a}_{i(2)} \, \cdots \, v_n \boldsymbol{a}_{i(n)} \,],$$

where $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ are all columns of A and $\{i(1), i(2), \cdots, i(n)\}$ is a permutation of $\{1, 2, \cdots, n\}$. Hence $c(A) = c(A \otimes V)$. □

Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$. Say that T is a

(1) *congruence operator* if there exist monomials $U \in \mathcal{M}_{m,m}(\mathbb{R}_{\max})$ and $V \in \mathcal{M}_{n,n}(\mathbb{R}_{\max})$ such that $T(A) = U \otimes A \otimes V$ for all A in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$;

(2) *transposition operator* if $m = n$ and $T(A) = A^t$, the transpose matrix of A, for all A in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$.

We define a sub-semimodule of $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ whose nonzero members have column rank 1 as a *column-rank-1-sub-semimodule.* Using Theorem 5.7, we can apply the results for ranks 1 and 2 in Theorem 4.8 to those column ranks 1 and 2. Thus we obtain the following Theorem 5.9.

Theorem 5.9. *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$, then the followings are equivalent;*

(1) *T is invertible and preserves column rank 1;*

(2) *T preserves column ranks 1 and 2 and preserves the weak dimension of all column-rank-1-sub-semimodules;*

(3) *T is a (U, V)-operator.*

Example 5.10. Let

$$B = \begin{bmatrix} 0 & c & 0 & 0 \\ 0 & 0 & e & 0 \\ a & 0 & f & 0 \\ b & d & 0 & 0 \end{bmatrix}$$

be a matrix in $\mathcal{M}_{4,4}(\mathbb{R}_{\max})$ with nonzero elements $a, b, c, d, e, f \in \mathbb{R}_{\max}$. Then $c(B) = 3$ since the first three columns of B constitute a weak basis of the column space of B. But the column rank of

$$B^t = \begin{bmatrix} 0 & 0 & a & b \\ c & 0 & 0 & d \\ 0 & e & f & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is 4 by (5.3) and Example 5.6. □

Lemma 5.11. *If T is a transposition operator on $\mathcal{M}_{m,m}(\mathbb{R}_{\max})$ with $m \geq 4$, then T does not preserve column rank r for $r \geq 3$ but preserves all ranks.*

Proof. Let B be the matrix in Example 5.10. Consider $C = B \oplus 0_{m-4} \in \mathcal{M}_{m,m}(\mathbb{R}_{\max})$. Then $c(C) = 3$ by (5.3) but $T(C) = C^t$ has column rank 4 by (5.3). Let

$$D = B \oplus I_k \oplus 0_{m-k-4} \in \mathcal{M}_{m,m}(\mathbb{R}_{\max}),$$

where I_k is the identity matrix of order k. Then $c(D) = 3 + k$ but $T(D) = D^t$ has column rank $4 + k$. Therefore T does not preserve column rank r

for $r \geq 3$, but it is obvious that T preserves all ranks. □

Lemma 5.11 shows that some rank preserver(a transposition operator) do not preserve any column ranks.

Theorem 5.12. *Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ with $n \geq m \geq 4$. Then the following are equivalent;*

(1) *T preserves column rank;*

(2) *T preserves column ranks $1, 2$ and 3;*

(3) *T is a congruence operator;*

(4) *T is bijective and preserves column ranks 1 and 3.*

Proof. (1) $\Longrightarrow$ (2) : Obviously. (2) $\Longrightarrow$ (3) : Assume (2). Then T preserves ranks 1 and 2 by Theorem 5.7. Theorems 4.12 and 4.11 implies that T is a $(U, V)-$operator. But the transposition operator does not preserve column rank 3 by Lemma 5.11. Hence T is a congruence operator. (3) $\Longrightarrow$ (1) : Assume (3). Then T preserves column rank by Lemma 5.8. (3) $\Longrightarrow$ (4) : Assume (3). Clearly T is bijective and preserves column ranks 1 and 3 by Lemma 5.8. (4) $\Longrightarrow$ (3) : Assume (4). Then T is invertible and preserves column rank 1. By Theorem 5.9, T is (U, V)-operator. But Lemma 5.11 implies that T is not a transposition operator. Hence T is a congruence operator. □

We have assumed that $n \geq m \geq 4$ in the Theorem 5.12. For the other cases, the linear operators which preserve column rank are the same as rank preservers in the Theorems 4.11 and 4.12. We show it in the following remark.

Remark 5.13. *Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ with $n \leq 3$. Then the following are equivalent;*

(1) *T preserves column rank;*

(2) *T preserves column ranks 1 and 2;*

(3) *T is a $(U, V)-$operator.*

Proof. (1) $\Longrightarrow$ (2) : Obviously. (2) $\Longrightarrow$ (3) : Assume (2). Then T preserves ranks 1 and 2 by Theorem 5.7. Thus T is a $(U, V)-$operator by Theorems 4.11 and 4.12. (3) $\Longrightarrow$ (1) : Assume (3). Then for any $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$, there exist monomials $U \in \mathcal{M}_{m,m}(\mathbb{R}_{\max})$ and $V \in \mathcal{M}_{n,n}(\mathbb{R}_{\max})$ such that either $T(A) = U \otimes A \otimes V$ or $m = n$, $T(A) = U \otimes A^t \otimes V$. For the case $T(A) = U \otimes A \otimes V$, T preserves all column ranks by Lemma 5.8. For the case $m = n$ and $T(A) = U \otimes A^t \otimes V$, we have $m = n \leq 3$ from the conditions on m and n. But Theorem 5.7 implies that $r(A) = c(A) \leq 3$ for $m = n \leq 3$. Then $T(A) = U \otimes A^t \otimes V$ preserves all ranks by Theorem 4.11 and hence it preserves all column ranks for $m = n \leq 3$. □

Thus we have characterized the linear operators that preserve column rank of matrices over max algebra.

6 Linear operator preserving maximal column rank over max algebra

6.1 Comparison of rank and maximal column rank

The *maximal column rank*, $mc(A)$, of $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$ is the maximal number of the columns of which are linearly independent over $\mathbb{R}_{\max}$. The maximal column rank of the zero matrix is zero.

It follows that

$$0 \le r(A) \le c(A) \le mc(A) \le n \tag{6.1}$$

for all matrices $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$.

The maximal column rank of a matrix may actually exceed its rank over $\mathbb{R}_{\max}$. For example, we consider the matrix A in Example 5.6;

$$A = \begin{bmatrix} 0 & 0 & a & b \\ c & 0 & 0 & d \\ 0 & e & f & 0 \end{bmatrix} \in \mathcal{M}_{3,4}(\mathbb{R}_{\max}). \tag{6.2}$$

Then Example 5.6 implies that $r(A) = 3$. But $mc(A) = 4$ because all columns of A are linearly independent over $\mathbb{R}_{\max}$.

Moreover the maximal column rank of a matrix may actually exceed its column rank over $\mathbb{R}_{\max}$. Consider a matrix X with nonzero elements $a, b, c, d, e, f, g \in \mathbb{R}_{\max}$;

$$X = \begin{bmatrix} a & 0 & 0 & b & 0 \\ 0 & c & 0 & 0 & d \\ 0 & 0 & e & f & g \end{bmatrix} \in \mathcal{M}_{3,5}(\mathbb{R}_{\max}).$$

Then $c(A) = 3$ since the first three columns of X constitute a weak basis of the column space of X. But $mc(X) = 4$ since the last four columns of X are linearly independent over $\mathbb{R}_{\max}$.

Lemma 6.1. *For any $m \times n$ matrix A over $\mathbb{R}_{\max}$, we have that $r(A) = 1$ if and only if $mc(A) = 1$.*

Proof. If $r(A) = 1$, then A can be factored as

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \otimes \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} a_1b_1 & \cdots & a_1b_i & \cdots & a_1b_n \\ a_2b_1 & \cdots & a_2b_i & \cdots & a_2b_n \\ \vdots & & \vdots & & \vdots \\ a_mb_1 & \cdots & a_mb_i & \cdots & a_mb_n \end{bmatrix}.$$

If there exist nonzero b_i and b_j for some $i \neq j$, then $b_j = \frac{b_j}{b_i} b_i$. Hence ith and jth columns of A are linearly dependent. This implies that any two columns of A are linearly dependent. Therefore $mc(A) = 1$. The converse is obvious from (6.1). □

Let $\beta(\mathbb{R}_{\max}, m, n)$ be the largest integer k such that for all $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$, $r(A) = mc(A)$ if $r(A) \leq k$. The matrix A in (6.2) shows that $\beta(\mathbb{R}_{\max}, 3, 4) < 3$. In general $0 \leq \beta(\mathbb{R}_{\max}, m, n) \leq n$. We also obtain that

$$r\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = r(A) \quad \text{and} \quad mc\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = mc(A) \qquad (6.3)$$

for all $m \times n$ matrices A over $\mathbb{R}_{\max}$.

Lemma 6.2. *If $mc(A) > r(A)$ for some $p \times q$ matrix A over $\mathbb{R}_{\max}$, then for all $m \geq p$ and $n \geq q$, $\beta(\mathbb{R}_{\max}, m, n) < r(A)$.*

Proof. Since $mc(A) > r(A)$ for some $p \times q$ matrix A, we have $\beta(\mathbb{R}_{\max}, p, q) < r(A)$ from the definition of β. Let $B = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ be an $m \times n$ matrix containing A as a submatrix. Then by (6.3),

$$r(B) = r(A) < mc(A) = mc(B).$$

So, $\beta(\mathbb{R}_{\max}, m, n) < r(A)$ for all $m \geq p$ and $n \geq q$. □

Lemma 6.3. *For any $A \in \mathcal{M}_{2,n}(\mathbb{R}_{\max})$ with $n \geq 2$, we have that $r(A) = 2$ if and only if $mc(A) = 2$.*

Proof. Let $r(A) = 2$. If $n = 2$, then (6.1) implies that $mc(A) = 2$. So we may assume that $n \geq 3$. Let

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \text{ and } \begin{pmatrix} e \\ f \end{pmatrix}$$

be any three columns of A. Then we claim that the three columns are linearly dependent. To show this, we consider three cases.

Case 1. There are at least two zero elements in $\{a, b, c, d, e, f\}$. Then it is obvious that the three columns are linearly dependent.

Case 2. There is only one zero element in $\{a, b, c, d, e, f\}$. Then, without loss of generality, we may take $b = 0$ and $\max(\frac{c}{d}, \frac{e}{f}) = \frac{e}{f}$. Thus we have

$$\frac{e}{a}\begin{pmatrix} a \\ 0 \end{pmatrix} \oplus \frac{f}{d}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} e \oplus \frac{f}{d} c \\ f \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}.$$

So the given three columns are linearly dependent.

Case 3. There is no zero element in $\{a, b, c, d, e, f\}$. Then, without loss of generality, we may assume that $\frac{a}{b} \leq \frac{c}{d} \leq \frac{e}{f}$. Thus we obtain

$$\frac{d}{b}\begin{pmatrix} a \\ b \end{pmatrix} \oplus \frac{c}{e}\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} \frac{d}{b} a \oplus c \\ d \oplus \frac{c}{e} f \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}.$$

Hence the given three columns are linearly dependent.

This shows that $mc(A) < 3$. Therefore $mc(A) = 2$ by (6.1). The converse follows from (6.1) and Lemma 6.1. □

Theorem 6.4. *For any $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$ with $m \geq 2$ and $n \geq 2$, we have that $r(A) = 2$ implies $mc(A) = 2$ and conversely.*

Proof. Let $r(A) = 2$. Then A can be factored as $A = B \otimes C$ for some $m \times 2$ matrix $B = [\boldsymbol{x}\ \boldsymbol{y}]$ and $2 \times n$ matrix C with $r(B) = r(C) = 2$. If $n = 2$, then (6.1) implies that $mc(A) = 2$. So we can assume that $n \geq 3$. Then any column of A has the form $\boldsymbol{ax} \oplus \boldsymbol{by}$ with a column $\begin{pmatrix} a \\ b \end{pmatrix}$ of C. Let $\boldsymbol{ax} \oplus \boldsymbol{by}$, $\boldsymbol{cx} \oplus \boldsymbol{dy}$ and $\boldsymbol{ex} \oplus \boldsymbol{fy}$ be any three columns of A. Then

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \text{ and } \begin{pmatrix} e \\ f \end{pmatrix}$$

are columns of C and hence they are linearly dependent by Lemma 6.3. Now we consider all three cases in the proof of Lemma 6.3. But it is sufficient to consider the case 3, that is, $\{a, b, c, d, e, f\}$ has no zero element with $\frac{a}{b} \leq \frac{c}{d} \leq \frac{e}{f}$. Then the proof of Lemma 6.3 implies that

$$c = \frac{d}{b}a \oplus \frac{c}{e}e \quad \text{and} \quad d = \frac{d}{b}b \oplus \frac{c}{e}f.$$

Thus we have

$$\boldsymbol{cx} \oplus \boldsymbol{dy} = \left(\frac{d}{b}a \oplus \frac{c}{e}e\right)\boldsymbol{x} \oplus \left(\frac{d}{b}b \oplus \frac{c}{e}f\right)\boldsymbol{y} = \frac{d}{b}(\boldsymbol{ax} \oplus \boldsymbol{by}) \oplus \frac{c}{e}(\boldsymbol{ex} \oplus \boldsymbol{fy}).$$

Therefore we have $mc(A) \leq 2$, which implies that $mc(A) = 2$ by (6.1). The converse is obvious from (6.1) and Lemma 6.1. □

Theorem 6.5. *For $m \times n$ matrices over max algebra, we have the values of β as follows ;*

$$\beta(\mathbb{R}_{\max}, m, n) = \begin{cases} 1 & \text{if } \min(m, n) = 1; \\ 3 & \text{if } m \geq 3, \text{ and } n = 3; \\ 2 & \text{otherwise}. \end{cases}$$

Proof. If $\min(m,n) = 1$, then we have $\beta(\mathbb{R}_{\max}, m, n) = 1$ from Lemma 6.1. Consider the matrix $A \in \mathcal{M}_{3,4}(\mathbb{R}_{\max})$ in (6.2). Then $r(A) = 3$ and $mc(A) = 4$. Thus we have $\beta(\mathbb{R}_{\max}, m, n) \leq 2$ for all $m \geq 3$ and $n \geq 4$ by Lemma 6.2. Suppose $m \geq 2$ and $n \geq 2$. Then we have $\beta(\mathbb{R}_{\max}, m, n) \geq 2$ for all $m \geq 2$ and $n \geq 2$ by Lemma 6.1 and Theorem 6.4. Moreover, for $A \in \mathcal{M}_{m,3}(\mathbb{R}_{\max})$ with $m \geq 3$, $r(A) = 3$ implies $mc(A) = 3$ from (6.1) and $mc(A) = 3$ implies $r(A) = 3$ from Lemma 6.1 and Theorem 6.4. Thus we have $\beta(\mathbb{R}_{\max}, m, 3) = 3$ for $m \geq 3$. Therefore we have determined the values of β, as required. □

6.2 Maximal column rank-preserving operator

In this section we obtain characterizations of the linear operators that preserve maximal column rank of matrices over max algebra.

A linear operator T on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ is said to *preserve maximal column rank* if $mc(T(A)) = mc(A)$ for all $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$. It *preserves maximal column rank* r if $mc(T(A)) = r$ whenever $mc(A) = r$.

Lemma 6.6. *The maximal column rank of a matrix over* $\mathbb{R}_{\max}$ *is preserved under pre- or post-multiplication by an invertible matrix.*

Proof. For the case of pre-multiplication, let A be any $m \times n$ matrix and U be an $m \times m$ invertible matrix over $\mathbb{R}_{\max}$. By Lemma 2.9, U is monomial. If $mc(A) = r$, then there exists r linearly independent columns $\boldsymbol{a}_{i(1)}, \cdots, \boldsymbol{a}_{i(r)}$ in A which are maximal. Then $U \otimes \boldsymbol{a}_{i(1)}, \cdots, U \otimes \boldsymbol{a}_{i(r)}$ are linearly independent columns of $U \otimes A$. Thus $mc(U \otimes A) \geq r$. Conversely, if $mc(U \otimes A) = r$, then there exists r linearly independent columns $\boldsymbol{b}_{i(1)}, \cdots, \boldsymbol{b}_{i(r)}$ in $U \otimes A$ which are maximal. Then $U^{-1} \otimes \boldsymbol{b}_{i(1)}, \cdots, U^{-1} \otimes \boldsymbol{b}_{i(r)}$ are linearly independent columns of $U^{-1} \otimes U \otimes A = A$. Hence $mc(A) \geq r$. Therefore we have $mc(A) = mc(U \otimes A)$.

For the case of post-multiplication, let V be an invertible matrix in $\mathcal{M}_{n,n}(\mathbb{R}_{\max})$. Then V is monomial. Let v_i be the nonzero entry of the ith column of V. Then we have

$$A \otimes V = [\, \boldsymbol{a}_1\, \boldsymbol{a}_2 \, \cdots \, \boldsymbol{a}_n \,] \otimes V = [\, v_1 \boldsymbol{a}_{i(1)}\, v_2 \boldsymbol{a}_{i(2)} \, \cdots \, v_n \boldsymbol{a}_{i(n)} \,],$$

where $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ are the columns of A and $\{i(1), \cdots, i(n)\}$ is a permutation of $\{1, \cdots, n\}$. If $\boldsymbol{a}_x, \boldsymbol{a}_y, \cdots, \boldsymbol{a}_z$ are linearly independent columns of A, then $v_x \boldsymbol{a}_{i(x)}, v_y \boldsymbol{a}_{i(y)}, \cdots, v_z \boldsymbol{a}_{i(z)}$ are linearly independent columns of $A \otimes V$, and conversely. Hence $mc(A) = mc(A \otimes V)$. □

We define a sub-semimodule of $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ whose nonzero members have maximal column rank 1 as a *maximal-column-rank-1-sub-semimodule.* Using Theorem 6.5, we can apply the results for ranks 1 and 2 in Theorem 4.8 to those maximal column ranks 1 and 2. Thus we obtain the following Theorem 6.7.

Theorem 6.7. *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ then the followings are equivalent;*

(1) *T is invertible and preserves maximal column rank 1;*

(2) *T preserves maximal column ranks 1 and 2, and preserves the weak dimension of all maximal-column-rank-1-sub-semimodules;*

(3) *T is a (U, V)-operator.*

Example 6.8. Let

$$B = \begin{bmatrix} 0 & c & 0 & 0 \\ 0 & 0 & e & 0 \\ a & 0 & f & 0 \\ b & d & 0 & 0 \end{bmatrix}$$

be the matrix in Example 5.10. Then $mc(B) = 3$ since the first three columns of B are linearly independent over $\mathbb{R}_{\max}$. But the maximal column rank of

$$B^t = \begin{bmatrix} 0 & 0 & a & b \\ c & 0 & 0 & d \\ 0 & e & f & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is 4 by (6.2) and (6.3). □

Lemma 6.9. *If T is a transposition operator on $\mathcal{M}_{m,m}(\mathbb{R}_{\max})$ with $m \geq 4$, then T does not preserve maximal column rank r for $r \geq 3$ but preserves all ranks.*

Proof. Let B be the matrix in Example 6.8. Consider $C = B \oplus 0_{m-4} \in \mathcal{M}_{m,m}(\mathbb{R}_{\max})$. Then $mc(C) = 3$ by (6.3) but $T(C) = C^t$ has maximal column rank 4 by (6.3). Let

$$D = B \oplus I_k \oplus 0_{m-k-4} \in \mathcal{M}_{m,m}(\mathbb{R}_{\max}),$$

where I_k is the identity matrix of order k. Then $mc(D) = 3 + k$ but $T(D) = D^t$ has maximal column rank $4 + k$. Therefore T does not preserve maximal column rank r for $r \geq 3$, but it is obvious that T preserves all ranks. □

Lemma 6.9 shows that some rank preservers(a transposition operator) do not preserve any maximal column ranks.

Theorem 6.10. *Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ with $n \geq m \geq 4$. Then the following are equivalent;*

(1) *T preserves maximal column rank;*

(2) *T preserves maximal column ranks 1, 2 and 3;*

(3) *T is a congruence operator;*

(4) *T is bijective and preserves maximal column ranks 1 and 3.*

Proof. (1) $\Longrightarrow$ (2) : Obviously. (2) $\Longrightarrow$ (3) : Assume (2). Then T preserves ranks 1 and 2 by Theorem 6.5. Theorems 4.12 and 4.11 implies that T is a (U, V)−operator. But the transposition operator does not preserve maximal column rank 3 by Lemma 6.9. Hence T is a congruence operator. (3) $\Longrightarrow$ (1) : Assume (3). Then T preserves column rank by Lemma 6.6. (3) $\Longrightarrow$ (4) : Assume (3). Clearly T is bijective and preserves maximal column ranks 1 and 3 by Lemma 6.6. (4) $\Longrightarrow$ (3) : Assume (4). Then T is invertible and preserves maximal column rank 1. By Theorem 6.7, T is (U, V)-operator. But Lemma 6.9 implies that T is not a transposition operator. Hence T is a congruence operator. □

We have assumed that $n \geq m \geq 4$ in the Theorem 6.10. For the other cases, the linear operators which preserve maximal column rank are the same as rank preservers in the Theorems 4.11 and 4.12. We show it in the following remark.

Remark 6.11. *Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ with $n \leq 3$. Then the following are equivalent;*

(1) *T preserves maximal column rank;*

(2) *T preserves maximal column ranks 1 and 2;*

(3) *T is a $(U, V)-$operator.*

Proof. (1) $\Longrightarrow$ (2) : Obviously. (2) $\Longrightarrow$ (3) : Assume (2). Then T preserves ranks 1 and 2 by Theorem 6.5. Thus T is a $(U, V)-$operator by Theorems 4.11 and 4.12. (3) $\Longrightarrow$ (1) : Assume (3). Then for any $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$, there exist monomials $U \in \mathcal{M}_{m,m}(\mathbb{R}_{\max})$ and $V \in \mathcal{M}_{n,n}(\mathbb{R}_{\max})$ such that either $T(A) = U \otimes A \otimes V$ or $m = n$, $T(A) = U \otimes A^t \otimes V$. For the case $T(A) = U \otimes A \otimes V$, T preserves all maximal column ranks by Lemma 6.6. For the case $m = n$ and $T(A) = U \otimes A^t \otimes V$, we have $m = n \leq 3$ from the conditions on m and n. But Theorem 6.5 implies that $r(A) = mc(A) \leq 3$ for $m = n \leq 3$. Then $T(A) = U \otimes A^t \otimes V$ preserves all ranks by Theorem 4.11 and hence it preserves all maximal column ranks for $m = n \leq 3$. □

Thus we have characterized the linear operators that preserve maximal column rank of matrices over max algebra.

7 Concluding remark

In this dissertation, we investigated the linear operator preserving column rank and maximal column rank over max algebra. In section 5, we obtained the relationships between rank and column rank, and characterized the linear operator that preserves column rank over max algebra. It turns out that the linear operator is a congruence operator, which equals the linear operator that preserves (binary) Boolean column rank. In section 6, we studied the relationships between rank and maximal column rank, and obtained the results like as column rank. Also we had the characterizations of the linear operator preserving maximal column rank over max algebra. Like column rank preserver, it is a congruence operator which equals the linear operator that preserves (binary) Boolean maximal column rank.

References

[1] M. Marcus and B. Moyls, *Transformations on tensor product spaces*, Pacific J. Math. **9** (1959), 1215-1221.

[2] R. Westwick, *Transformations on tensor spaces*, Pacific J. Math. **23** (1967), 613-620.

[3] C. Lautemann, *Linear transformations on matrices: rank preservers and determinant preservers*, Linear and Multilinear Algebra **10** (1981), 343-345.

[4] D. D. Caen, D. A. Gregory and N. J. Pullman, *The Boolean rank of zero-one matrices*, Proc. 3rd car. Conf. Comb. & Comp. (1981), 169-173.

[5] L. B. Beasley and N. J. Pullman, *Boolean rank-preserving operators and Boolean rank-1 spaces*, Linear Algebra Appl. **59** (1984), 55-77.

[6] L. B. Beasley, D. A. Gregory and N. J. Pullman, *Nonnegative Rank-Preserving Operators*, Linear Algebra and its Applications **65** (1985), 207-223.

[7] L. B. Beasley and N. J. Pullman, *Fuzzy rank-preserving operators*, Linear Algebra Appl. **73** (1986), 197-211.

[8] L. B. Beasley and N. J. Pullman, *Semiring rank versus column rank*, Linear Algebra Appl. **101** (1988), 33-48.

[9] L. B. Beasley and S. Z. Song, *A comparison of nonnegative real ranks and their preservers*, Linear and Multilinear Algebra **31** (1992), 37-46.

[10] S. Kirkland and N. J. Pullman, *Linear operators preserving invariants of nonbinary matrices*, Linear and Multilinear Algebra **33** (1992), 295-300.

[11] S. Z. Song, *Linear operators that preserves Boolean column ranks*, Proc. Amer. Math. Soc. **119** (1993), 1085-1088.

[12] S. G. Hwang, S. J. Kim and S. Z. Song, *Linear operators that preserve maximal column rank of Boolean matrices*, Linear and Multilinear Algebra **36** (1994), 305-313.

[13] S. Z. Song, *Linear operators that preserve column rank of fuzzy matrices*, Fuzzy Sets and Systems **62** (1994), 311-317.

[14] S. Z. Song and S. G. Lee, *Column ranks and their preserves of general Boolean matrices*, J. Korean Math. Soc. **32** (1995), 531-540.

[15] R. B. Bapat, D. P. Stanford and P. Van den Driessche, *Pattern properties and spectral inequalities in max algebra*, SIAM J. Matrix Analysis Appl. **16**(3) (1995), 964-976.

[16] S. Z. Song and S. G. Hwang, *Spanning Column Ranks and Their Preservers of Nonnegative Matrices*, Linear Algebra and its Applications **254** (1997), 485-495.

[17] S. Z. Song, *Linear opeators that preserve maximal column ranks of nonnegative integer matrices*, Proc. Amer. Math. Soc. **126** (1998), 2205-2211.

[18] R. B. Bapat, *A max version of the perron-Frebenius theorem*, Linear Algebra Appl. **275-276** (1998), 3-18.

[19] R. B. Bapat, Sukanta Pati and S. Z. Song, *Rank preservers of matrices over max Algebra*, Linear and Multilinear Algebra **48** (2000), 149-164.

[20] S. Z. Song and S. D. Yang, *Linear operators preserving maximal column ranks of nonbinary Boolean matrices*, General Algebra and Application **20** (2000), 255-265.

[21] S. Z. Song and S. R. Park, *Maximal column rank preservers of fuzzy matrices*, General Algebra and Applications **21** (2001), 207-218.

국 문 초 록

막스행렬 계수들의 비교와 그들의 선형보존자

몇 가지 반환 위의 행렬공간 상에서 행렬들의 계수, 열계수 그리고 극대열계수들에 관한 특성들이 많이 연구되어 왔고, 이들을 보존하는 선형연산자의 형태가 규명되어 왔다.

이 논문에서는 최근에 인기가 있는 막스대수 위의 행렬공간 상에서의 계수들의 성질과 그들을 보존하는 선형연산자의 형태를 규명하는 연구를 하였다.

곧, 막스행렬의 계수와 열계수의 관계를 행렬함수의 하나인 μ 함수를 이용하여 비교하였다. 그리고 이 함수값을 활용하여 막스행렬공간 상의 열계수를 보존하는 선형연산자의 형태를 규명하고 특성들을 밝혔다. 그 중 하나는 선형연산자가 막스행렬공간에서 모든 열계수를 보존할 필요충분조건은 그 선형연산자가 합동연산자의 형태임을 밝혔다.

또한, 막스행렬의 계수와 극대열계수 사이의 관계 비교는 행렬함수 β를 이용하였다. 그리고 이 함수를 활용하여 막스행렬공간 상의 극대열계수를 보존하는 선형연산자의 형태를 규명하고 그에 관한 특성들을 얻었다. 그 중 하나는 선형연산자가 막스행렬공간에서 모든 극대열계수를 보존할 필요충분조건은 그 선형연산자가 극대열계수 1, 2 그리고 3을 보존하면 된다는 사실을 밝혔다.

감사의 글

비로소 박사학위연구가 이 졸업논문으로 하여 결실을 맺게 되었습니다. 그 동안 저에게 많은 도움을 주신 분들에게 이 지면을 빌어 감사의 말씀을 드립니다.

먼저, 본 논문이 완성되기까지 부족한 저에게 항상 깊은 관심과 배려로서 보살펴 주시며 학문에 임하는 기본자세를 지도해 주신 송석준 교수님께 진심으로 감사의 말씀을 드립니다. 그 동안 교수님께서 보여주셨던 강하고 부드러운 학자의 모습은 저에겐 큰 귀감이 되었습니다. 다시 한 번 감사의 말씀을 드리며 항상 건강하시길 빕니다.

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