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LINEAR PRESERVERS OF TERM RANK OF FUZZY MATRIX PRODUCT

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LINEAR PRESERVERS OF TERM RANK OF FUZZY MATRIX PRODUCT

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⟨**Abstract**⟩

LINEAR PRESERVERS OF TERM RANK OF FUZZY MATRIX PRODUCT

In this thesis, we construct the sets of fuzzy matrix pairs. These sets are naturally occurred at the extreme cases for the (zero) term rank inequalities relative to the product of fuzzy matrices. These sets were constructed with the fuzzy matrix pairs which are related with the term ranks of the products and the zero term ranks of the products of two fuzzy matrices.

That is, we construct the following 5 sets;

$$\mathcal{T}_1(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | t(XY) = min\{r(X), c(Y)\}\};$$

$$\mathcal{T}_2(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | t(XY) = t(X) + t(Y) - n\};$$

$$\mathcal{T}_3(\mathcal{F}) = \{(X,Y,Z) \in \mathcal{M}_{m,n}(\mathcal{F})^3 | t(XYZ) + t(Y) = \rho(XY) + \rho(YZ)\};$$

$$\mathcal{Z}_1(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | z(XY) = 0\};$$

$$\mathcal{Z}_2(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | z(XY) = z(X) + z(Y)\};$$

For these 5 sets of fuzzy matrix pairs, we consider the linear operators that preserve them. We characterize those linear operators as $T(X) = PXQ$ or $T(X) = PX^tQ$ with appropriate invertible fuzzy matrices P and Q. We also prove that these linear operators preserve above 5 sets.

1 Introduction and Preliminaries

One of the most active and continuing subjects in matrix theory during the last century, is the study of those linear operators on matrices that leave certain properties or relations of subsets invariant. Such questions are usually called ”Linear Preserver Problems”. The earliest papers in our reference list are Frobenius(1897) and Kantor(1897). Since much effort has been devoted to this type of problem, there have been several excellent survey papers. For survey of these types of problems, we refer to the article of Song([11]) and the papers in [10]. The specified frame of problems is of interest both for matrices with entries from a field and for matrices with entries from an arbitrary semiring such as Boolean algebra, nonnegative integers, and fuzzy sets. It is necessary to note that there are several rank functions over a semiring that are analogues of the classical function of the matrix rank over a field. Detailed research and self-contained information about rank functions over semirings can be found in [1, 11].

There are some results on the inequalities for the rank function of matrices([1, 2, 3, 4]). Beasley and Guterman ([1]) investigated the rank inequalities of matrices over semirings. And they characterized the equality cases for some rank inequalities in [2]. The investigation of linear preserver problems of extreme cases of the rank inequalities of matrices over fields was obtained in [4]. The structure of matrix varieties which arise as extremal cases in the inequalities is far from being understood over fields, as well as semirings. A usual way to generate elements of such a variety is to find a matrix pairs which belongs to it and to act on this set by various linear operators that preserve this variety. Song and his colleagues ([3]) characterized the linear operators that preserve the extreme cases of column rank inequalities over semirings.

There are some results on the linear operators that preserve term rank([7, 8]) and zero-term rank([5]). But in these papers, the authors studied the term rank and zero-term rank function themselves.

In this thesis, we characterize linear operators that preserve the sets of matrix pairs which satisfy extreme cases for the term rank inequalities and zero-term rank inequalities for the product of matrices over fuzzy semirings.

Definition 1.1. ([3]) A *semiring* $\mathcal{S}$ consists of a set $\mathcal{S}$ and two binary operations, addition and multiplication, such that:

- $\mathcal{S}$ is an Abelian monoid under addition (identity denoted by 0);
- $\mathcal{S}$ is a semigroup under multiplication (identity, if any, denoted by 1);
- multiplication is distributive over addition on both sides;
- $s0 = 0s = 0$ for all $s \in \mathcal{S}$.

Definition 1.2. ([3]) A semiring is called *antinegative* if the zero element is the only element with an additive inverse.

Definition 1.3. ([5]) The Boolean semiring consists of the set $B = \{0, 1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$.

Definition 1.4. ([1]) A semiring is called *chain* if the set $\mathcal{S}$ is totally ordered with universal lower and upper bounds and the operations are defined by $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$.

It is straightforward to see that any chain semiring is commutative and antinegative.

Throughout we assume that $m \leq n$. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J, and O, respectively. The matrix $E_{i,j}$, called a *cell*, denotes the matrix with exactly one nonzero entry, that being a one in the (i, j) entry. Let R_i denote the matrix whose i^{th} row is all ones and is zero elsewhere, and C_j denote the matrix whose j^{th} column is all ones and is zero elsewhere. We let $|A|$ denote the number of nonzero entries in the matrix A.

Let $M_{m,n}(S)$ denote the set of $m \times n$ matrices with entries from the semiring S. If $m = n$, we use the notation $M_n(S)$ instead of $M_{m,n}(S)$.

Definition 1.5. ([12]) Let $\mathcal{R}$ be the field of reals, let $\mathcal{F}$=$\{\alpha \in \mathcal{R} \mid 0 \leq \alpha \leq 1\}$ denote a subset of reals. Define $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$ for all a,b in $\mathcal{F}$. Then $(\mathcal{F}, +, \cdot)$ is called a *fuzzy* semiring.

Let $\mathcal{M}_{m,n}(\mathcal{F})$ denote the set of all $m \times n$ matrices with entries in a *fuzzy* semiring $\mathcal{F}$. We call a matrix in $\mathcal{M}_{m,n}(\mathcal{F})$ as a *fuzzy* matrix.

Definition 1.6. ([4]) A *line* of a matrix A is a row or a column of the matrix A.

Definition 1.7. ([7]) A matrix $A \in \mathcal{M}_{m,n}(\mathcal{F})$ has *term rank* k ($t(A) = k$) if the least number of lines needed to include all nonzero elements of A is equal to k. Let us denote by $c(A)$ the least number of columns needed to include all nonzero elements of A and by $r(A)$ the least number of rows needed to include all nonzero elements of A.

Definition 1.8. ([5]) A matrix $A \in \mathcal{M}_{m,n}(\mathcal{F})$ has *zero-term rank* k ($z(A) = k$) if the least number of lines needed to include all zero elements of A is equal to k.

Example 1.9. Let

$$A = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & \frac{3}{4} \\ \frac{2}{3} & 0 & \frac{4}{5} \\ \frac{1}{2} & \frac{3}{4} & \frac{2}{3} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{2}{3} & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then $t(A) = 3$, $z(A) = 1$, $t(B) = 2$ and $z(B) = 3$ for $A, B \in M_3(\mathcal{F})$.

Definition 1.10. ([10]) A matrix $A \in \mathcal{M}_{m,n}(\mathcal{F})$ has *factor rank* k ($rank(A) = k$) if there exist matrices $B \in \mathcal{M}_{m,k}(\mathcal{F})$ and $C \in \mathcal{M}_{k,n}(\mathcal{F})$ such that $A = BC$ and k is the smallest positive integer such that such a factorization exists. By definition the only matrix with factor rank equal to 0 is the zero matrix, O.

If $\mathcal{S}$ is a subsemiring of a certain field then there is a usual rank function $\rho(A)$ for any matrix $A \in \mathcal{M}_{m,n}(\mathcal{S})$. It is easy to see that these functions are not equal in general but the inequality $rank(A) \geq \rho(A)$ always holds.

Example 1.11. Consider $\mathcal{Z}_+$, the set of nonnegative integers. The semiring $\mathcal{Z}_+$ is embedded in the real field $\mathcal{R}$. Then the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 3 & 3 & 3 \end{pmatrix}$$

has different values as, where $rank(A)$=3 and $\rho(A)$=2.

Definition 1.12. ([2]) Let $\mathcal{F}$ be a fuzzy semiring. An operator $T : \mathcal{M}_{m,n}(\mathcal{F}) \rightarrow \mathcal{M}_{m,n}(\mathcal{F})$ is called *linear* if $T(X+Y) = T(X)+T(Y)$ and $T(\alpha X) = \alpha T(X)$ for all $X, Y \in \mathcal{M}_{m,n}(\mathcal{F})$, $\alpha \in \mathcal{F}$.

Definition 1.13. ([3]) We say an operator, T, *preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ implies that $T(X) \in \mathcal{P}$, or, if $(X, Y) \in \mathcal{P}$ implies that $(T(X), T(Y)) \in \mathcal{P}$ when $\mathcal{P}$ is a set of ordered pairs.

Definition 1.14. ([7]) The matrix $X \circ Y$ denotes the *Hadamard* or *Schur product*, i.e., the (i, j) entry of $X \circ Y$ is $x_{i,j}y_{i,j}$.

Definition 1.15. ([7]) An operator T is called a (P, Q, B)-*operator* if there exist permutation matrices P and Q, and a matrix B with no zero entries, such that $T(X) = P(X \circ B)Q$ for all $X \in \mathcal{M}_{m,n}(\mathcal{F})$, or, if $m = n$, $T(X) = P(X \circ B)^t Q$ for all $X \in \mathcal{M}_{m,n}(\mathcal{F})$. The operator $T(X) = P(X \cdot B)Q$ is called nontransposing (P, Q, B)-operator. A (P, Q, B)-operator is called a (P, Q)-*operator* if $B = J$, the matrix of all ones.

It was shown in [2, 4, 9] that linear preserves for extremal cases of classical matrix inequalities over fields are types of (P, Q)-operators where P and Q are arbitrary invertible matrices. On the other side, linear preservers for various rank functions over semirings have been the object of much study during the last years, see for example [6, 7, 8, 10], in particular term rank and zero term rank were investigated in the last few years, see for example [5].

Definition 1.16. ([5]) We say that the matrix A *dominates* the matrix B if and only if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$.

Definition 1.17. If A and B are matrices and $A \geq B$ we let $A \backslash B$ denote the matrix C where

$$c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} \neq 0 \\ a_{i,j} & \text{otherwise} \end{cases}$$

The behaviour of the function ρ with respect to matrix multiplication and addition is given by the following inequalities:

Sylvester's laws:

$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min\{\rho(A), \rho(B)\},$$

and the *Frobenius inequality*:

$$\rho(AB) + \rho(BC) \leq \rho(ABC) + \rho(B),$$

where A, B, C are conformal matrices with coefficients from a field.

2 Term Rank Inequality Of Fuzzy Matrix Product

We obtain various inequalities for term rank of matrix product over fuzzy semirings. We also show that these inequalities are exact and best possible.

We denote by $A \bigoplus B$ the block-diagonal matrix of the form

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}.$$

Note that in this sense the operation $\bigoplus$ is not commutative.

Over a fuzzy semiring the Sylvester lower bound holds:

Proposition 2.1. *([1]) Let $\mathcal{F}$ be a fuzzy semiring. Then for any $A \in \mathcal{M}_{m,n}(\mathcal{F})$, $B \in \mathcal{M}_{n,k}(\mathcal{F})$ the following inequality holds:*

$$t(AB) \geq \begin{cases} 0 & \textit{if } t(A)+t(B) \leq n, \\ t(A)+t(B)-n & \textit{if } t(A)+t(B) > n. \end{cases}$$

This bound is exact and best possible.

Proof. Let $A \in \mathcal{M}_{m,n}(\mathcal{F})$, $B \in \mathcal{M}_{n,k}(\mathcal{F})$ be arbitray matrices, $t(A) = t_A$, $t(B) = t_B$. Then A and B have generalized diagonals with t_A and t_B nonzero elements, respectively. Denote them by D_A and D_B, respectively. Then $AB \geq D_A D_B$ since F is antinegative. Since the product of two generalized diagonal matrices, which have t_A and t_B nonzero entries, respectively, has at least $t_A + t_B - n$ nonzero entries, the inequality follows.

In order to show that this bound is exact and the best possible for each pair (r, s), $0 \leq r$, $s \leq n$ let us take $A_r = I_r \bigoplus O_{n-r}$, $B_s = O_{n-s} \bigoplus I_s$ in the case $m = n$. It is routine to generalize this example for the case $m \neq n$. $\square$

Example 2.2. Let $A, B \in \mathcal{M}_n(\mathcal{F})$. The inequality $t(AB) \leq \min(t(A), t(B))$ does not hold. It is enough to take $A = C_1$, $B = R_1$. Then

$$t(AB) = t(J_n) = n > 1.$$

However the following inequality is true.

Proposition 2.3. *([1]) Let $\mathcal{F}$ be a fuzzy semiring. Then for any $A \in \mathcal{M}_{m,n}(\mathcal{F})$, $B \in \mathcal{M}_{n,k}(\mathcal{F})$ the inequality $t(AB) \leq \min(t_r(A), t_c(B))$ holds. This is exact and the best possible bound.*

Proof. This inequality is a direct consequence of the definition of the term rank and antinegativity. The exactness follows from Example 2.2. In order to prove that this bound is the best possible, for each pair (r, s), $0 \leq r \leq m$, $0 \leq s \leq k$, consider the family of matrices $A_r = E_{1,1} + \ldots + E_{r,1}$ and $B_s = E_{1,1} + \ldots + E_{1,s}$. □

Example 2.4. For an arbitrary fuzzy semiring, the triple $(C_1, R_1, 0)$ is a counterexample to the term rank version of the Frobenius inequality, since $t(C_1R_1) + t(R_10) = n > t(C_1R_10) + t(R_1) = 1$. However if $\mathcal{F} = \{0, 1\}$ is a subsemiring of $\mathcal{R}^+$ the following obvious version is true :

$$\rho(AB) + \rho(BC) \leq t(ABC) + t(B).$$

3 Zero-Term Rank Inequality Of Fuzzy Matrix Product

We obtain inequalities for the zero-term rank product over fuzzy semirings. We also show that these inequalities are exact and best possible.

Proposition 3.1. *([1]) Let $\mathcal{F}$ be a fuzzy semiring. For $A \in \mathcal{M}_{m,n}(\mathcal{F})$, $B \in \mathcal{M}_{n,k}$ one has that*

$$0 \leq z(AB) \leq \min\{z(A) + z(B), k, m\}.$$

These bounds are exact and the best possible for $n > 2$.

Proof. The lower bound follows from the definition of the zero-term rank function. In order to show that this bound is exact and the best possible let us consider the family of matrices: for each pair (r, s), $0 \leq r \leq \min\{m, n\}$, $0 \leq s \leq \min\{k, n\}$, we take $A_r = J \backslash (\Sigma_{i=1}^{r} E_{i,i})$, $B_s = J \backslash (\Sigma_{i=1}^{s} E_{i,i+1})$ if $s < \min\{k, n\}$ and $B_s = J \backslash (\Sigma_{i=1}^{s-1} E_{i,i+1} + E_{s,1})$ if $s = \min\{k, n\}$. Then $z(A_r) = r$, $z(B_s) = s$ by definition and if $n > 2$ then $A_r B_s$ does not have zero elements by antinegativity. Thus $z(A_r B_s) = 0$.

The upper bound follows directly from the definition of zero-term rank and from the antinegativity of $\mathcal{F}$.

In order to show that this bound is exact and the best possible let us consider the family of matrices: for each pair (r, s), $0 \leq r \leq \min\{m, n\}$, $0 \leq s \leq \min\{k, n\}$, we take $A_r = J \backslash (\Sigma_{i=1}^{r} R_i)$ and $B_s = J \backslash (\Sigma_{i=1}^{s} C_i)$. □

Example 3.2. The triple (C_1, I, R_1) is a counterexample to the zero-term rank version of the Frobenius inequality, since

$$z(C_1) + z(R_1) = 2n - 2 > z(C_1 R_1) + z(I) = n$$

for $n > 2$.

4 Basic Results For Linear Operator Of Fuzzy Matrices

In this section, we obtain some basic results for our main theorems in the section 5 and 6. For a surjective linear operator, we have the followings.

Lemma 4.1. *Let $\mathcal{F}$ be a fuzzy semiring, $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ be an operator which maps lines to lines and is defined by $T(E_{i,j}) = E_{\sigma(i,j)}$, where σ is a permutation on the set $\{(i,j) \mid i = 1, 2, \cdots, m; j = 1, 2, \cdots, n\}$. Then T is a (P, Q)-operator.*

Proof. Since no combination of u rows and v columns can dominate J where $u+v = m$ unless $v = 0$ (or if $m = n$, if $u = 0$) we have that either the image of each row is a row and the image of each column is a column, or $m = n$ and the image of each row is a column and the image of each column is a row. Thus, there are permutation matrices P and Q such that $T(R_i) \leq PR_iQ$ and $T(C_j) \leq PC_jQ$ or, if $m = n$, $T(R_i) \leq P(R_i)^tQ$ and $T(C_j) \leq P(C_j)^tQ$. Since each cell lies in the intersection of a row and a column and T maps nonzero cells to nonzero (weighted) cells, it follows that $T(E_{i,j}) = PE_{i,j}Q$, or, if $m = n$, $T(E_{i,j}) = PE_{j,i}Q = P(E_{i,j})^tQ$.

Then $\mathcal{T}$ is a (P, Q)-operator. □

Lemma 4.2. *Let $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ be a $(P, Q) - operator$. Then T preserves all term rank and zero term rank.*

Proof. Let π be a permutation corresponding P, μ be a permutation corresponding Q.

Let $t(A) = r$ with $A \in M_{m,n}(F)$. Then there are r lines such that those r lines cover all nonzero entries of A, say $r_1, r_2, \cdots, r_s, c_1, c_2, \cdots, c_t$ with $s + t = r$, covers all nonzero entries of A. Then PAQ is covered by $r_{\pi(1)}, r_{\pi(2)}, \cdots, r_{\pi(s)}$ and $c_{\mu(1)}, c_{\mu(2)}, \cdots, c_{\mu(t)}$ with $s + t = r$.

Thus $t(PAQ) = r$ and hence $t(T(A)) = r$. Therefore T preserves term rank r, and hence T preserves all term rank.

Similarly T preserves all zero term rank.

□

Theorem 4.3. *Let $\mathcal{F}$ be a fuzzy semiring and $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ be a linear operator. Then the following are equivalent:*

1. *T is bijective.*

2. *T is surjective.*

3. *There exists a permutation σ on $\{(i,j) \mid i = 1, 2, \cdots, m; j = 1, 2, \cdots, n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$.*

Proof. That 1) implies 2) and 3) implies 1) is straightforward. We now show that 2) implies 3).

We assume that T is surjective. Then, for any pair (i,j), there exists some X such that $T(X) = E_{i,j}$. Clearly $X \neq O$ by the linearity of T. Thus there is a pair of indexes (r,s) such that $X = x_{r,s}E_{r,s} + X'$ where (r,s) entry of X' is zero and the following two conditions are satisfied: $x_{r,s} \neq 0$ and $T(E_{r,s}) \neq O$. Indeed, if in the contrary for all pairs (r,s) either $x_{r,s} = 0$ or $T(E_{r,s}) = O$ then $T(X) = 0$ which contradicts with the assumption $T(X) = E_{i,j} \neq 0$. Hence

$$T(x_{r,s}E_{r,s}) \leq T(x_{r,s}E_{r,s}) + T(X \setminus (x_{r,s}E_{r,s})) = T(X) = E_{i,j}$$

. That is, $x_{r,s}T(E_{r,s}) = T(x_{r,s}E_{r,s}) \leq E_{i,j}$. Thus $T(x_{r,s}E_{r,s}) = \alpha E_{i,j}$ for a certain $\alpha \in \mathcal{F}$. That is, there is some permutaion σ on $\{(i,j) \mid i = 1, 2, \cdots, m; j = 1, 2, \cdots, n\}$ such that for some scalars $b_{i,j}$, $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$. We now only need show that the $b_{i,j}$ are all units. Since T is surjective and $T(E_{r,s}) \nleq E_{\sigma(i,j)}$ for $(r,s) \neq (i,j)$,there is some α such that $T(\alpha E_{i,j}) = E_{\sigma(i,j)}$. But then, since T is linear, $T(\alpha E_{i,j}) = \alpha T(E_{i,j}) = \alpha b_{i,j}E_{\sigma(i,j)} = E_{\sigma(i,j)}$. That is, $\alpha b_{i,j} = 1$, or $b_{i,j}$ is a unit. But 1 is the only unit over fuzzy semiring. Thus $b_{i,j} = 1$ and $T(E_{i,j}) = E_{\sigma(i,j)}$. □

5 Term Rank Preservers Of Fuzzy Matrix Product

In this section, we obtain characterizations of the linear operators that preserve the set of matrix pairs which arise as the extremal cases in the inequalities of term rank of matrix products.

Below, we use the following notations in order to denote sets of matrices that arise as the extremal cases in the inequalities of term rank of matrix products listed in section 2 .

$\mathcal{T}_1(\mathcal{F}) = \{(X, Y) \in M_n(\mathcal{F})^2 | t(XY) = min\{r(X), c(Y)\}\}$,

$\mathcal{T}_2(\mathcal{F}) = \{(X, Y) \in M_n(\mathcal{F})^2 | t(XY) = t(X) + t(Y) - n\}$,

$\mathcal{T}_3(\mathcal{F}) = \{(X, Y, Z) \in M_n(\mathcal{F})^3 | t(XYZ) + t(Y) = \rho(XY) + \rho(YZ)\}$.

5.1 Linear Preservers of $\mathcal{T}_1(\mathcal{F})$

Consider the set of matrix pairs:

$\mathcal{T}_1(\mathcal{F}) = \{(X, Y) \in M_n(\mathcal{F})^2 | t(XY) = min\{r(X), c(Y)\}\}$.

This set contains (I, I) and hence it is not empty.

We characterize the linear operators that preserve set $\mathcal{T}_1(\mathcal{F})$.

Theorem 5.1. *Let $\mathcal{F}$ be a fuzzy semiring,$T : \mathcal{M}_n(\mathcal{F}) \to \mathcal{M}_n(\mathcal{F})$ be a surjective linear map. Then T preserves the set $\mathcal{T}_1(\mathcal{F})$ if and only if T is a (P, P^t)-operator, where P is a permutation matrix.*

Proof. By Theorem 4.3 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, σ is a permutation on the set of pairs (i, j).

For all k one has that $(E_{i,j}, E_{j,k}) \in \mathcal{T}_1(\mathcal{F})$ since

$t(E_{i,j}E_{j,k}) = t(E_{i,k}) = 1 = min\{1, 1\} = min\{r(E_{i,j}), c(E_{j,k})\}$. Thus $t(T(E_{i,j})T(E_{j,k})) = min\{r(T(E_{i,j})), c(T(E_{j,k}))\} = 1$ since T transforms cell to cells. But $T(E_{i,j})T(E_{j,k}) = E_{\sigma(i,j)}E_{\sigma(j,k)}$ so that $E_{\sigma(i,j)}$ is in the same row as $E_{\sigma(j,1)}$ for every k. That is, T maps rows to rows, similarly T maps columns to columns. That is, $T(X) = PXQ$ for some permutation matrices P and Q. Therefore, $T(E_{i,j}) = E_{\sigma(i)\tau(j)}$ where σ is the permutation corresponding to P and τ is the permutation corresponding to Q^t. But $(E_{1,i}, E_{i,1}) \in \mathcal{T}_1(\mathcal{F})$ implies

$(T(E_{1,i}), T(E_{i,1})) \in \mathcal{T}_1(\mathcal{F})$ by assumtion. Thus, $(E_{\sigma(1)\tau(i)}, E_{\sigma(i)\tau(1)}) \in \mathcal{T}_1(\mathcal{F})$, and hence $\sigma \equiv \tau$, that is, $Q = P^t$.

Conversely, (P, Q)-operators preserve term rank by Lemma 4.2.Hence (P, P^t)-operators preserve the term rank, c(A) and r(A), since fuzzy semiring $\mathcal{F}$ is antinegative. Therefore $(P, P^t$-operators preserve $\mathcal{T}_1(\mathcal{F})$.

□

5.2 Linear Preservers of $\mathcal{T}_2(\mathcal{F})$

Consider the set of matrix pairs:

$\mathcal{T}_2(\mathcal{F}) = \{(X, Y) \in M_n(\mathcal{F})^2 | t(XY) = t(X) + t(Y) - n\}$.

This set contains (I, I) and hence it is not empty.

We characterize the linear operators that preserve set $\mathcal{T}_2(\mathcal{F})$.

Lemma 5.2. *Let $\mathcal{F}$ be an abitrary fuzzy semiring and the linear transformation $T : M_n(\mathcal{F}) \to M_n(\mathcal{F})$ preserves the set $\mathcal{T}_2(\mathcal{F})$. Then $\mathcal{T}$ preserve the set of matrices with term-rank n.*

Proof. Let $A = 0$ and let B be any matrix of term rank n.

Then, $t(A) = 0$, $t(AB) = 0$.

Hence, $t(AB) = t(A)+t(B)-n$. It follows that $t(T(A)T(B)) = t(T(A))+t(T(B))-n$, since $\mathcal{T}$ preservers $\mathcal{T}_2(\mathcal{F})$.

That is, $0 = 0 + t(T(B)) - n$.

It follows that $t(T(B)) = n$. That is, T preserves term-rank n.

□

Lemma 5.3. *Suppose $\mathcal{F}$ is a fuzzy semiring, T is surjective linear transformation $T : M_n(\mathcal{F}) \to M_n(\mathcal{F})$. Transformation T preserves the set of matrices with term rank n if and only if T is a $(P, Q) - operator$, where$P$ and Q are permutation matrices of order n.*

Proof. By Theorem 4.3 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \le i \le m$, $1 \le j \le n$, σ is a permutation on the set of pairs (i, j).

Let us show that T^{-1} maps lines to lines. Assume that the preimage of a row is not dominated by any line. Then there are two cells in one line such that their preimages are not in one line. Let us consider the cells $E_{i,k}$ and $E_{i,l}$ such that $T^{-1}(E_{i,k}+E_{i,l}) \leq E_{r,s}+E_{p,q}, p \neq r, q \neq s$ and $T^{-1}(E_{i,k}+E_{i,l})$ is not dominated by each of the cells $E_{r,s}, E_{p,q}$.

By extending $E_{r,s}+E_{p,q}$ to a permutation matrix by adding $n-2$ cells, we find a matrix A such that $t(A) = n$.

Since T preservers term rank n by assumption one has that $t(T(A)) = n$.

On the other hand, $T(A)$ is dominated by $n-1$ lines by the choice of $E_{r,s}$ and $E_{p,q}$ and condition that the image of a cell is a cell, a contradiction with $t(T(A)) = n$. Thus the preimage of every row is a row or a column.

Similarly, the preimage of every column is a column or a row.

Moreover, since σ is bijective on the set of pairs (i,j) and each row intersects each column and does not intersect rows, T maps rows to rows and columns to columns, or , it is also possible T maps all rows to columns and all columns to rows. Thus there are permutation matrices P and Q such that $T(E_{i,j}) = PE_{i,j}Q$,or, $T(E_{i,j}) = PE_{j,i}Q = P(E_{i,j})^tQ$, i.e, T is a (P,Q)-operator where P and Q are permutation matrices of order n.

Therefore, we have that T is a (P,Q)-operator.

Conversely, let T be a (P,Q)-operator.Then T preserves all term rank by Lemma4.2 and hence T preserve the set of matrices with term rank n.

$\square$

Theorem 5.4. *Let F be a fuzzy semiring, $T : M_n(\mathcal{F}) \to M_n(\mathcal{F})$ be a linear surjective map. Then T preserves the set $\mathcal{T}_2(\mathcal{F})$ if and only if T is a nontransposing $(P,P^t)-operators$, where P is a permutation matrix.*

Proof. $(\Leftarrow)$ Let T be a nontransposing (P,P^t)-operators, on $M_n(\mathcal{F})$. By Lemma4.2, (P,Q)-operators preserve all term ranks. Thus $t(T(X)+t(T(Y))-n = t(X)+t(Y)-n$.

And $t(T(X)T(Y)) = t(PXP^tPYP^t) = t(PXYP^t) = t(XY)$.If $(X,Y) \in \mathcal{T}_2(\mathcal{F})$, Then $t(XY) = t(X)+t(Y)+n$. By above, $t(T(X)T(Y)) = t(T(X))+t(T(Y))-n$.

Therefore, $(T(X),T(Y)) \in \mathcal{T}_2(\mathcal{F})$.

Hence T preserves $\mathcal{T}_2(\mathcal{F})$.

($\Rightarrow$) Assume that linear preservers of the set $\mathcal{T}_2(\mathcal{F})$. Then T preserves the set of term rank n matrices by Lemma5.2 . Thus by applying Lemma 5.3 we obtain that T is a (P, Q)-operator.

That is, $T(X) = PXQ$ or $T(X) = PX^tQ$. But $T_1(X) = X^t$ does not preserve the set $\mathcal{T}_2(\mathcal{F})$. Indeed, the pair $(X = E_{i,j}, Y = I - E_{j,j}) \in \mathcal{T}_2(\mathcal{F})$ since $t(XY) = t(0) = 0 = 1+(n-1)-n = t(E_{i,j})+t(I-E_{j,j})-n$. However, $(X^t = E_{j,i}, Y^t = I-E_{j,j}) \notin \mathcal{T}_2(\mathcal{F})$ since $t(X^tY^t) = t(E_{j,i}) = 1 \neq 0 = t(X^t)+t(Y^t)-n$. Thus $T(X) = PX^tQ$ does not preserve the set $\mathcal{T}_2(\mathcal{F})$. Therefore, $T(X) = PXQ$ is a nontransposing (P, Q)-operator. Finally it remains to prove that $QP = I$, the identity matrix.

We have that a (P, Q)-operator preserves the set $\mathcal{T}_2(\mathcal{F})$ by Lemma 4.2. . . . . (1)

Thus $t(XY) = t(T(X)T(Y)) = t(PXQPYQ) = t(XQPY)$ for all pairs $(X, Y) \in \mathcal{T}_2(\mathcal{F})$. The matrix QP is permutation matrix as a product of two permutation matrices.

Assume that $QP = I$ and QP transforms i'th column into j'th column. . . . . (2)

Let $X = E_{i,i}$, $Y = I_n - E_{i,i}$. Then $t(X) = 1$, $t(Y) = n - 1$, $t(XY) = t(0) = 0$ and $t(X) + t(Y) - n = 1 + (n - 1) - n = 0$. i.e.,$(X, Y) \in \mathcal{T}_2(\mathcal{F})$.

On the other side, $XQP = E_{i,i}QP = E_{i,j}$ from (2). Then $XQPY = E_{i,j}(I_n - E_{i,i}) = E_{i,j} - 0 = E_{i,j}$.

Now, $t(T(X)T(Y)) = t(PXQPYQ) = t(XQPY) = 1$, and $t(T(X))+t(T(Y))-n = t(PXQ)+t(PYQ)-n = t(X)+t(Y)-n = 1+(n-1)-n = 0$. Hence, $t(T(X)T(Y)) \neq t(T(X)) + t(T(Y)) - n$. Thus, $(T(X), T(Y)) \notin \mathcal{T}_2(\mathcal{F})$. This contradicts the fact that (1). This contradiction comes from (2). Thus $QP = I$ and $PQ = I$. $i.e.$, $Q = P^t$. Thus T is a (P, P^t)-operator.

□

5.3 Linear Preservers of $\mathcal{T}_3(\mathcal{F})$

Consider the set of matrix pairs:

$\mathcal{T}_3(\mathcal{F}) = \{(X, Y, Z) \in M_n(\mathcal{F})^3 | t(XYZ) + t(Y) = \rho(XY) + \rho(YZ)\}$.

This set contains (I, I, I) and hence it is not empty.

We characterize the linear operators that preserve set $\mathcal{T}_3(\mathcal{F})$.

Theorem 5.5. *Let F be a fuzzy semiring, $T : M_n(\mathcal{F}) \rightarrow M_n(\mathcal{F})$ be a linear surjective map. Then T preserves the set $\mathcal{T}_3(\mathcal{F})$ if and only if T is a nontransposing $(P, P^t) - operators$, where P is a permutation matrix.*

Proof. By Theorom 4.3 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i, j \leq n$, where σ is a permutation on the set of pairs (i, j). Then $(E_{i,j}, E_{j,k}, E_{k,l}) \in \mathcal{T}_3(\mathcal{F})$ for all l and for abitrary fixed i, j, k since $t(E_{i,j}E_{j,k}E_{k,l}) + t(E_{j,k}) = 1 + 1$ and $\rho(E_{i,j}E_{j,k}) + \rho(E_{j,k}E_{k,l}) = 1 + 1$.

Since T preserves the set $\mathcal{T}_3(\mathcal{F})$, $t(T(E_{i,j})T(E_{j,k})T(E_{k,l}))+t(T(E_{j,k})) = \rho(T(E_{i,j})T(E_{j,k}))+ \rho(T(E_{j,k})T(E_{k,l}))$.

By Theorem 4.3 it follows that $T(E_{i,j}) = E_{p,q}$, $T(E_{j,k}) = E_{r,s}$, $T(E_{k,l}) = E_{u,v}$. Since $t(E_{r,s}) = 1 \neq 0$ it follows from the last equality that either $r = q$ or $s = u$ or both. Let us assume that only one of the equalities hold for a certain l.

Without loss of generality, assume that $s = u$ and $r \neq q$. Thus for arbitrary m, $1 \leq m \leq n$ one has that $(E_{i,j}, E_{j,k}, E_{k,m}) \in \mathcal{T}_3(\mathcal{F})$.

By Theorem 4.3, we have $T(E_{i,j}) = E_{p,q}, T(E_{j,k}) = E_{r,s}, T(E_{k,m}) = E_{w,z}$. Since $r \neq q$ and $(E_{p,q}, E_{r,s}, E_{w,z}) \in \mathcal{T}_3(\mathcal{F})$, it follows that $s = w$, and hence T maps k'th row into s'th row. Thus in this case we obtain : rows are transformed to rows.

By the same arguments, columns are transformed to columns.

Assume that $s \neq u$ and $r = q$. $(E_{a,j}, E_{j,k}, E_{k,l}) \in \mathcal{T}_3(\mathcal{F})$. $T(E_{a,j}) = E_{x,y}, T(E_{j,k}) = E_{r,s}, T(E_{k,l}) = E_{u,v}$. And hence $q = y = r$.

It follows that there exists a permutation P and Q such that $T(X) = PXQ$ for all $X \in M_n(\mathcal{F})$.

In order to show that the transposition transformation does not preserve T_3 it suffices to show an example that $(E_{1,2}, E_{2,3}, C_3) \in \mathcal{T}_3(\mathcal{F})$ and $(E_{2,1}, E_{3,2}, R_3) \notin \mathcal{T}_3(\mathcal{F})$.

In order to show that $Q = P^t$ it suffices to show that $(E_{i,j}, E_{j,k}, E_{i,j}) \in \mathcal{T}_3(\mathcal{F})$. In fact, $t(E_{i,j}E_{j,k}E_{i,j}) + t(E_{j,k}) = 0 + 1$, $\rho(E_{i,j}E_{j,k}) + \rho(E_{j,k}E_{i,j}) = 1 + 0$.

Let σ is corresponding P and τ is corresponding Q^t.

By assumption, $(T(E_{i,j}), T(E_{j,k}), T(E_{i,j})) = (E_{\sigma(i)\tau(j)}, E_{\sigma(j)\tau(i)}, E_{\sigma(i)\tau(j)}) \in \mathcal{T}_3(\mathcal{F})$.

Thus $\sigma \equiv \tau$. Hence $Q = P^t$ and T is a (P, P^t)-operator.

Conversely, nontransposing (P, P^t)-operator preserves term rank by Lemma 4.2. Moreover, nontransposing (P, P^t)-operator preserves real rank since P is an invertible matrix over real field $\mathbb{R}$. Hence (P, P^t)-operator preserves $\mathcal{T}_3(\mathcal{F})$.

$\square$

6 Zero-Term Rank Preservers Of Fuzzy Matrix Product

In this section, we obtain characterizations of the linear operators that preserve the set of matrix pairs which arise as the extremal cases in the inequalities of zero-term rank of matrix products.

Below, we use the following notations in order to denote sets of matrices that arise as the extremal cases in the inequalities of zero-term rank of matrix products listed in section 3 .

$\mathcal{Z}_1(\mathcal{F}) = \{(X,Y) \in M_n(F)^2 | z(XY) = 0\}$,

$\mathcal{Z}_2(\mathcal{F}) = \{(X,Y) \in M_n(F)^2 | z(XY) = z(X) + z(Y)\}$.

6.1 Linear Preservers of $\mathcal{Z}_1(\mathcal{F})$

Consider the set of matrix pairs:

$\mathcal{Z}_1(\mathcal{F}) = \{(X,Y) \in M_n(\mathcal{F})^2 | z(XY) = 0\}$.

This set contains (J, J), where J is the $n \times n$ matrix with $1's$ as its all entries. Thus $\mathcal{Z}_1(\mathcal{F})$ is not empty.

We characterize the linear operators that preserve set $\mathcal{Z}_1(\mathcal{F})$.

Theorem 6.1. *Let F be a fuzzy semiring, $T : M_n(\mathcal{F}) \to M_n(\mathcal{F})$ be a linear surjective map. Then T preserves the set $\mathcal{Z}_1(\mathcal{F})$ if and only if T is a nontransposing $(P, P^t) - operators$, where P is a permutation matrix.*

Proof. By Theorem 4.3 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \le i \le m$, $1 \le j \le n$, σ is a permutation on the set of pairs (i, j).

Let us show that T maps lines to lines.

Suppose that the images of two cells are in the same line, but the cells are not in the same line, say, $E_{i,j}, E_{i,k}$ are the cells such that $T^{-1}(E_{i,j}), T^{-1}(E_{i,k})$ are not in the same line.

Let us consider $A = T^{-1}(J \setminus R_i)$. Thus there are no zero rows of A since T is a permutation on the set of cells and not all elements of the preimage of the i'th row of J lie in one row by the choice of i. Hence AJ does not have zero elements by the additions and multiplications in $\mathcal{F}$ and $z(AJ) = 0$.

Thus $(A, J) \in Z_1(\mathcal{F})$ as far as $(T(A), T(J)) = (T(T^{-1}(J \setminus R_i)), T(J)) = (J \setminus R_i, T(J)) \notin Z_1(F)$, since $z((J \setminus R_i)(T(J))) = z(J \setminus R_i) = 1$, a contradiction to the assumption that T preserves the set $\mathcal{Z}_1(\mathcal{F})$.

Moreover, since σ is bijective on the set of pairs (i, j) and each row intersects each column and does not intersect rows, T maps rows to rows and columns to columns, or , it is also possible T maps all rows to columns and all columns to rows. Thus there are permutation matrices P and Q such that $T(E_{i,j}) = PE_{i,j}Q$, or, $T(E_{i,j}) = PE_{j,i}Q = P(E_{i,j})^tQ$, i.e, T is a (P, Q)-operator where P and Q are permutation matrices of order n. Let us show that $Q = P^t$. Assume on the contrary that $QP \neq I$. Thus there exists indexes i, j such that QP transforms i'th column into j'th column. In this case we take matrices $A = J\backslash(E_{1,1} + \cdots + E_{1,n}) + E_{1,i}$, $B = J\backslash E_{j,1}$. Thus AB has no zero elements, i.e, $z(AB) = 0$.

However, the $(1, j)^{th}$ element of $T(A)T(B)$ is zero, i.e, $z(T(A)T(B)) \neq 0$.

This contradiction implies that $QP = I$. Thus $Q = P^t$. Hence T is a $(P, P^t) - operator$.

Conversely (P, Q)-operators preserve zero term rank by Lemma 4.2. Thus $(P, P^t) - operators$ preserve the set $\mathcal{Z}_1(\mathcal{F})$.

□

Example 6.2. Let $\mathcal{F}$ be a fuzzy semiring, $T : M_4(\mathcal{F}) \to M_4(\mathcal{F})$ be a surjective map such that $T(A) = PAQ$, where $P = I_4$, $Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and $QP = Q \neq I$.

Then T maps rows to themselves. But T maps 1_{st} column of A to itself, 2_{nd} column of A to 3_{rd} column, 3_{rd} column of A to 2_{nd} column and 4_{th} column of A to 4_{th} column.

Consider $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

Thus $AB = J$ and hence $z(AB) = 0$. Thus $(A, B) \in Z_1(\mathcal{F})$.

But $T(A) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, $T(B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

Then $T(A)T(B) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ and hence $z(T(A)T(B)) = 1$. Thus $(T(A), T(B)) \notin Z_1(\mathcal{F})$

This example shows that linear operator T does not preserve $Z_1(\mathcal{F})$, where $\mathcal{T}$ is not $(P, P^t) - operator$.

6.2 Linear Preservers of $\mathcal{Z}_2(\mathcal{F})$

Consider the set of matrix pairs:

$\mathcal{Z}_2(\mathcal{F}) = \{(X, Y) \in M_n(F)^2 | z(XY) = z(X) + z(Y)\}$.

This set contains (J, J), where J is the $n \times n$ matrix with $1's$ as its all entries. Thus $\mathcal{Z}_1(\mathcal{F})$ is not empty.

We characterize the linear operators that preserve set $\mathcal{Z}_2(\mathcal{F})$.

Theorem 6.3. *Let F be a fuzzy semiring, $T : M_n(\mathcal{F}) \to M_n(\mathcal{F})$ be a linear surjective map. Then T preserves the set $\mathcal{Z}_2(\mathcal{F})$ if and only if T is a nontransposing $(P, Q) - operators$, where P is a permutation matrix.*

Proof. By Theorem 4.3 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, σ is a permutation on the set of pairs (i, j).

Let us show that T maps lines to lines. Suppose that the images of two cells are not in the same line, but the cells are in the same line, say, $E_{i,j}, E_{i,k}$ are the cells such that $T(E_{i,j}), T(E_{i,k})$ are not in the same line.

Note that $z((J \setminus R_i)J) = z(J \setminus R_i) = 1 = 1 + 0 = z(J \setminus R_i) + z(J)$. Thus $(J \setminus R_i, J) \in \mathcal{Z}_2(\mathcal{F})$. Now, $T(J \setminus R_i$ has no zero rows by above argument, and $T(J) = J$ over $M_n(\mathcal{F})$.

Hence $T(J \setminus R_i)T(J) = T(J \setminus R_i)J = J$ on $M_n(\mathcal{F})$ by the sums and products over $\mathcal{F}$. Thus $z(T(J \setminus R_i)T(J)) = 0$. On the other hand, $(T(J \setminus R_i), T(J)) \notin \mathcal{Z}_2(\mathcal{F})$. This contradiction shows that T maps lines to lines.

It follows from Lemma4.1 that T is a (P, Q)-operator where P and Q are permutation matrices of order n.

To show that transposition operator does not preserve $\mathcal{Z}_2(\mathcal{F})$, it suffices to take the pair of matrices $A = J \setminus R_i$, $B = J \setminus C_i$. Consider $A = J \setminus R_1$, $B = J \setminus C_1$. Then $z(AB) = 2 = 1 + 1 = z(A) + z(B)$, hence $(A, B) \in \mathcal{Z}_2(\mathcal{F})$. But $z(A^t B^t) = z(J) = 0$ and $z(A^t) = z(B^t) = 1$. Hence $z(A^t B^t) \neq z(A^t) + z(B^t)$, that is, $(A^t, B^t) \notin \mathcal{Z}_2(\mathcal{F})$. Thus $(T(A), T(B)) \notin \mathcal{Z}_2(\mathcal{F})$. This show that transposing operator does not preserve $\mathcal{Z}_2(\mathcal{F})$.

Therefore T is a nontransposing (P, Q)-operator.

Let us show that $Q = P^t$ now.

Assume on the contrary that $QP \neq I$. Thus there exists indexes i, j such that QP transforms i'th column into j'th column. But then consider $A = J\ C_i$, $B = R_i$. We have $z(AB)$ = $z(0)$ = n =1 + n = $z(A) + z(B)$. Hence $(A, B) \in \mathcal{Z}_2(\mathcal{F})$. But $z(AQPB)$ = $z((J - C_j)R_i)$ = $z(J)$ = 0 and $z(AQP) + z(B) = 1 + (n-1) = n$. Thus $(T(A), T(B)) \notin \mathcal{Z}_2(\mathcal{F})$, which contradicts the fact that $\mathcal{T}$ preserves $\mathcal{Z}_2(\mathcal{F})$. Hence $QP = I$, and $Q = P^t$. We have $\mathcal{T}$ is a nontransposing (P, P^t)-operator.

Conversely, (P, Q)operator preserve zero term rank by Lemma 4.2. Thus (P, P^t)-operators preserve the set $\mathcal{Z}_2(\mathcal{F})$.

$\square$

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⟨국문초록⟩

퍼지 행렬의 항별 계수 곱의 선형보존자

본 논문에서는 퍼지 행렬의 짝들로 구성되는 집합들을 구성하였다. 이 집합들은 두 퍼지 행렬들의 곱의 항별 계수와 영항 계수와 관련된 부등식의 극치인 경우들에서 자연스럽게 나타나는 퍼지 행렬 짝들의 집합들이다. 이 퍼지 행렬 짝들의 집합들은 두 퍼지 행렬의 항별 계수들의 곱과 영항 계수들의 곱과 관련된 부등식들에서 극치인 경우들로 구성하였다.

곧, 다음과 같은 5가지 집합을 구성하였다;

$$\mathcal{T}_1(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | t(XY) = min\{r(X), c(Y)\}\};$$

$$\mathcal{T}_2(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | t(XY) = t(X) + t(Y) - n\};$$

$$\mathcal{T}_3(\mathcal{F}) = \{(X,Y,Z) \in \mathcal{M}_{m,n}(\mathcal{F})^3 | t(XYZ) + t(Y) = \rho(XY) + \rho(YZ)\};$$

$$\mathcal{Z}_1(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | z(XY) = 0\};$$

$$\mathcal{Z}_2(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | z(XY) = z(X) + z(Y)\};$$

이상의 퍼지 행렬 짝들의 집합을 선형연산자로 보내어 그 집합의 성질들을 보존하는 선형연산자를 연구하여 그 형태를 규명하였다. 곧, 이러한 퍼지 행렬 짝들의 집합을 보존하는 선형연사자의 형태는 $T(X) = PXQ$ 또는 $T(X) = PX^tQ$로 나타남을 보이고, 이들을 증명하였다. 그리고 이 선형연산자가 위의 5가지 집합들을 보존함을 증명하였다.

감사의 글

학부시절 공부를 열심히 하지 않고 졸업을 하는 것이 많이 걱정이 되었습니다. 수학과를 나온 만큼 수학적인 사고를 갖고 문제를 풀어나가야 하는데 그러한 수학적 사고가 많이 부족하다는 것을 느끼고 좀 더 공부해보고 싶다는 생각이 많이 들었습니다. 그래서 많은 고민끝에 대학원 진학을 결심하게 되었습니다. 하지만 학부 열심히 하지않아 대학원 생활이 결코 쉽지 않았습니다. 기초가 많이 부족할 뿐더러 하면 할수록 부족함이 많이 느껴져 포기하고 싶은 순간도 많았습니다. 하지만 2년동안의 석사과정을 무사히 마칠 수 있었고 이렇게 졸업논문이 나오게 되었습니다. 송석준 교수님이 계셨기에 무사히 논문도 쓰고 졸업을 할 수 있었습니다. 많이 부족한 저에게 많은 격려와 도움을 주셨기에 여기까지 올 수 있었습니다. 교수님, 감사드립니다. 하나하나 신경써서 많은 내용들을 가르쳐주신 양영오 교수님, 방은숙 교수님, 정승달 교수님, 윤용식 교수님, 유상욱 교수님, 진현성 교수님, 정말 감사드립니다.

많은 충고와 좋은 얘기들을 해주신 이지순 선생님께도 감사드립니다.

혼자 공부하기 힘들었는데 연정언니가 옆에 있어줘서 많은 도움이 된 것 같습니다. 언니일에 바빴을텐데 논문쓰는 마지막까지 하나하나 챙겨준 연정언니, 고마습니다. 1년동안 이지만 함께 수업도 받고 같이 다녔던 혜정이와 희란이, 많은 조언을 해준 민주언니, 금란언니가 있어서 든든했던 것 같습니다.

제가 하고자하는 일에 항상 믿음으로 지켜봐주시는 부모님과 동생에게도 감사의 말을 전합니다.

아직도 부족한 점이 많은 저 이지만 항상 자신감을 갖고 열심히 살겠습니다. 고맙습니다.

2008년 12월