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Globally Asymptotic Stability for Ordinary Differential Equations

濟州大學校大學院
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<국문초록 >

常微分 方程式의 漸近적 安定性

본 논문에서는 常微分 方程式系 $x'(t) = f(t,x)$ 의 0해의 漸近적 安定性과 一樣漸近적 安定性을 보장하는 리아프노프 함수 $v(t,x)$에 관한 條件들을 얻었다.

CONTENTS

I. Introduction

In the late nineteenth century a Russian mathematician, A.M.Liapunov, introduced a valuable tool-now known as Liapunov's direct or second method-for studying certain qualitative properties of solutions to systems of ordinary differential equations. This method essentially was in a dormant state for a few decades and was revived by several Russian mathematicians in the 1930s. However, it did not begin to receive much international attention until the late 1940s. The method then began to flourish in the 1950s and has been an important part of scientific literature, in general, and mathematics literature, in particular, ever since.

The basic idea behind Liapunov's direct method involves the study of an auxiliary function along solutions to a system of differential equations, and several steps are involved in order to use this approach. First, one must construct the auxiliary function-usually called a Liapunov function - which satisfies certain properties in compliance with the theory that has been developed. Then, the system of equations itself and the derivative of the Liapunov function along solutions to the system are examined for various attributes. Among the qualitative properties of solutions that one often can investigate using this technique are stability, uniform stability, asymptotic stability, uniform asymptotic stability.

One of the goals of this dissertation is to improve and supplement previous theorems in the literature regarding globally asymptotically stable and globally uniformly asymptotically stable for ordinary differential equations. In particular, we concentrate on two main directions ; namely, we seek to (i) present sufficient conditions to ensure the globally uniform asymptotic stability of the zero solution of differential equation, (ii) present the examples to apply our results.

In chapter II, we present the concept of an equililbrium point and definitions of stability. Chapter III is dedicated to obtain new theorems involving Liapunov functions.

II. The Concept of an Equilibrium Point and Definition of Stability

2.1. The Basic Notation and Concept of an Equilibrium Point

We concern ourselves with systems of equations

(E) $$x' = f(t, x),$$

where $x \in R^n$ and R^n denotes Euclidean n-space. When discussing global results, such as global asymptotic stability, we shall always assume that $f : R^+ \times R^n \to R^n$ [$R^+ = [0, \infty)$] is continuous. On the other hand, when considering local results, we shall usually assume that $f : R^+ \times B(h) \to R^n$ [$B(h) = \{x \in R^n \mid 0 \le |x| < h, \text{for some } h > 0\}$]. On some occassions we may assume that $t \in R$, rather than $t \in R^+$. Unless otherwise stated, we shall assume that for every $(t_0, \xi), t_0 \in R^+$, the initial value problem

(I) $$x' = f(t, x), \qquad x(t_0) = \xi$$

possesses a unique solution $\phi(t, t_o, \xi)$ which depends continuously on the initial data (t_0, ξ). Since it is very natural in this chapter to think of t as representing time, we shall use the symbol t_0 in (I) to represent the initial time (rather than using τ as was done earlier). Furthermore, we shall frequently use the symbol x_0 in place of ξ to represent the initial state.

Definition 2.1.1. A point $x_e \in R^n$ is called an **equilibrium point** of (E) (at time $t^* \in R^+$) if

$$f(t, x_e) = 0 \qquad \text{for all} \quad t \ge t^*.$$

Other terms for equilibrium point include ***stationary point***, ***singular point***, ***critical point***, and ***rest position***. Note that if x_e is an equilibrium point of (E) at t^*, then it is an equilibrium point at all $\tau \geq t^*$. Note also that in the case of autonomous systems

$$\text{(A)} \qquad x' = f(x)$$

and in the case of T-periodic systems

$$\text{(P)} \qquad x' = f(t,x), \qquad f(t,x) = f(t+T,x),$$

a point $x_e \in R^n$ is an equilibrium at some time t^* if and only if it is an equilibrium point at all times. Also note that if x_e is an equilibrium (at t^*) of (E), then the transformation $s = t - t^*$ reduces (E) to

$$dx/ds = f(s+t^*, x),$$

and x_e is an equilibrium (at s=0) of this system. For this reason, we shall henceforth assume that $t^* = 0$ in Definition 2.1.1 and we shall not mention t^* further. Note also that if x_e is an equilibrium point of (E), then for any $t_0 \geq 0$

$$\phi(t, t_0, x_e) = x_e \qquad \text{for all} \quad t \geq t_0,$$

i.e., x_e is a unique solution of (E) with initial data given by $\phi(t_0, t_0, x_e) = x_e$.

Example 2.1.2. *Considered the simple pendulum described by the equations*

$$x_1' = x_2,$$
$$x_2' = -k \sin x_1, \qquad k > 0$$

Physically, the pendulum has two equilibrium points. However, the model of this pendulum has countably infinitely many equilibrium points which are located in R^2 at the points $(\pi n, 0)$, $n = 0, \pm 1, \pm 2, \cdots$.

Definition 2.1.3. An equilibrium point x_e of (E) is called an **isolated equilibrium point** if there is an $r > 0$ such that $B(x_e, r) = \{x \in R^n \mid |x_e - x| < r\} \subset R^n$ contains no equilibrium points of (E) other than x_e itself.

Example 2.1.4. *The linear homogeneous system*

$$\text{(LH)} \qquad x' = A(t)x$$

where $A(t) = [a_{ij}(t)]$ is a real $n \times n$ matrix function, has a unique equilibrium which is at the origin if $A(t_0)$ is nonsingular for all $t_0 \geq 0$.

Example 2.1.5. *Assume that for*

$$x' = f(x),$$

f is continuously differentiable with respect to all of its arguments, and let

$$J(x_e) = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_e},$$

where $\partial f/\partial x$ is the $n \times n$ **Jacobian matrix** *defined by*

$$\partial f/\partial x = [\partial f_i/\partial x_j].$$

If $f(x_e) = 0$ and $J(x_e)$ is nonsingular, then x_e is an isolated equilibrium of (E).

Unless stated otherwise, we shall assume throughout this chapter that a given equilibrium point is an isolated equilibrium. Also, we shall usually find *it extremely useful to assume that in a give discussion, the equilibrium of interest is located at the origin of R^n. This assumption can be made without any loss of generality. To see this, assume that $x_e \neq 0$ is an equilibrium point of*

$$x' = f(t, x),$$

i.e., $f(t, x_e) = 0$ for all $t \geq 0$. Let $w = x - x_e$. Then

$$\omega' = F(t, \omega), \tag{2.1}$$

where

$$F(t, \omega) = f(t, \omega + x_e). \tag{2.2}$$

Since (2.2) establishes a one-to-one correspondence between the solutions of (E) and (2.1), we may assume henceforth that (E) possesses the equilibrium of interest located at the origin. This equilibrium $x = 0$ will sometimes be referred to as the **trivial solution** *of (E).*

2.2. Definitions of Stability

Definition 2.2.1. The equilibrium $x = 0$ of (E) is **stable** if for every $\epsilon > 0$ and any $t_0 \in R^+$ there exists a $\delta(\epsilon, t_0) > 0$ such that

$$|\phi(t, t_0, \xi)| < \epsilon \qquad \text{for all} \quad t \geq t_0$$

whenever

$$|\xi| < \delta(\epsilon, t_0).$$

Definition 2.2.2. The equilibrium $x = 0$ of (E) is said to be **uniformly stable** if δ is independent of t_0 in Definition 2.2.1, i.e., if $\delta = \delta(\epsilon)$.

Definition 2.2.3. The equilibrium $x = 0$ of (E) is **asymptotically stable** if

(i) it is stable, and

(ii) for every $t_0 \geq 0$ there exists an $\eta(t_0) > 0$ such that

$$\lim_{t\to\infty} \phi(t, t_0, \xi) = 0 \qquad \text{whenever} \quad |\xi| < \eta.$$

The set of all $\xi \in R^n$ such that $\phi(t, t_0, \xi) \to 0$ as $t \to \infty$ for some $t_0 \geq 0$ is called the **domain of attraction** of the equilibrium $x = 0$ of (E). Also, if for (E) condition (ii) is true, then the equilibrium $x = 0$ is said to be **attractive**.

Deflnion 2.2.4. The equilibrium $x = 0$ of (E) is **uniformly asymptotically stable** if

(i) it is uniformly stable, and

(ii) there is a $\delta_0 > 0$ such that for every $\epsilon > 0$ and for any $t_0 \in R^+$, there exists a $T(\epsilon) > 0$, independent of t_0, such that

$$|\phi(t, t_0, \xi)| < \epsilon, \qquad \text{for all} \quad t \geq t_0 + T(\epsilon)$$

$$\text{whenever} \quad |\xi| < \delta_0.$$

Definition 2.2.5. The equilibrium $x = 0$ of (E) is **globally asymptotically stable** if it is stable, and if every solution of (E) tends to zero as $t \to \infty$.

Definition 2.2.6. The equilibrium $x = 0$ of (E) is **globally uniformly asymptotically stable** if

(i) it is uniformly stable, and

(ii) for any $\alpha > 0$ any $\epsilon > 0$, and $t_0 \in R^+$, there exists $T(\epsilon, \alpha) > 0$, independent of t_0, such that

$$\text{if } |\xi| < \alpha, \qquad \text{then } |\phi(t, t_0, \xi)| < \epsilon \qquad \text{for all} \quad t \geq t_0 + T(\epsilon, \alpha).$$

III. Principal Liapunov Stability

3.1. Liapunov Functions

We shall present stability results for the equilibrium $x = 0$ of a system

$$\text{(E)} \qquad x' = f(t, x).$$

Such results involve the existence of real valued functions $v : D \to R$. In the case of local results (e.g., stability, asymptotic stability), we shall usually only require that $D = B(h) \subset R^n$ for some $h > 0$, or $D = R^+ \times B(h)$. On the other hand, in the case of global results (e.g., globally asymptotic stability), we have to assume that $D = R^n$ or $D = R^+ \times R^n$. Unless stated otherwise, we shall always assume that $v(t, 0) = 0$ for all $t \in R^+$ [resp., $v(0) = 0$].

Now let ϕ be an arbitrary solution of (E) and consider the function $t \longmapsto v(t, \phi(t))$. If v is continuously differentiable with respect to all of its arguments, then we obtain (by the chain rule) the derivative of v with respect to t along the solutions of (E), $v'_{(E)}$, as

$$v'_{(E)}(t, \phi(t)) = \frac{\partial v}{\partial t}(t, \phi(t)) + \nabla v(t, \phi(t))^T f(t, \phi(t)).$$

Here ∇v denotes the gradient vector of v with respect to x. For a solution $\phi(t, t_0, \xi)$ of (E), we have

$$v(t, \phi(t)) = v(t_0, \xi) + \int_{t_0}^{t} v'_{(E)}(\tau, \phi(\tau, t_0, \xi))d\tau.$$

Definition 3.1.1. Let $v : R^+ \times R^n \to R$ [resp., $v : R^+ \times B(h) \to R$] be continuously differentiable with respect to all of its arguments and let ∇v denote the **gradient** of v with respect to x. Then $v'_{(E)} : R^+ \times R^n \to R$ [resp., $v'_{(E)} : R^+ \times B(h) \to R$] is defined by

$$\begin{aligned} v'_{(E)}(t,x) &= \frac{\partial v}{\partial t}(t,x) + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i}(t,x) f_i(t,x) \\ &= \frac{\partial v}{\partial t}(t,x) + \nabla v(t,x)^T f(t,x). \end{aligned} \tag{3.1}$$

We call $v'_{(E)}$ **the derivative of** v (with respect to t) **along the solutions of** (E) [or along the trajectories of (E)].

Occasionally we shall only require that v be continuous on its domain of definition and that it satisfy locally a Lipschitz condition with respect to x. In such case we call v a **Liapunov function** and we define the **upper right–hand derivative of** v **with respect to** t **along the solutions of** (E) by

$$\begin{aligned} v'_{(E)}(t,x) &= \lim_{\theta \to 0^+} \sup (1/\theta)\{v(t+\theta, \phi(t+\theta,t,x)) - v(t,x)\} \\ &= \lim_{\theta \to 0^+} \sup (1/\theta)\{v(t+\theta, x+\theta \cdot f(t,x)) - v(t,x)\}. \end{aligned} \tag{3.2}$$

When v is continuously differentiable, then (3.2) reduces to (3.1).

Definition 3.1.2. A continuous function $\omega : R^n \to R$ [resp., $\omega : B(h) \to R$] is said to be **positive definite** if

(i) $\omega(0) = 0$, and

(ii) $\omega(x) > 0$ for all $x \neq 0$ [resp., $0 < |x| \leq r$ for some $r > 0$].

Definition 3.1.3. A continuous function $\omega : R^n \to R$ is said to be **radially unbounded** if

(i) $\omega(0) = 0$,

(ii) $\omega(x) > 0$ for all $x \in R^n - \{0\}$, and

(iii) $\omega(x) \to \infty$ as $|x| \to \infty$.

Definition 3.1.4. A function ω is said to be **negative definite** if $-\omega$ is a positive definite function.

Definition 3.1.5. A continous function $\omega : R^n \to R$ [resp., $\omega : B(h) \to R$] is said to be **positive semidefinite** if

(i) $\omega(0) = 0$, and

(ii) $\omega(x) \geq 0$ for all $x \in B(r)$ and for some $r > 0$.

Definition 3.1.6. A function ω is said to be **negative semidefinite** if $-\omega$ is positive semidefinite.

Next, we consider the case $v : R^+ \times R^n \to R$ [resp., $v : R^+ \times B(h) \to R$].

Definition 3.1.7. A continuous function $v : R^+ \times R^n \to R$ [resp., $v : R^+ \times B(h) \to R$] is said to be **positive definite** if there exists a positive definite function $\omega : R^n \to R$ [resp., $\omega : B(h) \to R$] such that

(i) $v(t, 0) = 0$ for all $t \geq 0$, and

(ii) $v(t, x) \geq \omega(x)$ for all $t \geq 0$ and for all $x \in B(r)$

for some $r > 0$.

Definition 3.1.8. A continuous function $v : R^+ \times R^n \to R$ is **radially unbounded** if there exists a radially unbounded function $\omega : R^n \to R$ such that

(i) $v(t,0)=0$ for all $t \geq 0$, and

(ii) $v(t,x) \geq \omega(x)$ for all $t \geq 0$ and for all $x \in R^n$.

Definition 3.1.9. A continuous function $v : R^+ \times R^n \to R$ [resp., $v : R^+ \times B(h) \to R$] is said to be **decrescent** if there exists a positive definite function $\omega : R^n \to R$ [resp., $\omega : B(h) \to R$] such that

$$|v(t,x)| \leq \omega(x) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad \text{for all} \quad x \in B(r)$$

$$\text{for some} \quad r > 0.$$

Definition 3.1.10. A continuous function $W : R^+ \to R^+$ is called a **wedge** if $W(0) = 0$ and W is strictly increasing on R^+.

Theorem 3.1.11 ([11, Theorem 5.7.12]). A continuous function $v : R^+ \times R^n \to R$ [resp., $v : R^+ \times B(h) \to R$] is **positive definite** if and only if

(i) $v(t,0)=0$ for all $t \geq 0$, and

(ii) for any $r > 0$ [resp., some $r > 0$] there exists a wedge W such that

$$v(t,x) \geq W(|x|) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad \text{for all} \quad x \in B(r).$$

Proof. If $v(t,x)$ is positive definite, then there is a function $\omega(x)$ satisfying the conditions of Definition 3.1.2 such that

$$v(t,x) \geq \omega(x) \quad \text{for } t \geq 0 \quad \text{and} \quad |x| \leq r.$$

Define $\psi_0(s) = \inf\,\{\omega(x) : s \leq |x| \leq r\}$ for $0 < s \leq r$. Clearly ψ_0 is a positive and nondecreasing function such that

$$\psi_0(|x|) \leq \omega(x) \qquad \text{on } 0 < |x| \leq r.$$

Since ψ_0 is continuous, it is Riemann integrable. Define the function ψ by $\psi(0) = 0$ and

$$\psi(u) = u^{-1} \int_0^u (s/r)\psi_0(s)ds, \qquad 0 \le u \le r.$$

Clearly $0 < \psi(u) \le \psi_0(u) \le \omega(x) \le v(t,x)$ if $t \ge 0$ and $|x| = u$. Moreover, ψ is continuous and increasing by construction.

Theorem 3.1.12 ([11, Theorem 5.7.13]). A continuous function $v : R^+ \times R^n \to R$ is **radially unbounded** if and only if

(i) $v(t,0) = 0$ for all $t \ge 0$, and

(ii) there exists a wedge W such that

$$v(t,x) \ge W(|x|) \quad \text{for all} \quad t \ge 0 \quad \text{and} \quad \text{for all} \quad x \in R^n,$$

$$\text{where } \lim_{r\to\infty} W(r) = \infty.$$

Theorem 3.1.13 ([11, Theorem 5.7.14]). A continuous function $v : R^+ \times R^n \to R$ [resp., $v : R^+ \times B(h) \to R$] is **decrescent** if and only if there exists a wedge W such that

$$|v(t,x)| \le W(|x|) \quad \text{for all} \quad t \ge 0 \quad \text{and} \quad \text{for all} \quad x \in B(r),$$

$$\text{for some } r > 0.$$

Example 3.1.14. *(a) The function $\omega : R^3 \to R$ given by $\omega(x) = x^Tx = x_1^2 + x_2^2 + x_3^2$ is positive definite and radially unbounded.*

(b) The function $\omega : R^3 \to R$ given by $\omega(x) = x_1^2 + (x_2 + x_3)^2$ is positive semidefinite. It is not positive definite since it vanishes for all $x \in R^3$ such that $x_1 = 0$ and $x_2 = -x_3$.

(c) The function $\omega : R^2 \to R$ given by $\omega(x) = x_1^2 + x_2^2 - (x_1^2 + x_2^2)^3$ is positive definite (in the interior of the unit circle given by $x_1^2 + x_2^2 < 1$). However, it is not radially unbounded. In fact, if $x^T x > 1$, then $\omega(x) < 0$.

(d) The function $\omega : R^3 \to R$ given by $\omega(x) = x_1^2 + x_2^2$ is positive semidefinite. It is not positive definite.

(e) The function $\omega : R^2 \to R$ given by $\omega(x) = x_1^4/(1 + x_1^4) + x_2^4$ is positive definite but not radially unbounded.

Theorem 3.1.15 ([11, Theorem 5.9.1]). If there exists a continuously differentiable, positive definite function v with a negative semidefinite (or identically zero) derivative $v'_{(E)}$, then the equilibrium $x = 0$ of (E) is **stable**.

Proof. According to Definition 2.2.1, we fix $\epsilon > 0$ and $t_0 \geq 0$ and we seek a $\delta > 0$ such that Definition 2.2.1 and Definition 2.2.3 are satisfied. Without loss of generality, we can assume that $\epsilon < h_1$. Since $v(t, x)$ is positive definite, then by Theorem 3.1.11 there is a wedge W such that

$$v(t, x) \geq W(|x|) \quad \text{for} \quad 0 \leq |x| \leq h_1, \ t \geq 0.$$

Pick $\delta > 0$ so small that $v(t_0, x_0) < W(\epsilon)$ if $|x_0| \leq \delta$. Since $v'_{(E)}(t, x) \leq 0$, then $v(t, \phi(t, t_0, x_0))$ is monotone nonincreasing and $v(t, \phi(t, t_0, x_0)) < W(\epsilon)$ for all $t \geq t_0$. Thus, $|\phi(t, t_0, x_0)|$ cannot reach the value ϵ, since this would imply that

$$v(t, \phi(t, t_0, x_0)) \geq W(|\phi(t, t_0, x_0)|) = W(\epsilon).$$

Theorem 3.1.16 ([12, Theorem 8.5]). If there exists a continuously differentiable, positive definite function v with $v'_{(E)}(t, x) \leq 0$, if $v'_{(E)}(t, x) \leq -c(|x|)$, where $c(r)$ is continuous on $[0, h]$ and positive definite, and if $F(t, x)$ is bounded, then the solution $x(t) = 0$ of (E) is **asymptotically stable**.

Theorem 3.1.17 ([11, Theorem 5.9.2]). If there exists a continuously differentiable, positive definite, decrescent function v with a negative semi-definite derivative $v'_{(E)}$, then the equilibrium $x = 0$ of (E) is **uniformly stable**.

Proof. By Theorem 3.1.11 and Theorem 3.1.13, there are two wedges W_1 and W_2 such that

$$W_1(|x|) \leq v(t,x) \leq W_2(|x|)$$

for all $t \geq 0$ and for all x with $|x| \leq h_1$. Fix ϵ in the range $0 < \epsilon < h_1$. Pick $\delta > 0$ so small that $W_2(\delta) < W_1(\epsilon)$. If $t_0 \geq 0$ and if $|x_0| \leq \delta$, then $v(t_0, x_0) \leq W_2(\delta) < W_1(\epsilon)$. Since $v'_{(E)}$ is nonpositive, then $v(t, \phi(t, t_0, x_0))$ is monotone nonincreasing. Thus $v(t, \phi(t, t_0, x_0)) < W_1(\epsilon)$ for all $t \geq t_0$. Hence, $W_1(|\phi(t, t_0, x_0)|) < W_1(\epsilon)$ for all $t \geq t_0$. Since W_1 is strictly increasing, then $|\phi(t, t_0, x_0)| < \epsilon$ for all $t \geq t_0$.

Theorem 3.1.18 ([11, Theorem 5.9.6]). If there exists a continuously differentiable, positive definite, decrescent function v with a negative definite derivative $v'_{(E)}$, then the equilibrium $x = 0$ of (E) is **uniformly asymptotically stable**.

Proof. By Theorem 3.1.17 the equilibrium $x = 0$ is uniformly stable. It remains to be shown that Definition 2.2.4(ii) is also satisfied.

The hypotheses of this theorem along with Theorems 3.1.11–3.1.13 imply that there are wedges W_1, W_2, and W_3 such that

$$W_1(|x|) \leq v(t,x) \leq W_2(|x|) \qquad \text{and} \qquad v'_{(E)}(t,x) \leq -W_3(|x|)$$

for all $(t, x) \in R^+ \times B(r_1)$ for some $r_1 > 0$. Pick $\delta_1 > 0$ such that $W_2(\delta_1) < W_1(r_1)$. Choose ϵ such that $0 < \epsilon \leq r_1$. Choose δ_2 such that $0 < \delta_2 < \delta_1$ and such that $W_2(\delta_2) < W_1(\epsilon)$. Define $T = W_1(r_1)/W_3(\delta_2)$. Fix $t_0 \geq 0$ and x_0 with $|x_0| < \delta_1$.

We now claim that $|\phi(t^*, t_0, x_0)| < \delta_2$ for some $t^* \in [t_0, t_0 + T]$. For if this were not true, we would have $|\phi(t, t_0, x_0)| \geq \delta_2$ for all $t \in [t_0, t_0 + T]$. Thus

$$\begin{aligned} 0 < W_1(\delta_2) &\leq v(t, \phi(t, t_0, x_0)) \\ &\leq v(t_0, x_0) + \int_{t_0}^{t} v'_{(E)}(s, \phi(s, t_0, x_0))ds \\ &\leq W_2(\delta_1) - \int_{t_0}^{t} W_3(\delta_2)ds. \end{aligned}$$

Now at $t = t_0 + T$ we find that

$$0 < W_2(\delta_1) - TW_3(\delta_2) = W_2(\delta_1) - W_1(r_1) < 0.$$

a contradiction. Hence, t^* exists.

Now for $t \geq t^*$ we have

$$\begin{aligned} W_1(|\phi(t, t_0, x_0)|) &\leq v(t, \phi(t, t_0, x_0)) \leq v(t^*, \phi(t^*, t_0, x_0)) \\ &\leq W_2(|\phi(t^*, t_0, x_0)|) \leq W_2(\delta_2) < W_1(\epsilon). \end{aligned}$$

Since W_1 is strictly increasing, it follows that $|\phi(t, t_0, x_0)| < \epsilon$ for all $t \geq t^*$ and hence for all $t \geq t_0 + T$.

Theorem 3.1.19 ([11, Theorem 5.9.7]). If there exists a continuously differentiable, positive definite, decrescent, and radially unbounded function v such that $v'_{(E)}$ is negative definite for all $(t, x) \in R^+ \times R^n$, then the equilibrium $x = 0$ of (E) is **globally uniformly asymptotically stable.**

Proof. The trivial solution of (E) is uniformly asymptotically stable by Theorem 3.1.18. It remains to be shown that the domain of attraction of $x = 0$ is all of R^n.

Fix $(t_0, x_0) \in R^+ \times R^n$. Then $v(t, \phi(t, t_0, x_0))$ is nonincreasing and so has a limit $\eta \geq 0$. If $|x_0| \leq \alpha$, then

$$W_2(\alpha) \geq v(t, \phi(t, t_0, x_0)) \geq W_1(|\phi(t, t_0, x_0)|),$$

and so $|\phi(t, t_0, x_0)| \leq \alpha_1 = W_1^{-1}(W_2(\alpha))$.

Suppose that no $T(\alpha, \epsilon)$ exists. Then for some x_0, $\eta > 0$. By Theorem 3.1.11, for $|x| \leq \alpha_1$ find a wedge W_3 such that $v'_{(E)}(t, x) \leq -W_3(|x|)$. Thus, for $t \geq t_0$ we have $W_2^{-1}(\eta) \leq |\phi(t, t_0, x_0)|$ and

$$\begin{aligned} \eta \leq v(t, \phi(t, t_0, x_0)) &\leq v(t_0, x_0) - \int_{t_0}^{t} W_3(|\phi(s, t_0, x_0)|)ds \\ &\leq v(t_0, x_0) - \int_{t_0}^{t} W_3(W_2^{-1}(\eta))ds. \end{aligned}$$

Thus, the right-hand side of this inequality becomes negative for t sufficiently large. But this is impossible when $\eta > 0$. Hence, $\eta = 0$.

Example 3.1.20. *Consider the simple pendulum*

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -k \sin x_1, \end{aligned} \tag{3.3}$$

where $k > 0$ is a constant. The system (3.3) has an isolated equilibrium at $x = 0$. The total energy for the pendulum is the sum of the Kinetic energy and potential energy, given by

$$v(x) = \frac{1}{2}x_2^2 + k\int_o^{x_1} \sin \eta d\eta = \frac{1}{2}x_2^2 + k(1 - \cos x_1).$$

Note that this function is continuously differentiable, that $v(0) = 0$, and that v is positive definite. Also, note that v is automatically decrescent, since it does not depend on t. Along the solutions of (3.3) we have

$$v'_{(3.3)}(x) = (k\sin x_1)x'_1 + x_2x'_2 = (k\sin x_1)x_2 + x_2(-k\sin x_1) = 0.$$

In accordance with Theorem 3.1.15, the equilibrium $x = 0$ of (3.3) is stable, and in accordance with Theorem 3.1.17, the equilibrium $x = 0$ of (3.3) is uniformly stable.

Example 3.1.21. *Consider the system*

$$\begin{aligned} x'_1 &= x_2 + cx_1(x_1^2 + x_2^2), \\ x'_2 &= -x_1 + cx_2(x_1^2 + x_2^2), \end{aligned} \tag{3.4}$$

where c is a real constant. Note that $x = 0$ is the only equilibrium. Choosing $v(x) = x_1^2 + x_2^2$, we obtain

$$v'_{(3.4)}(x) = 2c(x_1^2 + x_2^2)^2.$$

If $c = 0$, then Theorem 3.1.15 and 3.1.17 are applicable and the equilibrium $x = 0$ of (3.4) is uniformly stable. If $c < 0$, then Theorem 3.1.19 is applicable and the equilibrium $x = 0$ of (3.4) is globally uniformly asymptotically.

3.2. Main Results

Theorem 3.2.1. Let a function $v : R^+ \times R^n \to R$ be continuous and locally Lipschitz in $x \in R^n$ and let $\eta : R^+ \to R^+$ be a measurable function such that $\int_0^\infty \eta(s)ds = \infty$.

Suppose that there exist wedges W_1, W_2 and W_3 such that for all $t \in R^+$ and $x \in R^n$,

(i) $W_1(|x|) \le v(t,x) \le W_2(|x|)$ and

(ii) $v'_{(E)}(t,x) \le -\eta(t)W_3(|x|)$,

where

$$\lim_{r\to\infty} W_1(r) = \lim_{r\to\infty} W_2(r) = \infty.$$

Then the zero solution of (E) is **uniformly stable** and **globally asymptotically stable.**

Proof. Let $\epsilon > 0$ be given. Then there exists a $\delta = \delta(\epsilon) > 0$ such that $W_2(\delta) < W_1(\epsilon)$. Let $\phi(t, t_0, x_0)$ be a solution of (E) such that $t \ge t_0 \ge 0$ and $|x(t_0)| = |x_0| < \delta$. Then we have

$$\begin{aligned} W_1(|\phi(t,t_0,x_0)|) &\le v(t,\phi(t,t_0,x_0)) \le v(t_0,x_0) \\ &\le W_2(|x_0|) < W_2(\delta) < W_1(\epsilon), \end{aligned}$$

which implies that $|\phi(t, t_0, x_0)| < \epsilon$ if $t \ge t_0$ and $|x_0| < \delta$. This proves the uniform stability of the zero solution of (E).

Now we show that the domain of attraction of $x = 0$ of (E) is all of R^n. Fix $(t_0, x_0) \in R^+ \times R^n$. Then $v(t, \phi(t, t_0, x_0))$ is nonincreasing and so has a limit $\quad r \ge 0, \quad$ where $|x_0| < \alpha$ for any $\alpha > 0$.

Suppose that

$$\phi(t, t_0, x_0)) \not\longrightarrow 0 \qquad \text{as} \qquad t \to \infty.$$

Then there exists $r > 0$ such that $\lim_{t\to\infty} v(t, \phi(t, t_0, x_0)) = r$. This implies

$$r \le v(t, \phi(t, t_0, x_0) \le W_2(|\phi(t)|)$$

and

$$|\phi(t)| \ge W_2^{-1}(r) \qquad \text{for all } t \ge t_0.$$

By integrating v' along $\phi(t, t_0, x_0)$, we obtain

$$\begin{aligned} v(t, \phi(t, t_0, x_0)) &\le v(t_0, x_0) - \int_{t_0}^{t} \eta(s) W_3(|\phi(s)|)ds \\ &\le v(t_0, x_0) - W_3(W_2^{-1}(r)) \int_{t_0}^{t} \eta(s)ds \to -\infty \quad \text{as } t \to \infty, \end{aligned}$$

which is a contradiction. Thus the proof is complete.

Example 3.2.2. Consider a scalar equation

$$x' = -a(t)g(x) \tag{3.5}$$

where $a : R^+ \to R^+$. Suppose that there exists a wedge W^* such that $xg(x) \ge W^*(|x|)$ for any $x \in R$, and that $\int_0^\infty a(s)da = \infty$. Then the zero solution $x = 0$ of (3.5) is **uniformly stable** and **globally asymptotically stable.**

Proof. Consider the function $v(t, x) = \frac{1}{2}x^2$. Then

$$\begin{aligned} v'_{(3.5)}(t, x) &= x(-a(t)g(x)) \\ &= -a(t)xg(x) \\ &\le -a(t)W^*(|x|). \end{aligned}$$

Therefore, all conditions in Theorem 3.2.1 are satisfied. Hence the zero solution $x = 0$ of (3.5) is uniformly stable and globally asymptotically stable.

Example 3.2.2 revisted Consider a scalar equation

$$x' = -a(t)g(x) \tag{3.5}$$

where $a : R^+ \to R^+$ and $g : R \to R$ are continuous such that

$$xg(x) > 0 \quad \text{for any } x \in R - \{0\}.$$

Then

(1) the zero solution $x = 0$ of (3.5) is **unique to the right,**

(2) the zero solution $x = 0$ of (3.5) is **uniformly stable,** and

(3) if $\int_0^\infty a(t)dt = \infty$, then the zero solution $x = 0$ of (3.5) is **globally asymptotically stable.**

Proof. (1) If $x > 0$, then $|x|' = x' = -a(t)g(t) \le 0$. If $x < 0$, then

$$|x|' = (-x)' = -x' = a(t)g(x) \le 0.$$

That is, $|x|' \le 0$ for all $x \in R - \{0\}$. Thus $|\phi(t)|$ is nonincreasing for any solution $\phi(t)$ of (3.5). Therefore, $\phi(t) = 0$ for all $t \ge t_0$ if there exists $t_0 \ge 0$ such that $\phi(t_0) = 0$.

(2) Let $\epsilon > 0$ be given. Then $|\phi(t, t_0, x_0)| \le |x_0| < \epsilon$ if $t_0 \ge 0$, $t \ge t_0$ and $|x_0| = |x(t_0)| = |\phi(t_0)| < \epsilon$. Put $\delta = \epsilon$. Then the zero solution $x = 0$ of (3.5) is uniformly stable.

(3) Let $\phi(t, t_0, x_0)$ be a solution of (3.5). Then $\phi(t, 0, x_0) \ge 0$ for all $t \ge 0$ if $\phi(0) = x_0 \ge 0$, and $\phi(t, 0, x_0) \le 0$ for all $t \ge 0$ if $\phi(0) = x_0 \le 0$, since the zero solution of (3.5) is unique to the right.

Case 1). Let $\phi(0) = x_0 \geq 0$. Then $\phi(t, 0, x_0)$ is nonincreasing, since $|\phi(t)|' \leq 0$ for any $t \geq 0$.

Now we claim that $\lim_{t\to\infty} \phi(t) = 0$. Suppose not. Then there is a $P > 0$ such that $\lim_{t\to\infty} \phi(t) = P$. By assumption on g $1/g(x)$ is bounded for all $x \in (P, \phi(0))$. Thus we have

$$\int_P^{\phi(0)} \frac{1}{g(x)} dx = \int_\infty^0 -a(t)dt = \int_0^\infty a(t)dt = \infty,$$

which is a contradiction.

Case 2). Let $\phi(0) = x_0 < 0$. Then $\phi(t, 0, x_0)$ is nondecreasing, since $|\phi(t)|' \leq 0$ for any $t \geq 0$.

Now we claim that $\lim_{t\to\infty} \phi(t) = 0$. Suppose not. Then there is a $Q < 0$ such that $\lim_{t\to\infty} \phi(t) = Q$. By assumption on g $1/g(x)$ is bounded for all $x \in (\phi(0), Q)$. Thus we have

$$\int_{\phi(0)}^{Q} \frac{1}{g(x)} dx = \int_0^\infty -a(t)dt = -\int_0^\infty a(t)dx = -\infty,$$

which is a contradiction. Hence the proof is complete.

In the process of the above proof of Example 3.2.2 revisted we do not use the result of Theorem 3.2.1. Suppose that we replace the condition that $xg(x) > 0$ for any $x \in R - \{0\}$ with the condition that $xg(x) \geq W^*(|x|)$ for some wedge W^* and any $x \in R$ in Example 3.2.2 revisted. Then we apply Theorem 3.2.1 to prove the Example 3.2.2 revisted.

Remark 3.2.3. We can easily find a function $g(x)$ such that $xg(x) > 0$ for $x \in R - \{0\}$ implies that there exists a wedge W^* such that $W^*(|x|) \leq g(x)x$

for any $x \in R$. Consider $g(x) = Mx^n$, where $M > 0$ and n is a positive odd number. Then $xg(x) = Mx^{n+1} > 0$ if $x \in R - \{0\}$. Furthermore, we can consider $W^*(|x|) = xg(x) = Mx^{n+1}$.

Lemma 3.2.4. Let $\eta : R^+ \to R^+$ be a measurable function such that $\lim_{S\to\infty} \int_t^{t+S} \eta(s)ds = \infty$ uniformly with respect to $t \in R^+$. Then for any $M > 0$ there exists a $\delta = \delta(M) > 0$ such that

$$\int_t^{t+\delta} \eta(s)ds > M \qquad \text{for any} \quad t \in R^+.$$

Proof. Suppose that $\lim_{S\to\infty} \int_t^{t+S} \eta(s)ds = \infty$ uniformly with respect to $t \in R^+$. Then for 1 there exists $\delta_0 > 0$ such that $\int_t^{t+\delta_0} \eta(s)ds > 1$ for any $t \in R^+$.

Let $M > 0$ be given. If $M \leq 1$, then we may take δ as $\delta = \delta_0$. if $M > 1$, then there exists a positive integer N with $M \leq N$. Thus we have

$$\begin{aligned} M \leq N &< \int_t^{t+\delta_0} \eta(s)ds + \int_{t+\delta_0}^{t+2\delta_0} \eta(s)ds + \cdots + \int_{t+(N-1)\delta_0}^{t+N\delta_0} \eta(s)ds \\ &= \int_t^{t+N\delta_0} \eta(s)ds \end{aligned}$$

for any $t \in R$. That is, $\delta = N\delta_0$. Hence the proof is complete.

Theorem 3.2.5. Let a function $v : R^+ \times R^n \to R$ be continuous and locally Lipschitz in $x \in R^n$ and let $\eta : R^+ \to R^+$ be a measurable function such that $\lim_{S\to\infty} \int_t^{t+S} \eta(s)ds = \infty$ uniformly with respect to $t \in R^+$.

Suppose that there exist wedges W_1, W_2 and W_3 such that for all $t \in R^+$ and $x \in R^n$,

(i) $W_1(|x|) \leq v(t, x) \leq W_2(|x|)$ and

(ii) $v'_{(E)}(t,x) \le -\eta(t)W_3(|x|)$,

where

$$\lim_{r\to\infty} W_1(r) = \lim_{r\to\infty} W_2(r) = \infty.$$

Then the zero solution of (E) is **globally uniformly asymptotically stable.**

Proof. Let $\epsilon > 0$ be given. Then there exists a $\delta = \delta(\epsilon) > 0$ such that $W_2(\delta) < W_1(\epsilon)$. Let $\phi(t,t_0,x_0)$ be a solution of (E) such that $t \ge t_0 \ge 0$ and $|x(t_0)| = |x_0| < \delta$. Then we have

$$\begin{aligned} W_1(|\phi(t,t_0,x_0)|) &\le v(t,\phi(t,t_0,x_0)) \le v(t_0,x_0) \\ &\le W_2(|x_0|) < W_2(\delta) < W_1(\epsilon), \end{aligned}$$

which implies that $|\phi(t,t_0,x_0)| < \epsilon$ if $t \ge t_0$ and $|x_0| < \delta$. This proves the uniform stability of the zero solution of (E).

For 1 take $\delta_0 = \delta_0(1)$ of the uniform stability. By the Lemma 3.2.4 there is an $L = L(\epsilon)$ such that

$$\int_{t_1}^{t_1+L} \eta(s)ds > W_2(\delta_0)/W_3(\delta) \quad \text{for all} \quad t_1 \in R^+.$$

Let $\phi(t,t_0,x_0)$ be a solution of (E) with $|x_0| < \delta_0$. Now we claim that $|\phi(t^*,t_0,x_0)| < \delta$ for some $t^* \in [t_0,t_0+L]$. For if this were not true, we would have

$$|\phi(t,t_0,x_0)| \ge \delta \qquad \text{for all} \quad t \in [t_0,t_0+L].$$

Thus

$$0 < W_1(\delta) \le v(t, \phi(t, t_0, x_0)) \le v(t_0, x_0) + \int_{t_0}^{t} v'_{(E)}(s, \phi(s))ds$$
$$\le v(t_0, x_0) - \int_{t_0}^{t} \eta(s) W_3(|\phi(s)|)ds$$
$$< W_2(\delta_0) - W_3(\delta) \int_{t_0}^{t} \eta(s)ds$$
$$< W_2(\delta_0) - W_3(\delta) W_2(\delta_0)/W_3(\delta) = 0,$$

which is a contradiction if $t = t_0 + L$. Therefore, for $t \ge t_0 + L$ and some $t^* \in [t_0, t_0 + L]$ with $|\phi(t^*)| < \delta$,

$$W_1(|\phi(t)|) \le v(t, \phi(t)) \le v(t^*, \phi(t^*))$$
$$\le W_2(|\phi(t^*)|) < W_2(\delta) < W_1(\epsilon),$$

which implies that the zero solution of (E) is uniformly asymptotically stable.

Finally, we show that the domain of attraction of $x = 0$ of (E) is all of R^n. Fix $(t_0, x_0) \in R^+ \times R^n$. Then $v(t, \phi(t, t_0, x_0))$ is nonincreasing and so has a limit $A \ge 0$, where $|x_0| < \alpha$ for any $\alpha > 0$.

Suppose that no $T = T(\alpha, \epsilon) > 0$ exists. Then

$$\lim_{t \to \infty} v(t, \phi(t, t_0, x_0)) = A > 0 \qquad \text{for some } x_0 \text{ with } |x_0| < \alpha.$$

Thus, for $t \ge t_0$ we have

$$A \le v(t, \phi(t, t_0, x_0))$$
$$\le v(t_0, x_0) - \int_{t_0}^{t} \eta(s) W_3(|\phi(s, t_0, x_0)|)ds$$
$$\le v(t_0, x_0) - W_3(W_2^{-1}(A)) \int_{t_0}^{t} \eta(s)ds.$$

Therefore, the right-hand side of this inequality becomes negative for t sufficiently large. But this is impossible when $A > 0$. Hence the proof is complete.

Remark 3.2.6. Theorem 3.2.5 generalizes Theorem 3.1.19. For if η is a constant, then $\lim_{S\to\infty}\int_t^{t+S}\eta(s)ds = \infty$ uniformly with respect to $t \in R^+$.

Remark 3.2.7. Consider a scalar differential equation

$$x' = -\frac{1}{t+1}x \qquad \text{on } [0,\infty). \tag{3.6}$$

Then the zero solution $x = 0$ of (3.6) is **uniformly stable** and **globally asymptotically stable** (by Theorem 3.2.1). But it is well known that the zero solution of (3.6) is not globally uniformly asymptotically stable. In fact, $\eta(t) = \frac{1}{t+1}$ does not satisfy the condition in Theorem 3.2.5

Example 3.2.8. Consider a scalar differential equation

$$x' = -|\sin t|x^n \qquad \text{on } [0,\infty), \tag{3.7}$$

where n is a positive odd integer. Then the zero solution of (3.7) is **globally uniformly asymptotically stable.**

Proof. Consider the function $v(t,x) = \frac{1}{2}x^2$. Then

$$\begin{aligned} v'_{(3.7)}(t,x(t)) &= xx' = x(-|\sin t|x^n) \\ &= -|\sin t|x^{n+1}. \end{aligned}$$

Let $\eta(t) = |\sin t|, W_1(t) = \frac{1}{3}t^2, W_2(t) = t^2$ and $W_3(t) = \frac{1}{2}t^{n+1}$. Then all conditions in Theorem 3.2.5 are satisfied. Hence the zero solution of (3.7) is globally uniformly asymptotically stable.

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< Abstract >

Globally Asymptotic Stability For Ordinary Differential Equations

In this thesis, we consider a system of differential equations $x' = f(t, x)$ and obtain conditions on a Liapunov function $v(t, x)$ to ensure the globally asymptotic stability and the globally uniform asymptotic stability of the zero solution of $x' = f(t, x)$.

감사의 글

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