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Liouville type theorem
for
p-harmonic maps and morphisms

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文　東　柱

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p-harmonic maps and morphisms

Dong Joo Moon

(Supervised by professor Seoung Dal Jung)

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審査委員長

委　　員

委　　員

委　　員

委　　員

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<Abstract>

Liouville type theorem for p-harmonic maps and p-harmonic morphisms

The classical Liouville theorem for harmonic maps is that any bounded harmonic functions on the whole plane must be constant. This Liouville theorem has been studied by many authors. In this thesis, we study Liouville type theorems for p-harmonic maps and p-harmonic morphisms with finite p-energy. Any p-harmonic maps from a complete Riemannian manifold M of a Ricci curvature bounded from the below by negative constant depending on p to a complete Riemannian manifold N of non-positive sectional curvature is shown to be constant if it has finite p-energy. Moreover, we prove any p-harmonic morphisms from a complete Riemannian manifold of the Ricci curvature bounded below by a p-dependent negative constant to a complete Riemannian manifold of non-positive scalar curvature is constant if it has finite p-energy.

1 Introduction

Let (M, g) and (N, h) be smooth Riemannian manifolds and let $\phi : M \to N$ be a smooth map. For a compact domain $\Omega \subset M$, the *p-energy* E of ϕ over Ω is defined by

$$E_p(\phi; \Omega) = \frac{1}{p} \int_\Omega |d\phi|^p \mu_M, \tag{1.1}$$

where the differential $d\phi$ is a section of the bundle $T^*M \otimes \phi^{-1}TN \to M$ and $\phi^{-1}TN$ denotes the pull-back bundle via the map ϕ at the point $x \in M$, μ_M is the volume element on M and the p-energy density $|d\phi|^p$ on M is defined by

$$|d\phi|^p = \left(\sum_{i=1}^{m} \langle d\phi(e_i), d\phi(e_i) \rangle \right)^{\frac{p}{2}} .$$

Then the bundle $T^*M \otimes \phi^{-1}TN \to M$ carries the connection ∇ induced by the Levi-Civita connections on M and N. A map $\phi : (M, g) \to (N, h)$ is called *p-harmonic* if ϕ is a critical point of the energy functional defined by (1.1) on any compact domain $\Omega \subset M$.

Equivalently, p-harmonic maps are solutions of the following systems (harmonicity equation) of PDEs:

$$\begin{aligned} \tau_p(\phi) :&= |d\phi|^{p-2}\tau_2(\phi) + (p-2)|d\phi|^{p-3} d\phi(\mathrm{grad}_g |d\phi|) \\ &= 0, \end{aligned} \tag{1.2}$$

where tr_g denotes the trace with respect to the metric g. Note that when $|d\phi| \neq 0$, we can write

$$\tau_p(\phi) = |d\phi|^{p-2} \{ \tau_2(\phi) + (p-2) d\phi(\mathrm{grad}_g (ln|d\phi|)) \}. \tag{1.3}$$

In particular, $\tau_2(\phi)$ is called the *tension field* of ϕ, i.e. $\tau_2(\phi)$ is the trace of the second fundamental form of ϕ.

Note that 2-harmonic maps are well-known to be harmonic maps. Several studies are given for harmonic maps (see [7]). For these harmonic maps, there are Liouville type theorems, which states that a harmonic map ϕ is constant under some conditions.

The classical Liouville theorem says that any bounded harmonic function defined on the whole plane must be constant. In 1975, S. T. Yau ([17]) generalized the Liouville theorem to harmonic function on Riemannian manifolds of non-negative Ricci curvature. After that, the Liouville theorem was extended to several cases of manifolds. First, we consider the following conditions on M and N:

$(C1)$ *M is a complete Riemannian manifold of non-negative Ricci curvature.*

$(C2)$ *The sectional curvature of a complete Riemannian manifold N is non-positive.*

In 1976, R. M. Schoen and S. T. Yau ([14]) proved the following theorem.

Theorem 1.1 *Under the above assumptions $(C1)$ and $(C2)$, any harmonic map $\phi : M \to N$ of $E_2(\phi) < \infty$ is constant.*

In 1998, N. Nakauchi ([12]) showed the following theorem.

Theorem 1.2 *Under the above assumptions $(C1)$ and $(C2)$, any p-harmonic map $\phi : M \to N$ of $E_p(\phi) < \infty$ $(p > 2)$ is constant.*

Let μ_0 be the least eigenvalue of the Laplacian acting on L^2-function on M. Then we assume the following weaker condition than $(C1)$ on M.

$(WC1)$ *M is a complete Riemannian manifold such that $Ric^M \geq -\mu_0$ at all point $x \in M$ and either $Ric^M > -\mu_0$ at some point x_0 or $Vol(M)$ is infinite.*

In 1997, S. D. Jung ([8]) improved Theorem 1.1 to harmonic maps on a complete Riemannian manifold M which satisfies the condition $(WC1)$. Namely

Theorem 1.3 *Under the above assumptions $(WC1)$ and $(C2)$, any harmonic map $\phi : M \to N$ of $E_2(\phi) < \infty$ is constant.*

Now, we consider the generalized weak condition:

$(GWC1)$ *M is a complete Riemannian manifold such that $Ric^M \geq -\frac{4(p-1)}{p^2}\mu_0$ for all point $x \in M$ and $Ric^M > -\frac{4(p-1)}{p^2}\mu_0$ at some point x_0.*

In Chapter 3, we study the Liouville type theorem for p-harmonic maps under the generalized weak condition. Namely,

Theorem 1.4 *Under the above assumptions $(GWC1)$ and $(C2)$, any p-harmonic map $\phi : M \to N$ of $E_p(\phi) < \infty$ is constant.*

A map $\phi : (M, g) \to (N, h)$ is a *p-harmonic morphism* if it pulls back (local) p-harmonic function on N to (local) p-harmonic function on M, i.e., for any function $f : V \subset N \to \mathbb{R}$ if $\tau_p(f) = 0$, then $\tau_p(f \circ \phi) = 0$. It is well known ([12]) that a non-constant map is a p-harmonic morphism if and only if it is a horizontal weakely conformal p-harmonic map. Now we consider the following weaker condition than (C2) on N.

$(WC2)$ *The scalar curvature of a complete Riemannian manifold N is non-positive.*

In 2001, G. D. Choi and G. J. Yun ([3]) proved the following theorem.

Theorem 1.5 *Under the assumptions* $(C1)$ *and* $(WC2)$*, any* 2*-harmonic morphism* $\phi : M \to N$ *of* $E_2(\phi) < \infty$ *is constant.*

In Chapter 4, we extend Theorem 1.5 under the weak condition of M. That is, we have the following theorem.

Theorem 1.6 *Under the assumptions* $(WC1)$ *and* $(WC2)$*, any harmonic morphism* $\phi : M \to N$ *of* $E_2(\phi) < \infty$ *is constant.*

In 2003, G. D. Choi and G. J. Yun([4]) extended Theorem 1.5 to any arbitrary p-harmonic morphism. Namely, we have

Theorem 1.7 *Under the assumptions* $(C1)$ *and* $(WC2)$*, any* p*-harmonic morphism* $\phi : M \to N$ *of* $E_p(\phi) < \infty$ *is constant.*

Moreover we also improve Theorem 1.7 to p-harmonic morphism on a complete Riemannian manifold M which satisfies the condition $(GWC1)$. Namely, we have

Theorem 1.8 *Under the assumptions* $(GWC1)$ *and* $(WC2)$*, any* p*-harmonic morphism* $\phi : M \to N$ *of* $E_p(\phi) < \infty$ *is constant.*

2 Weitzenböck formulas and cut off functions

2.1 Weitzenböck formulas

In this section, we review the Weitzenböck formula ([9,12,16]). Let (M^m, g) and (N^n, h) be Riemannian manifolds and let $\nabla^M$and ∇^N be their Levi-Civita connections respectively. Let $\phi : M \to N$ be a smooth map and $E = \phi^{-1}TN$ be the induced bundle over M. Then E has a naturally induced metric connection $\nabla \equiv \phi^{-1}\nabla^N$. Trivially $d\phi$ is a cross section of *Hom*(TM,E) over M. Since *Hom*(TM,E) is canonically identified with $T^*M \otimes E$, $d\phi$ is regarded as an E-valued 1-form on M. Let $d_\nabla : A^r(E) \to A^{r+1}(E)$ be an anti derivation and δ_∇ the formal adjoint of d_∇, where $A^r(E)$ is the space of E-valued r-forms with an inner product $\langle \cdot, \cdot \rangle$ on M. Let $\{e_i\}_{i=1,\cdots,m}$ and $\{v_a\}_{a=1,\cdots,n}$ be local orthonomal frame fields on M and N respectively, and let $\{w^i\}_{i=1,\cdots,m}$ and $\{\theta^a\}_{a=1,\cdots,n}$ be their dual coframe fields on M and N respectively. Locally, the operators d_∇ and δ_∇ are expressed by

$$d_\nabla = \sum_{i=1}^m w^i \wedge \nabla_{e_i} \qquad \text{and} \qquad \delta_\nabla = -\sum_{j=1}^m i(e_j)\nabla_{e_j}$$

respectively, where $i(X)$ denotes the *interior product*, i.e. if η is a r-form, then $i(X)\eta$ is the $(r-1)$-form defined by $\{i(X)\eta\}(Y_1 \cdots Y_{r-1}) = \eta(X, Y_1 \cdots Y_{r-1})$. The Laplacian $\triangle$ on $A^*(E)$ is defined by

$$\triangle = d_\nabla \delta_\nabla + \delta_\nabla d_\nabla. \tag{2.1}$$

We now give the computation of the Weitzenböck formula.

Theorem 2.1 (Weitzenböck formula) *On an oriented Riemannian manifold M of dimension m, we have*

$$\triangle = -\sum_{i}^{m} \nabla^2_{e_i e_i} + \sum_{k,j}^{m} w^k \wedge i(e_j) R(e_j, e_k), \tag{2.2}$$

where $\nabla^2_{XY} = \nabla_X \nabla_Y - \nabla_{\nabla^M_X Y}$ *and* $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ *for any* $X, Y \in TM$.

Proof. Since the right side of the formula for $\triangle$ is independent of the choice of $\{e_i\}$, it suffices to check this formula at a point $x \in M$ with $\{e_i\}$ chosen to be normal at x. Then we have at $x \in M$

$$\begin{aligned}
\delta_\nabla d_\nabla &= -\sum_{i,j=1}^{m} i(e_j)\nabla_{e_j}\left(w^i \wedge \nabla_{e_i}\right) \\
&= -\sum_{i,j=1}^{m} i(e_j)\nabla_{e_j} w^i \wedge \nabla_{e_i} - \sum_{i,j=1}^{n} i(e_j)\left(w^i \wedge \nabla_{e_j}\nabla_{e_i}\right) \\
&= -\sum_{i,j=1}^{n} i(e_j) w^i \wedge \nabla_{e_j}\nabla_{e_i} + \sum_{i,j=1}^{m} w^i \wedge i(e_j)\nabla_{e_j}\nabla_{e_i} \\
&= -\sum_{i=1}^{n} \nabla_{e_i}\nabla_{e_i} + \sum_{i,j=1}^{m} w^i \wedge i(e_j)\nabla_{e_j}\nabla_{e_i}.
\end{aligned}$$

To compute $d_\nabla \delta_\nabla$, we note that at x, the identity

$$i(e_j)\nabla_{e_k} = \nabla_{e_k} i(e_j) \tag{2.3}$$

is valid on forms for all j, k. Thus a direct calulation with (2.3) gives

$$\begin{aligned}
d_\nabla \delta_\nabla &= \sum_{i=1}^{m} w^i \wedge \nabla_{e_i}\left(-\sum_{j=1}^{m} i(e_j)(\nabla_{e_j})\right) \\
&= -\sum_{i,j=1}^{m} w^i \wedge \nabla_{e_i}\left(i(e_j)\nabla_{e_j}\right) \\
&= -\sum_{i,j=1}^{m} w^i \wedge i(e_j)\nabla_{e_i}\nabla_{e_j}.
\end{aligned}$$

So the Laplacian $\triangle$ is given by

$$\begin{aligned}\triangle &= -\sum_{i=1}^{n} \nabla^2_{e_i e_i} + \sum_{i,j=1}^{n} w^i \wedge i(e_j)[\nabla_{e_j}\nabla_{e_i} - \nabla_{e_i}\nabla_{e_j}] \\ &= -\sum_{i=1}^{n} \nabla^2_{e_i e_i} + \sum_{i,j=1}^{n} w^i \wedge i(e_j) R(e_j, e_i). \qquad \square\end{aligned}$$

Corollary 2.2 *On functions, as well as on forms of degree n, we have*

$$\triangle = -\sum_i \nabla^2_{e_i e_i}. \tag{2.4}$$

Proof. Since $R(e_i, e_j)$ is a derivation on forms, $R(e_i, e_j)1 = 0$. Moreover since R is a tensor field, we have

$$R(e_i, e_j)f = fR(e_i, e_j)1 = 0.$$

Thus the assertion is clear for functions. For a form ψ of degree n,

$$\omega^i \wedge i(e_j)\psi = \delta^i_j \psi.$$

Since $R(e_i, e_j)\psi$ is also of degree n, we have

$$\sum_{i,j} \omega^i \wedge i(e_j) R(e_i, e_j)\psi = \sum_i R(e_i, e_i)\psi = 0.$$

Hence the proof is completed. $\square$

From Theorem 2.1, we have the following scalar Weitzenböck formula.

Proposition 2.3 *For any $\Phi \in A^r(E)$, we have*

$$-\frac{1}{2}\triangle^M |\Phi|^2 = |\nabla \Phi|^2 - \langle \triangle \Phi, \Phi \rangle + \sum_{k,j} \langle \omega^k \wedge i(e_j) R(e_j, e_k)\Phi, \Phi \rangle. \tag{2.5}$$

For applications, if we put $\Phi = |d\phi|^{p-2} d\phi$, then we have

Proposition 2.4 *Let $\phi : (M^m, g) \to (N^n, h)$ be an arbitrary smooth map. Then we have*

$$-\frac{1}{2}\triangle^M |d\phi|^{2p-2} = |\nabla(|d\phi|^{p-2}d\phi)|^2 - \langle |d\phi|^{p-2}d\phi, \triangle(|d\phi|^{p-2}d\phi)\rangle + F(\phi), \tag{2.6}$$

where

$$\begin{aligned} F(\phi) &= |d\phi|^{2p-4}\sum_{k=1}^{m}\langle Ric^M(d\phi(e_k)), d\phi(e_k)\rangle \\ &- |d\phi|^{2p-4}\sum_{k,j=1}^{m}\langle R^N(d\phi(e_j), d\phi(e_k))d\phi(e_k), d\phi(e_j)\rangle. \end{aligned}$$

Proof. Let R^E be the curvature tensor of ∇ on E. Then R^E is related to the curvature tensor R^N of ∇^N in the following way: Let $X, Y \in T_xM$ and $s \in \Gamma E$, then

$$R^E(X,Y)s = R^N(d\phi_x(X), d\phi_x(Y))s. \tag{2.7}$$

When a function f is given on N, we shall identify it throughout this paper with the function $f \circ \phi$ induced on M. Let $f^a \equiv \phi^*\theta^a$. Then $d\phi$ is expressed by

$$d\phi = \sum_{a=1}^{n} f^a \otimes v_a. \tag{2.8}$$

Since a direct calculation gives

$$R(e_j, e_k)d\phi = \sum_a R^M(e_j, e_k)f^a \otimes v_a + \sum_a f^a \otimes R^E(e_j, e_k)v_a, \tag{2.9}$$

we have

$$\begin{aligned} &\sum_{k,j}\langle w^k \wedge i(e_j)R(e_j, e_k)d\phi, d\phi\rangle \\ &= \sum_{k,j,a,b}\langle w^k \wedge i(e_j)R^M(e_j, e_k)f^a \otimes v_a, f^b \otimes v_b\rangle \\ &+ \sum_{k,j,a,b} g(w^k \wedge i(e_j)f^a, f^b)h(R^E(e_j, e_k)v_a, v_b). \end{aligned}$$

Since $d\phi(e_\ell) = \sum_a f^a(e_\ell)v_a$, we have

$$\sum_{k,j,a} g(w^k \wedge i(e_j)R^M(e_j, e_k)f^a, f^a) = \sum_k h(d\phi(Ric^M(e_k)), d\phi(e_k)). \tag{2.10}$$

From (2.7) and (2.10), we have

$$\begin{aligned}\sum_{k,j}\langle w^k \wedge i(e_j)R(e_j, e_k)d\phi, d\phi\rangle = &\sum_k h(d\phi(Ric^M(e_k)), d\phi(e_k)) \\ &+ \sum_{k,j} h(R^N(d\phi(e_j), d\phi(e_k))d\phi(e_j), d\phi(e_k)),\end{aligned}$$

which prove (2.6). □

2.2 Cut off functions

Let x_0 be a point of M and fix it. For each point $y \in M$, we denote by $\rho(y)$ the geodesic distance from x_0 to y. Let $B(\ell) = \{y \in M | \rho(y) < \ell\}$ for $\ell > 0$. Then there exists a Lipschitz continuous function ω_ℓ on M satisfying the following properties:

$$\begin{aligned}&0 \le \omega_\ell(y) \le 1 \qquad for\ \ any \quad y \in M, \\ &\mathrm{supp}\omega_\ell \subset B(2\ell), \\ &\omega_\ell(y) = 1 \qquad for\ \ any \quad y \in B(\ell), \\ &\lim_{\ell\to\infty} \omega_\ell = 1, \\ &|d\omega_\ell| \le \frac{C}{\ell} \qquad almost\ \ everywhere\ \ on \quad M,\end{aligned}$$

Where $C(> 0)$ is a constant independent of ℓ([1]). Then we have

Lemma 2.5 ([1]) *For any $\Phi \in A^r(E)$, there exists a positive constant A independent of ℓ such that*

$$\|d\omega_\ell \wedge \Phi\|^2_{B(2\ell)} \leq \frac{A}{\ell^2}\|\Phi\|^2_{B(2\ell)},$$
$$\|d\omega_\ell \wedge *\Phi\|^2_{B(2\ell)} \leq \frac{A}{\ell^2}\|\Phi\|^2_{B(2\ell)},$$

where $\|\Phi\|^2_{B(2\ell)} = \int_{B(2\ell)}\langle\Phi,\Phi\rangle$ and $*$ is the Hodge-star operator.

Now, we remark that, for $\Phi \in L_2^r(E) \cap A^r(E)$, $\omega_\ell\Phi$ has compact support and $\omega_\ell\Phi \to \Phi(\ell \to \infty)$ in the strong sense. From $d_\nabla(S_a\eta^a) = \nabla S_a \wedge \eta^a + S_a(d\eta^a)$ for $S_a \in E$ and $\delta_\nabla\Phi = (-1)^{n(r+1)+1} * d_\nabla * \Phi$ for any $\Phi \in A^r(E)$, we have

$$d_\nabla(\omega_\ell^2\Phi) = \omega_\ell^2 d_\nabla\Phi + 2\omega_\ell d\omega_\ell \wedge \Phi,$$
$$\delta_\nabla(\omega_\ell^2\Phi) = \omega_\ell^2\delta_\nabla\Phi - *(2\omega_\ell d\omega_\ell \wedge *\Phi).$$

By using the inequality $|\langle a,b\rangle| \leq \frac{1}{t}|a|^2 + t|b|^2$ for any positive real number t, we have

$$| \ll \omega_\ell\delta_\nabla\Phi, *(d\omega_\ell \wedge *\Phi) \gg_{B(2\ell)} | \leq \frac{1}{4}\|\omega_\ell\delta_\nabla\Phi\|^2_{B(2\ell)} + 4\| * (d\omega_\ell \wedge *\Phi)\|^2_{B(2\ell)}.$$

From Lemma 2.5, we have

$$| \ll \omega_\ell\delta_\nabla\Phi, *(d\omega_\ell \wedge *\Phi) \gg_{B(2\ell)} | \leq \frac{1}{4}\|\omega_\ell\delta_\nabla\Phi\|^2_{B(2\ell)} + \frac{4A}{\ell^2}\|\Phi\|^2_{B(2\ell)}. \tag{2.11}$$

Similarly we have

$$| \ll \omega_\ell d_\nabla\Phi, d\omega_\ell \wedge \Phi \gg_{B(2\ell)} | \leq \frac{1}{4}\|\omega_\ell d_\nabla\Phi\|^2_{B(2\ell)} + \frac{4A}{\ell^2}\|\Phi\|^2_{B(2\ell)}. \tag{2.12}$$

3 Harmonic maps

3.1 Harmonic functions on Euclidean spaces

Definition 3.1 *Harmonic functions* on an open domain Ω of $\mathbb{R}^m$ are solutions of the Laplace equation

$$\triangle f = 0, \tag{3.1}$$

where $\triangle := -\frac{\partial^2}{(\partial x_1)^2} - \cdots - \frac{\partial^2}{(\partial x_m)^2}$ and $(x_1, \cdots, x_m) \in \Omega$. The operator $\triangle$ is called the *Laplace operator* or *Laplacian.*

Theorem 3.2 *The harmonic functions are critical points of the Dirichlet functional*

$$E_2(f; \Omega) = \frac{1}{2}\int_\Omega |df|^2 dx. \tag{3.2}$$

Proof. For any smooth function g with compact support in Ω, the first variation gives

$$\begin{aligned} \frac{d}{dt}E_2(f_t; \Omega)\bigg|_{t=0} &: = lim_{t\to 0}\{E_2(f_t; \Omega) - E_2(f; \Omega)\}/t \\ &= \int_\Omega \sum_a^m \frac{\partial f}{\partial x_a}\frac{\partial g}{\partial x_a}dx \\ &= \int_\Omega (\triangle f) g dx, \end{aligned}$$

where $f_t = f + tg$. Hence if we choose $g = \triangle f$, then the proof is completed. □

3.2 Harmonic maps between Riemannian manifolds

In this section, we review the harmonic maps. See [2] for details. Let (M, g) and (N, h) be smooth Riemannian manifolds and let $\phi : (M, g) \to (N, h)$ be a smooth map.

Definition 3.3 Let $\phi : (M, g) \to (N, h)$ be a smooth map. Let Ω be a domain of M. The *energy* or *Dirichlet integral* of ϕ over Ω is defined by

$$E_2(\phi; \Omega) = \frac{1}{2} \int_\Omega |d\phi|^2 dM, \tag{3.3}$$

where $|d\phi_x|^2 = \sum_{i=1}^m h(d\phi_x(e_i), d\phi_x(e_i))$ and $\{e_i\}$ is an orthonomal basis for T_xM. A smooth map ϕ is called *harmonic* if it is a critical point of the energy integral (3.3).

Let $\{\phi_t\}$ be all smooth one-parameter of ϕ and v the variation vector field of ϕ_t defined by $v = \frac{d\phi_t}{dt}\big|_{t=0}$. The *tension field* $\tau(\phi)$ of ϕ is defined by

$$\tau(\phi) := tr_g \nabla d\phi = div(d\phi) = \sum_{i=1}^m (\nabla_{e_i} d\phi)(e_i). \tag{3.4}$$

Then we have the following.

Theorem 3.4 (First variation of the energy) *Let $\phi : M \to N$ be a smooth map and let $\{\phi_t\}$ be a smooth variation of ϕ supported in Ω. Then*

$$\frac{d}{dt} E_2(\phi_t; \Omega)\bigg|_{t=0} = -\int_\Omega \langle \tau(\phi), v \rangle dM. \tag{3.5}$$

where $v = \frac{d\phi_t}{dt}\big|_{t=0}$ denotes the variation vector field of $\{\phi_t\}$.

Proof. Let Ω be a compact domain of M and let $\{\phi_t\}$ be a variation of ϕ supported in Ω with variation vector field $v \in \Gamma(\phi^{-1}TN)$. Let $\{e_i\}$ be a local orthonormal frame on M. Define $\Phi : M \times (-\varepsilon, \varepsilon) \to N$ by $\Phi(x,t) = \phi_t(x)((x,t) \in M \times (-\varepsilon, \varepsilon))$ and set $E = \Phi^{-1}TN \to M \times (-\varepsilon, \varepsilon)$. Let ∇^{Φ} denote the pull-back connection on E. Note that, for any vector field X on M considered as a vector field on $M \times (-\varepsilon, \varepsilon)$, we have $\left[\frac{\partial}{\partial t}, X\right] = 0$. If we use $\nabla^{\phi}_X(d\phi(Y)) - \nabla^{\phi}_Y(d\phi(X)) = d\phi([X,Y])$ $(X, Y \in \Gamma TM)$, then

$$\begin{aligned}\frac{d}{dt}E_2(\phi_t;\Omega)\bigg|_{t=0} &= \int_\Omega \sum_{i=1}^m \langle \nabla^{\Phi}_{\frac{\partial}{\partial t}} d\Phi(e_i), d\Phi(e_i)\rangle\bigg|_{t=0} \\ &= \int_\Omega \sum_{i=1}^m \langle \nabla^{\Phi}_{e_i} d\Phi(\frac{\partial}{\partial t}), d\Phi(e_i)\rangle\bigg|_{t=0} \\ &= \int_\Omega \sum_{i=1}^m \langle \nabla^{\phi}_{e_i} v, d\phi(e_i)\rangle,\end{aligned}$$

where the last equality holds because $d\Phi(\frac{\partial}{\partial t}) = v$ and $d\Phi(e_i) = d\phi(e_i)$ when $t = 0$. Define a 1-form ψ on M by $\psi(-) = \langle v, d\phi(-)\rangle$. Then

$$\begin{aligned}\mathrm{div}\psi &= \sum_{i=1}^m \{e_i(\psi(e_i)) - \psi(\nabla^M_{e_i} e_i)\} \\ &= \sum_{i=1}^m \{e_i(\langle v, d\phi(e_i)\rangle) - \langle v, d\phi(\nabla^M_{e_i} e_i)\rangle\} \\ &= \sum_{i=1}^m \{\langle \nabla^{\phi}_{e_i} v, d\phi(e_i)\rangle + \langle v, \nabla^{\phi}_{e_i}(d\phi(e_i)) - d\phi(\nabla^M_{e_i} e_i)\rangle\}\end{aligned}$$

By the divergence theorem, the left hand side is zero. Hence from (3.4), the proof is completed. □

Corollary 3.5 *Let $\phi : M \to N$ be a smooth map. Then ϕ is harmonic if and only if*

$$\tau(\phi) = 0. \tag{3.6}$$

Remark 1 *Let $f : M \to \mathbb{R}$ be a smooth function. Then the Laplace-Beltrami operator $\triangle^M$ is given by*

$$\triangle^M f = \delta df = -\mathrm{tr}(\nabla df) = -\tau(f), \tag{3.7}$$

Hence $f : M \to \mathbb{R}$ *is a harmonic function if* $\triangle^M f = 0$.

Examples. We list some well-known examples of harmonic maps. (For details, See [2])

(1) Constant maps: $\phi : (M, g) \to (N, h)$ and identity maps $I_d : (M, g) \to (M, g)$ are clearly always harmonic.

(2) Isometries are harmonic maps. Further, composing a harmonic map with an isometry on its domain or codomain preserves harmonicity.

(3) Harmonic maps between Euclidean spaces: A smooth map $\phi : A \to \mathbb{R}^n$ from an open subset A of $\mathbb{R}^m$ is harmonic if and only if $\triangle\phi = 0$; here $\triangle$ is the usual (vector-valued) Laplacian on $\mathbb{R}^m$, thus ϕ is harmonic if and only if $\sum_{i=1}^{m} \frac{\partial^2 \phi^\alpha}{\partial x_i^2} = 0$ for all $\alpha \in \{1, 2, \cdots \in\}$, at all points $(x_1, \cdots, x_m) \in A$.

(4) Harmonic maps to a Euclidean spaces: A smooth map $\phi : (M, g) \to \mathbb{R}^n$ is harmonic if and only if each of its components is a harmonic function on (M, g), i.e.,.$\triangle^M \phi^\alpha = 0 \quad (\alpha = 1, \cdots, n)$. Note that, in the last two examples, the harmonic equation is *linear*. However, when the domain is not flat, this is no longer the case as sown by our next few examples.

(5) Harmonic maps to the circle: S^1 are given by integrating harmonic 1-form with integral periods. Hence, when the domain M is compact, there are non-constant harmonic maps to the circle if and only if the first Betti number of M is non zero. In fact, there is a harmonic map in every homotopyclass.

(6) Geodesics: For a smooth map (curve) $\phi : A \to N$ from an open subset A of $\mathbb{R}$ or from the circle S^1, the tension field is just the acceleration vector of the curve; Hence ϕ is *harmonic if and only if it defined a geodesic* parametrized linearly (i.e., parametrized by a constant multiful of arc length). More generally, a map $\phi : M \to N$ is called *totally geodesic* if it maps linearly parametrized geodesic of M to linearly parametrized geodesic of N, such maps are chracterized by the vanishing of their second fundmental form.

(7) Holomorphic maps: Holomorphic maps $\phi : (M, g, J^M) \to (N, h, J^N)$ between Kähler manifolds are harmonic. Indeed, when M is compact, the energy integral decomposes into

$$E(\phi) = E^{'}(\phi) + E^{''}(\phi)$$

where

$$E^{'}(\phi) = \int_M |\partial\phi|^2 \omega_g \quad and \quad E^{''}(\phi) = \int_M |\overline{\partial}\phi|^2 \omega_g$$

Here $\partial\phi$ (resp. $\overline{\partial}\phi$) denotes the (1,0) (resp. (0,1) part of $d\phi$; this vanishes precisely when ϕ is antiholomorphic (reps. holomorphic).

(8) Isometric immersions: Let $\phi : (N, h) \to (P, k)$ be an isometric immersion. Then its second fundamental form $\beta(\phi)$ of ϕ has values in the normal space and coincides with the usual second fundamental form $A \in \Gamma(S^2T^*N \otimes NN)$ of N as an (immersed) submanifolds of P defined on vector fields X, Y on M by $A(X, Y) = -$normal component of $\nabla_X Y$ (Here, by S^2T^*N we denotes the symmetrized tensor product of T^*N with itself and NN is the normal bundle of N in P) In particular, the tension field $\tau(\phi)$ is m times the mean curvature of M in N so that ϕ is harmonic if and only if M is a minimal submanifold of N.

(9) Compositions: The composition of two harmonic maps is not, in integral, harmonic. In fact, the tension field of the composition of two smooth maps $\phi : (M, g) \to (N, h)$ and $f : (N, h) \to (P, k)$ is given by

$$\begin{aligned}\tau(f \circ \phi) =& df(\tau(\phi)) + \beta(f)(d\phi, d\phi) \\ =& df(\tau(\phi)) + \sum_{i=1}^{m} \beta(f)(d\phi(e_i), d\phi(e_i)),\end{aligned} \tag{3.8}$$

where $\{e_i\}$ is an orthonomal frame on N. From this we see that *if ϕ is harmonic and f totally geodesic, then $f \circ \phi$ is harmonic.*

3.3 Liouville type theorem for harmonic maps

Let (M^m, g) and (N^n, h) be Riemannian manifolds with $\dim M = m \geq n = \dim N$. Let $\{e_i\}_{i=1,\cdots,m}$ be a local orthonomal frame field and $\{w^i\}_{i=1,\cdots,m}$ the dual coframe field of $\{e_i\}$. Let $\phi : M \to N$ be a harmonic map. It is trivial from (3.4) that

$$\delta_{\nabla}(d\phi) = 0. \tag{3.9}$$

From Proposition 2.4, we have the following proposition.

Proposition 3.6 *Let $\phi : (M, g) \to (N, h)$ be a harmonic map. Then*

$$\begin{aligned}-\frac{1}{2}\triangle^M |d\phi|^2 = |\nabla |d\phi||^2 - < d\phi, \delta_{\nabla} d_{\nabla}(d\phi) > + \sum_{k=1}^{m} h(d\phi(Ric^M(e_k)), d\phi(e_k)) \\ - \sum_{k,j=1}^{m} h(R^N(d\phi(e_j), d\phi(e_k))d\phi(e_k), d\phi(e_j)).\end{aligned} \tag{3.10}$$

From Proposition 3.6, we can prove the following theorem.

Theorem 3.7 ([3]) *Let M be a complete Riemannian manifold non-negative Ricci cuvature and N be a complete Riemannian manifold of non-positive sectional curvature. Then any harmonic map $\phi : M \to N$ of $E_2(\phi) < \infty$ is constant.*

Let μ_0 is the infimum of the spectrum of the Laplacian $\triangle^M$ on L^2-function on M.

Theorem 3.8 ([8]) *Let* $\phi : M \to N$ *be a harmonic map from a complete Riemannian manifold* M *to a Riemannian manifold* N *with non-positive sectional curvature. Assume* $Ric^M \geq -\mu_0$ *at all* $x \in M$ *and either* $Ric^M > -\mu_0$ *at some point* $x_0 \in M$*. Then any harmonic map* $\phi : M \to N$ *of* $E_2(\phi) < \infty$ *is constant.*

3.4 Liouville type theorem for p-harmonic maps

Let (M^m, g) and (N^n, h) be Riemannian manifolds with dim$M = m \geq n = dimN$. Let $\{e_i\}_{i=1,\cdots,m}$ be a local orthonomal frames on M. Let $\phi : (M, g) \to (N, h)$ be a p-harmonic map. Then we have from (3.6)

$$\delta_\nabla(|d\phi|^{p-2}d\phi) = 0. \tag{3.11}$$

From Proposition 2.4, we have the following lemma.

Lemma 3.9 *Let* $\phi : (M^m, g) \to (N^n, h)$ *be a* p*-harmonic map. Then the Weitzenbočk formula is given by*

$$-\frac{1}{2}\Delta^M|d\phi|^{2p-2} = |\nabla(|d\phi|^{p-2}d\phi)|^2 - \langle|d\phi|^{p-2}d\phi, \delta_\nabla d_\nabla(|d\phi|^{p-2}d\phi)\rangle + F(\phi), \tag{3.12}$$

where

$$\begin{aligned} F(\phi) &= |d\phi|^{2p-4}\sum_{k=1}^{m} h(d\phi(Ric^M(e_k)), d\phi(e_k)) \qquad (3.13)\\ &- |d\phi|^{2p-4}\sum_{k,j=1}^{m} h(R^N(d\phi(e_j), d\phi(e_k))d\phi(e_k), d\phi(e_j)). \end{aligned}$$

From Lemma 3.9 we have the following proposition.

Proposition 3.10 *Let M be a complete Riemannian manifold such that for some constant $C \geq 0$, $Ric^M \geq -C$ at all $x \in M$ and let N be a Riemannian manifold of non-positive sectional curvature. If $\phi : (M, g) \to (N, h)$ is a p-harmonic map, then*

$$\begin{aligned} &|d\phi| \triangle^M |d\phi|^{p-1} - \langle d\phi, \delta_\nabla d_\nabla (|d\phi|^{p-2} d\phi) \rangle \\ &\leq -|d\phi|^{p-2} \sum_{i=1}^m h(d\phi(Ric^M(e_i)), d\phi(e_i)) \\ &\leq C|d\phi|^p. \end{aligned} \tag{3.14}$$

Proof. Since $\frac{1}{2}\Delta^M |d\phi|^{2p-2} = |d\phi|^{p-1}\Delta^M |d\phi|^{p-1} - |\nabla^M |d\phi|^{p-1}|^2$, from (3.12) and (3.11) we have

$$\begin{aligned} |d\phi|^{p-1} \triangle^M |d\phi|^{p-1} &= \left|\nabla^M |d\phi|^{p-1}\right|^2 - \left|d_\nabla (|d\phi|^{p-2} d\phi)\right|^2 - F(\phi) \\ &+ \langle |d\phi|^{p-2} d\phi, \delta_\nabla d_\nabla (|d\phi|^{p-2} d\phi) \rangle. \end{aligned} \tag{3.15}$$

By the first Kato's inequality([1]), i.e.,$|d_\bigtriangledown (|d\phi|^{p-2} d\phi)|^2 \geq \left|\bigtriangledown^M |d\phi|^{p-1}\right|^2$, the equation (3.15) implies

$$|d\phi|^{p-1} \triangle^M |d\phi|^{p-1} - \langle |d\phi|^{p-2} d\phi, \delta_\triangledown d_\triangledown (|d\phi|^{p-2} d\phi) \rangle + F(\phi) \leq 0 \tag{3.16}$$

On the other hand, under the conditions that the sectional curvature of N is non-positive and $Ric^M \geq -C$, (3.13) implies

$$F(\phi) \geq |d\phi|^{2p-4} \sum_{i=1}^m h\left(d\phi(Ric^M(e_i)), \phi(e_i)\right) \geq -C|d\phi|^{2p-2} \tag{3.17}$$

Hence (3.14) is obtained from (3.16) and (3.17). □

Theorem 3.11 ([9]) *Let M be a complete Riemannian manifold such that $Ric^M \geq -\frac{4(p-1)}{p^2}\mu_0$ for all $x \in M$ and $Ric^M > -\frac{4(p-1)}{p^2}\mu_0$ at some* *point x_0. Let N be a complete Riemannian manifold with a non-positive sectional curvature. Then any p-harmonic map with $E_p(\phi) < \infty$ is constant.*

Proof. Let x_0 be a point of M and fix it. We choose a Lipschitz continuous function ω_ℓ on M such that $0 \leq \omega_\ell(y) \leq 1$ for any $y \in M$. Multiplying (3.14) by ω_ℓ^2 and integrating by parts, we get

$$\begin{aligned}
&\int_M \langle \omega_\ell^2 |d\phi|, \triangle^M\, |d\phi|^{p-1}\rangle - \int_M \langle \omega_\ell^2 d\phi, \delta_\nabla d_\nabla (|d\phi|^{p-2} d\phi)\rangle \\
&\qquad \leq -\sum_{i=1}^{m} \int_M \omega_\ell^2 |d\phi|^{p-2} h(d\phi(Ric^M(e_i)), d\phi(e_i)) \\
&\qquad \leq C \int_M \omega_\ell^2 |d\phi|^p.
\end{aligned} \tag{3.18}$$

Since the inequality $|\langle V, W\rangle| \leq |V||W|$, By a direct calculation we have

$$\begin{aligned}
\int_M \langle \omega_\ell^2 |d\phi|, \triangle^M |d\phi|^{p-1}\rangle &= \int_M \langle d(\omega_\ell^2 |d\phi|), d|d\phi|^{p-1}\rangle \\
&= A_1 \int_M \langle |d\phi|^{\frac{p}{2}} d\omega_\ell, \omega_\ell d|d\phi|^{\frac{p}{2}}\rangle \\
&\quad + \frac{A_1}{p} \int_M \omega_\ell^2 \left| d|d\phi|^{\frac{p}{2}} \right|^2, \\
&\geq -A_1 \int_M \omega_\ell |d\phi|^{\frac{p}{2}} |d\omega_\ell| \left| d|d\phi|^{\frac{p}{2}} \right| \\
&\quad + \frac{A_1}{p} \int_M \omega_\ell^2 \left| d|d\phi|^{\frac{p}{2}} \right|^2,
\end{aligned} \tag{3.19}$$

where $A_1 = \frac{4(p-1)}{p}$. Since $\left|\langle |d\phi|^{\frac{p}{2}} d\omega_\ell, \omega_\ell d|d\phi|^{\frac{p}{2}}\rangle\right| \leq \frac{1}{a} |d\phi|^p |d\omega_\ell|^2 + a|\omega_\ell d|d\phi|^{\frac{p}{2}}|^2$ for any real number $a > 0$, the equation (3.19) implies

$$\begin{aligned}
\int_M \langle \omega_\ell^2 |d\phi|, \triangle^M |d\phi|^{p-1}\rangle &\geq -\frac{A_1}{a} \int_M |d\phi|^p |d\omega_\ell|^2 \\
&\quad + \left(\frac{1}{p} - a\right) A_1 \int_M \left| \omega_\ell d|d\phi|^{\frac{p}{2}} \right|^2.
\end{aligned} \tag{3.20}$$

It is well-known ([9]) that for a function f on M and for some constant $b > 0$,

$$|d_{\triangledown}(f d\phi)| \leq b|df||d\phi|. \tag{3.21}$$

Hence we have

$$\begin{aligned}
|\int_M \langle w_\ell^2 d\phi, &\delta_\triangledown d_\triangledown(|d\phi|^{p-2}d\phi)\rangle| \\
&= \left|\int_M \langle d_\triangledown(w_\ell^2 d\phi), d_\triangledown(|d\phi|^{p-2}d\phi)\rangle\right| \\
&\leq \int_M \left|d_\triangledown(w_\ell^2 d\phi)\right| \left|d_\triangledown(|d\phi|^{p-2}d\phi)\right| \\
&\leq 2b^2 \int_M w_\ell dw_\ell \left|d|d\phi|^{p-2}\right| |d\phi|^2 \\
&\leq A_2 \int_M w_\ell |dw_\ell||d\phi|^{\frac{p}{2}} \left|d|d\phi|^{\frac{p}{2}}\right| \\
&\leq \alpha A_2 \int_M w_\ell^2 \left|d|d\phi|^{\frac{p}{2}}\right|^2 + \frac{A_2}{\alpha}\int_M |dw_\ell|^2|d\phi|^p
\end{aligned}$$

for any real number $\alpha > 0$, where $A_2 = \frac{4(p-2)}{p}b^2$. From (3.20) and (3.21), we have

$$\begin{aligned}
&\int_M \langle \omega_\ell^2|d\phi|, \triangle^M|d\phi|^{p-1}\rangle - \int_M \langle \omega_\ell^2 d\phi, \delta_\triangledown d_\triangledown(|d\phi|^{p-2}d\phi)\rangle \qquad (3.22) \\
&\geq -(A_1 + A_2)\int_M \omega_\ell|d\omega_\ell||d\phi|^{\frac{p}{2}}\left|d|d\phi|^{\frac{p}{2}}\right| + \frac{A_1}{p}\int_M \left|\omega_\ell d|d\phi|^{\frac{p}{2}}\right|^2.
\end{aligned}$$

From (3.18) and the Fauto's inequality, it is trivial that $d|d\phi|^{\frac{p}{2}} \in L^2$. Hence by the Hölder inequality

$$\int_M \omega_\ell|d\omega_\ell||d\phi|^{\frac{p}{2}}\left|d|d\phi|^{\frac{p}{2}}\right| \leq \left(\int_M |d\omega_\ell|^2|d\phi|^p\right)^{\frac{1}{2}}\left(\int_M \omega_\ell^2\left|d|d\phi|^{\frac{1}{2}}\right|^2\right)^{\frac{1}{2}}.$$

If we let $\ell \to \infty$, then $\int_M \omega_\ell|d\omega_\ell||d\phi|^{\frac{p}{2}}\left|d|d\phi|^{\frac{p}{2}}\right| \to 0$. From (3.18) and (3.22), we have

$$\frac{A_1}{p}\int_M \left|d|d\phi|^{\frac{p}{2}}\right|^2 \leq -\sum_{i=1}^m \int_M |d\phi|^{p-2}h\left(d\phi(Ric^M(e_i)), d\phi(e_i)\right) \leq C\int_M |d\phi|^p. \tag{3.23}$$

On the other hand, by the Rayleigh theorem, i.e., $\int_M \langle df, df\rangle / \int_M f^2 \geq \mu_0$ for any smooth function f such that $supp(f) \subset \Omega$, a compact domain, and the Hölder inequality, if we put $f = \omega_\ell |d\phi|^{\frac{p}{2}}$, then we have

$$\mu_0 \int_M |d\phi|^p \leq \int_M \left| d|d\phi|^{\frac{p}{2}} \right|^2 . \tag{3.24}$$

From (3.23) and (3.24), we have

$$\frac{A_1}{p}\mu_0 \int_M |d\phi|^p \leq -\sum_{i=1}^m \int_M |d\phi|^{p-2} h\left(d\phi(Ric^M(e_i)), d\phi(e_i)\right) \leq C \int_M |d\phi|^p. \tag{3.25}$$

Since $C = \frac{4(p-1)}{p^2}\mu_0$, (3.25) implies that

$$\sum_{i=1}^m \int_M |d\phi|^{p-2} h\left(d\phi((Ric^M + C)(e_i)), d\phi(e_i)\right) = 0. \tag{3.26}$$

So if $Ric^M + C > 0$ at some point x, then $d\phi = 0$. This implies that ϕ is constant. □

For any q with $2 \leq q \leq p$, it is trial that $-\frac{4(p-1)}{p^2} \geq -\frac{4(q-1)}{q^2}$. So we have the following corollary.

Corollary 3.12 *Let M be a complete Riemannian manifold such that $Ric^M \geq -\frac{4(p-1)}{p^2}\mu_0$ at all $x \in M$ and $Ric^M > -\frac{4(p-1)}{p^2}\mu_0$ at some point x_0. Let N be a complete Riemannian manifold with a non-positive sectional curvature. Then any q-harmonic map $\phi : M \to N$ with $2 \leq q \leq p$ of $E_q(\phi) < \infty$ is constant.*

4 Harmonic morphisms

4.1 Horizontally weakly conformal maps

Definition 4.1 A smooth map $\phi : (M^m, g) \to (N^n, h)$ is called *horizontally (weakly) conformal* if for each $x \in M$ at which $d\phi_x \neq 0$, the restriction $d\phi_x|_{H_x} : H_x \to T_{\phi(x)}N$ is conformal and surjective, where $H_x = Ker(d\phi_x)^\perp$, the horizontal space of ϕ at x.

If we put $V_x = Ker(d\phi_x)$, then $T_xM = H_x \oplus V_x$. Let $C_\phi = \{x \in M \mid d\phi_x = 0\}$. Then we have the following.

Theorem 4.2 ([3]) *A smooth map $\phi : (M, g) \to (N, h)$ is horizontally weakly conformal if and only if there exists a function $\lambda : M - C_\phi \to \mathbb{R}^+$ such that*

$$h(d\phi(X), d\phi(Y)) = \lambda^2 g(X, Y) \quad \forall X, Y \in H_x. \tag{4.1}$$

Note that at the point $x \in C_\phi$ we can let $\lambda(x) = 0$ and obtain a continuous function $\lambda : M \to \mathbb{R}^+ \cup \{0\}$, which is called the *dilation* of a horizontally weakly conformal the map ϕ. Let $\{e_i\}_{i=1,\cdots,m}$ be a local orthonomal frames on M such that $\{e_i\} \in H_x(i = 1, \cdots, n)$ and $\{e_{n+i}\} \in V_x(i = 1, \cdots, m - n)$. On taking the trace in (4.1) at a regular or critical point x, we obtain

$$\lambda^2 = \frac{1}{n}|d\phi|^2. \tag{4.2}$$

Proposition 4.3 ([2]) *Let $\phi : M \to N$ be a horizontally weakly conformal map. if* $\dim M < \dim N$, *then ϕ is constant.*

When gradλ is vertical, a horizontally weakly conformal map is called a *horizontally homothetic* map. For example, a Riemannian submersion is horizontally homothetic.

4.2 Harmonic morphisms

Definition 4.4 A continuous map $\phi : (M, g) \to (N, h)$ is called a *harmonic morphism* if for any harmonic function $f : U \to \mathbb{R}$ on an open subset $U \subset N$ with $\phi^{-1}(U)$ non-empty, the composition $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is also a harmonic function on $\phi^{-1}(U)$. Namely, if $\tau(f) = 0$ for any f, then $\tau(\phi \circ f) = 0$.

Theorem 4.5 ([7]) *A smooth map $\phi : M \to N$ is a harmonic morphism if and only if it is harmonic and horizontally weakly conformal.*

Proposition 4.6 ([2]) *$\phi : M^m \to N^n$ be horizontally weakly conformal with dilation λ. Then, at a regular point,*

$$\tau(\phi) = d\phi(-(n-2)\text{grad}(ln\lambda)^H - (m-n)d\phi(\mu^\nu) = 0, \qquad (4.3)$$

where μ^ν denotes the mean curvature of the fibers.

Thus ϕ is harmonic, and so a harmonic morphism, if and only if, at regular points,

$$(n-2)\text{grad}(ln\lambda)^H + (m-n)\mu^\nu = 0, \qquad (4.4)$$

where $\text{grad}(ln\lambda)^H$ denotes the orthogonal projection of the gradient of the function $ln\lambda$ onto the horizontal distribution H.

Corollary 4.7 *If $n = 2$ or* $\mathrm{grad}\lambda$ *is vertical at regular points, a horizontally weakly conformal map is harmonic, and so is a harmonic morphism, if and only if its fibers are minimal at regular points.*

Corollary 4.8 *A Riemannian submersion is a horizontally conformal map with dilation 1. So a Riemannian submersion is a harmonic morphism if and only if its fibers are minimal.*

Examples. For more examples, see ([2]).

1. **Constants and identity maps** : are clearly harmonic morphisms.
2. **Harmonic morphisms between surfaces** : A smooth map between oriented surfaces is a harmonic morphism if and only if it is holomorphic or anti-holomorphic.
3. **Compositions** : the composition of two harmonic morphism is a harmonic morphism.
4. **A Riemannian submersion** :is harmonic, and so a harmonic morphism, if and only if its fibers are minimal. The *Hopf fiberations* have minimal(in fact, totally geodesic)fibers, and so are harmonic morphism,
5. **Warped product** :The natural projection of a *warped product* $F \times_{f^2} N \to N$ onto its second factor is a horizontal distribution. In particular, it is a harmonic morphism.

4.3 Liouville type theorem for harmonic morphisms

From Proposition 3.6 and (4.2) we have the following lemma.

Lemma 4.9 ([6]) *If* $\phi : M \to N$ *is a harmonic morphism, then*

$$-\frac{n}{2}\triangle^M \lambda^2 = |\nabla d\phi|^2 + \lambda^2 \mathrm{tr} Ric^M\big|_H - \lambda^4 r_N \circ \phi, \tag{4.5}$$

where λ *denotes the dilation,* $\mathrm{tr}Ric^M|_H$ *the trace of the Ricci tensor of* M *on the horizontal distribution* H*, and* r_N *the scalar curvature of* N*.*

Let μ_0 be the least eigenvalue of $\triangle^M$ acting on L^2-function on M. Then we have the following proposition.

Proposition 4.10 *Let* M *be a complete Riemannian manifold such that* $Ric^M \geq -\mu_0$ *at all* $x \in M$ *and let* N *be a Riemannian manifold of nonpositive scalar curvature. If* $\phi : M \to N$ *is a harmonic morphism, then*

$$n\triangle^M \lambda \leq -\lambda \mathrm{tr} Ric^M\big|_H \leq n\mu_0\lambda. \tag{4.6}$$

Proof. Since $\triangle^M\lambda^2 = 2\lambda\triangle^M\lambda - 2|\nabla^M\lambda|^2$, we have from (4.5),

$$n\lambda\triangle^M\lambda = n|\nabla^M\lambda|^2 - |\nabla d\phi|^2 - \lambda^2 \mathrm{tr}Ric^M\big|_H + \lambda^4\gamma_N \circ \phi. \tag{4.7}$$

Since $|d\phi|^2 = n\lambda^2$, we have $|d\phi|\nabla^M|d\phi| = n\lambda\nabla^M\lambda$ and

$$\left|\nabla^M|d\phi|\right|^2 = n|\nabla^M\lambda|^2. \tag{4.8}$$

By the first Kato's inequality([1]), i.e.,$\left|\nabla^M|d\phi|\right|^2 \leq n|\nabla d\phi|^2$, (4.8) yields

$$n|\nabla^M\lambda|^2 \leq |\nabla d\phi|^2. \tag{4.9}$$

Since the scalar curvature γ_N of N is nonpositive, the first inequality of (4.6) follows from (4.7) and (4.9). The second inequality of (4.6) is trivial from $Ric^M \geq -\mu_0$. □

Theorem 4.11 *Let M be a complete Riemannian manifolds such that* $\mathrm{Ric}^M \geq -\mu_0$ *at all point $x \in M$ and either* $\mathrm{Ric}^M > -\mu_0$ *at some point x_0 or* $\mathrm{Vol}(M)$ *is infinite. Let N be a complete Riemannian manifolds with the non-positive scalar curvature. Then any harmonic morphism $\phi : M \to N$ with $E_2(\phi) < \infty$ is constant.*

Proof. Let x_0 be a point of M and fix it. We choose a Lipschitz continuous function ω_ℓ on M such that $0 \leq \omega_\ell(y) \leq 1$ for any $y \in M$. Multiplying (4.6) by $\omega_\ell^2 \lambda$ and integrating by parts, we obtain

$$n \int_M \langle d\lambda, d(\omega_\ell^2 \lambda) \rangle \leq - \int_M \omega_\ell^2 \, \lambda^2 \mathrm{tr} Ric^M \big|_H \leq n\mu_0 \int_M (\omega_\ell \lambda)^2. \tag{4.10}$$

By a direct calculation, we have

$$\langle d\lambda, d(\omega_\ell^2 \lambda) \rangle = 2\omega_\ell \lambda \langle d\lambda, d\omega_\ell \rangle + |\omega_\ell d\lambda|^2 = |d(\omega_\ell \lambda)|^2 - \lambda^2 |d\omega_\ell|^2. \tag{4.11}$$

From (4.10) and (4.11), we have

$$\begin{aligned} \int_M |d(\omega_\ell \lambda)|^2 &\leq -\frac{1}{n} \int_M \omega_\ell^2 \, \lambda^2 \mathrm{tr} Ric^M \big|_H + \int_M \lambda^2 |d\omega_\ell|^2 \\ &\leq \mu_0 \int_M (\omega_\ell \lambda)^2 + \int_M \lambda^2 |d\omega_\ell|^2. \end{aligned} \tag{4.12}$$

Since μ_0 is the infimum of the spectrum of the Laplacian $\triangle^M$ acting on L^2-functions on M, the Rayleigh theorem implies

$$\int_M |d(\omega_\ell \lambda)|^2 \geq \mu_0 \int_M (\omega_\ell \lambda)^2. \tag{4.13}$$

If we let $\ell \to +\infty$ in (4.12) with (4.13),then we have

$$\mu_0 \int_M \lambda^2 \leq -\frac{1}{n} \int_M \lambda^2 \mathrm{tr} Ric^M \big|_H \leq \mu_0 \int_M \lambda^2. \tag{4.14}$$

This means that

$$\int_M \left(n\mu_0 + \mathrm{tr} Ric^M \big|_H \right) \lambda^2 = 0. \tag{4.15}$$

(i) First case: If $Ric^M \geq -\mu_0$ at all x and $Ric^M > -\mu_0$ at some x_0, then $n\mu_0 + \mathrm{tr}Ric^M|_H \geq 0$ for all x and $n\mu_0 + \mathrm{tr}Ric^M|_H > 0$ at some point x_0, respectively. The unique continuation property for section implies $|d\phi| = 0$, i.e., ϕ is constant.

(ii) Second case: Now we study Theorem 4.11 under the assumption $Ric^M \geq -\mu_0$ and $Vol(M) = \infty$. We first note that for any real number $\delta > 0$

$$\left| 2\int_M \omega_\ell \lambda \langle d\lambda, d\omega_\ell \rangle \right| \leq \delta^2 \int_M \omega_\ell^2 |d\lambda|^2 + \frac{1}{\delta^2}\int_M \lambda^2 |d\omega_\ell|^2. \tag{4.16}$$

From (4.10) , (4.11) and (4.16), we have

$$\begin{aligned} (1-\delta^2)\int_M \omega_\ell^2 |d\lambda|^2 - \frac{1}{\delta^2}\int_M \lambda^2 |d\omega_\ell|^2 &\leq -\frac{1}{n}\int_M \omega_\ell^2 \lambda^2 \, \mathrm{tr}Ric^M\big|_H \\ &\leq \mu_0 \int_M (\omega_\ell \lambda)^2. \end{aligned} \tag{4.17}$$

If we choose $\delta = \frac{1}{\sqrt{\ell}}$ and let $\ell \to +\infty$, then

$$\int_M |d\lambda|^2 \leq -\frac{1}{n}\int_M \lambda^2 \mathrm{tr}Ric^M\big|_H \leq \mu_0 \int_M \lambda^2. \tag{4.18}$$

On the other hand, from (4.11) and (4.17) we similarly obtain

$$(1+\delta^2)\int_M \omega_\ell^2 |d\lambda|^2 \geq \int_M |d(\omega_\ell \lambda)|^2 - \left(1 + \frac{1}{\delta^2}\right)\int_M \lambda^2 |d\omega_\ell|^2. \tag{4.19}$$

If we put $\delta = \frac{1}{\sqrt{\ell}}$ and let $\ell \to +\infty$, then we have from (4.13)

$$\int_M |d\lambda|^2 \geq \mu_0 \int_M \lambda^2. \tag{4.20}$$

From (4.18) and (4.20), we have $\int_M (\triangle^M \lambda - \mu_0 \lambda)\lambda = 0$. Hence (4.6) implies that $\triangle^M \lambda = \mu_0 \lambda$. This means that λ is nonnegative L^2-subharmonic function. By the maximum principle ([16,18]), λ is constant. Since $Vol(M) = \infty$, it is trivial that $\lambda = 0$, which yields that ϕ is constant.

4.4 Liouville type theorem for p-harmonic morphisms

Let $\phi : (M^m, g) \to (N^n, h)(m \geq n)$ be a p-harmonic morphism with dilation λ. Let $\{e_i\}_{i=1,\cdots,m}$ be a local orthonomal frame field on M such that $\{e_i\}_{i=1,\cdots,n} \in H_x$ and $\{e_i\}_{i=n+1,\cdots,m} \in V_x$. Then it is trial from (4.1) that

$$|d\phi|^2 = n\lambda^2. \tag{4.21}$$

Moreover, it is easy to see that

$$\begin{aligned}\sum_{i=1}^{m} h\left(d\phi(Ric^M(e_i)), d\phi(e_i)\right) &= \sum_{i=1}^{n} \lambda^2 g\left(Ric^M(e_i), e_i\right) \\ &= \lambda^2 \operatorname{tr} Ric^M\big|_H \end{aligned} \tag{4.22}$$

and

$$\begin{aligned}&\sum_{i,j=1}^{m} h\left(R^N(d\phi(e_i), d\phi(e_j))d\phi(e_j), d\phi(e_i)\right) \\ &= \sum_{i,j=1}^{m} h\left(R^N(\lambda(v_i), \lambda(v_j))\lambda(v_j), \lambda(v_i)\right) \\ &= \lambda^4 \gamma_N \circ \phi \\ &= \lambda^4 scal_N \circ \phi \end{aligned} \tag{4.23}$$

From (4.21), (4.22) and (4.23), we have the following lemma.

Lemma 4.12 *Let $\phi : (M, g) \to (N, h)$ be a p-harmonic morphism with dilation λ. Then we have the following.*

$$\begin{aligned}-\frac{1}{2} n \triangle^M \lambda^{2p-2} &= \left|\nabla(\lambda^{p-2} d\phi)\right|^2 - \left\langle \lambda^{p-2} d\phi, \delta_\nabla d_\nabla(\lambda^{p-2} d\phi)\right\rangle \\ &\quad + \lambda^{2p-2} \operatorname{tr}\left(Ric^M\big|_H\right) - \lambda^{2p} scal_N \circ \phi.\end{aligned} \tag{4.24}$$

From Lemma 3.10 and (4.21), we have the following lemma.

Lemma 4.13 *Let M be a complete Riemannian manifolds such that $Ric^M \geq -C$ $(C > 0)$ at all $x \in M$ and let N be a Riemannian manifolds of non-positive scalar curvature. If $\phi : (M, g) \to (N, h)$ is a p-harmonic morphism, then*

$$n\lambda \triangle^M \lambda^{p-1} - \left\langle d\phi, \delta_\nabla d_\nabla(\lambda^{p-2} d\phi)\right\rangle \leq -\lambda^p \mathrm{tr}\left(Ric^M\big|_H\right) \leq nC\lambda^p. \quad (4.25)$$

Theorem 4.14 *Let M is a complete Riemannian manifold such that $Ric^M \geq -\frac{4(p-1)}{p^2}\mu_0$ for all x and $Ric^M > -\frac{4(p-1)}{p^2}\mu_0$ at some point x_0. Let N be a complete Riemannian manifold with non-positive scalar curvature. Then any p-harmonic morphism $\phi : M \to N$ of $E_p(\phi) < \infty$ is constant.*

Proof. Let us put $C = \frac{4(p-1)}{p^2}\mu_0$ in Lemma 4.13. By the same process as in the proof of Theorem 4.11, we have that

$$\int_M \lambda^p \left(\mathrm{tr}Ric^M + C\right)\big|_H = 0. \quad (4.26)$$

So $Ric^M > -C$ at some point x_0 implies that $\lambda = 0$. Hence ϕ is constant. □

Corollary 4.15 *Let M be a complete Riemannian manifold such that $Ric^M \geq -\frac{4(p-1)}{p^2}\mu_0$ at all $x \in M$ and $Ric^M > -\frac{4(p-1)}{p^2}\mu_0$ at some point x_0. Let N be a complete Riemannian manifold with a non-positive scalar curvature. Then any q-harmonic map $\phi : M \to N$ with $2 \leq q \leq p$ of $E_q(\phi) < \infty$ is constant.*

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<국문초록>

p-조화사상에 대한 리우빌 형식의 정리

조화함수에 대한 고전적인 리우빌정리는 “평면상에서 유계된 조화함수는 상수 뿐이다.” 이다. 본 논문에서는 유한의 p-에너지를 갖는 p-조화함수에 대한 리우빌형식의 정리를 연구하였다. 즉, 리치곡률 $Ric^M \geq -\frac{4(p-1)}{p^2}\mu_0$을 만족하는 완비인 리만다양체로부터 양수가 아닌 단면곡률을 갖는 리만다양체로의 p-조화함수가 유한 p-에너지를 가지면 p-조화함수는 상수이다. 또한 양수가 아닌 scalar 곡률을 갖는 완비인 리만다양체로의 p-조화사상이 유한 p-에너지를 가지면 p-조화사상은 상수이다.