A Thesis for the Degree of M. E.

Local Approximation of Functions by Linear Functions.

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감 사 의 글

이 논문이 완성되기 까지 바쁘신 가운데도 많은 지도를 하여 주신 현진오 교수님께 감사를 드리며, 아울러 그동안 많은 도움을 주신 고봉수 교수님과 수학교육과의 여러 교수님께 심심한 사의를 표합니다.

그리고 그동안 저에게 격려를 하여 주신 가족, 친지 및 주위의 많은 분들에게 감사를 드립니다.

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부 광 훈

CONTENTS

1. Introduction

The subject matter of this paper is nothing else but elementary theorems of Calculus, which however are presented in a way which will probably be new to most students. That presentation, which throughout adheres strictly to our general "geometric" outlook on Analysis, aims at keeping as close as possible to the fundamental idea of Calculus, namely the "local" approximation of functions by linear functions. In the classical teaching Calculus, this idea is immediately obscured by the accidental fact that, on a one-dimensional vector space, there is a one-to-one correspondence between linear forms and numbers, and therefore the derivative at a point is defined as a number instead of a linear form. This slavish subservience to the shibboleth of numerical interpretation at any cost becomes much worse when dealing with functions of several variables.

In order to arrive at a definition of the derivative of a function whose domain is Banach space E (or an open subset of E), let us take another look at the familiar case $E = R$, and let us see how to interpret the derivative in that case in a way which will naturally extend to Banach space.

If f is a real function with domain $(a, b) \subset R$ and if $x \in (a, b)$, then $f'(x)$ is usually defined to be the real number

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided, of course, that this limit exists. Thus

(1) $$f(x+h) - f(x) = f'(x)h + r(h),$$

where the "remainder" r(h) is small, in the sense that

$$\lim_{h \to 0} \frac{r(h)}{h} = 0.$$

Note that (1) expresses the difference $f(x \times h) - f(x)$ as the sum of the linear function that takes h to $f'(x)h$, plus a small remainder. We can therefore regard the derivative of f at x, not as a real number, but as the linear function on R that takes h to $f'(x)h$.

In section 2, we shall study derivatives of a continuous functions and seperately continuous functions.

In section 3, we shall prove formal rules of derivation (In particular, for composite function of differentiable functions) and inverse functions.

In section 4, we shall study derivatives in spaces of continuous linear functions.

In section 5, finally, we shall work applications for local approximations of function by linear functions.

2. Derivative of a continuous function.

Definition 2-1. Let E, F be real Banach spaces and A an open subset of E. Let f, g be two functions of A into F; we say that f and g are tangent at a point $x_0 \in A$ if

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{\| f(x) - g(x) \|}{\| x - x_0 \|} = 0.$$

Theorem 2-1. Among all functions tangent at x_0 to a function f, there is at most one function of the form $x \mapsto f(x_0) + u(x - x_0)$ where u is linear.

Proof. If two such functions $x \mapsto f(x_0) + u_1(x - x_0)$, $x \mapsto f(x_0) + u_2(x - x_0)$ are tangent at x_0, this means, for $v = u_1 - u_2$; that

$$\lim_{\substack{y \to 0 \\ y \neq 0}} \frac{\| v(y) \|}{\| y \|} = 0.$$

But this implies $v = 0$, for if, given $\varepsilon > 0$, there is $r > 0$ such that $\| y \| \leq r$ implies $\| v(y) \| \leq \varepsilon \| y \|$, then this last inequality is valid for any $x \neq 0$, by applying it to $y = \frac{rx}{\| x \|}$; as ε is arbitrary, we see that $v = 0$ for any x.

Definition 2-2. We say that a continuous function f of A into F is differentiable at the point $x_0 \in A$ if there is a linear function u of E into F such that $x \mapsto f(x_0) + u(x - x_0)$ is tangent to f at x_0. And u is called the derivative f at the point x_0, and written $f'(x_0)$.

Theorem 2-2. If the continuous function f of A into F is differentiable at the point x_0, the derivative $f'(x_0)$ is a continuous linear function of E into F.

Proof. Let $u = f'(x_0)$. Given $\varepsilon > 0$, there is $r > 0$ such that $0 < r < 1$ and that $\| t \| \leq r$ implies $\| f(x_0 + t) - f(x_0) \| \leq \frac{\varepsilon}{2}$ and

$$\| f(x_0 + t) - f(x_0) - u(t) \| \leq \frac{\varepsilon \| t \|}{2};$$

hence $\| t \| \leq r$ implies $\| u(t) \| \leq \varepsilon$, which proves u is continuous by linearity.

Corollary. The derivative of a continuous linear function u of E into F exists at every point $x \in E$ and $u'(x) = u$.

Proof. For by definition $u(x_0) + u(x - x_0) = u(x)$.

Lemma 2-1. Let E, F, G be three normed spaces, u a bilinear function of $E \times F$ into G. In order that u be continuous, a necessary and sufficient condition is the existence of a number $a > 0$ such that, for any $(x, y) \in E \times F$,

$$\| u(x, y) \| \leq a \cdot \| x \| \cdot \| y \| .$$

Proof.

(1). Sufficiency. To prove u is continuous at any point (x, y), we write

$$u(x', y') - u(x, y) = u(x' - x, y') + u(x, y' - y),$$

hence

$$\| u(x', y') - u(x, y) \| \leq a(\| x' - x \| \cdot \| y' \| + \| x \| \cdot \| y' - y \|).$$

For any δ such that $0 < \delta < 1$, suppose $\| x' - x \| \leq \delta$, $\| y' - y \| \leq \delta$, hence $\| y' \| \leq \| y \| + 1$. We therefore have

$$\| u(x', y') - u(x, y) \| \leq a(\| x \| + \| y \| + 1)$$

which is arbitrary small with δ.

(2). Necessity. If u is continuous at the point $(0, 0)$, there exists a ball $B: \sup(\|x\| \cdot \|y\|) \leq r$ in $E \times F$ such that the relation $(x, y) \in B$ implies $\|u(x, y)\| \leq 1$. Let (x, y) be arbitrary: suppose first $x \neq 0$, $y \neq 0$; then if $z_1 = rx/\|x\|$, $z_2 = ry/\|y\|$, we have $\|z_1\| = \|z_2\| = r$, and therefore $\|u(z_1, z_2)\| \leq 1$. But $u(z_1, z_2) = r^2 u(x, y)/\|x\| \cdot \|y\|$, and therefore $\|u(x, y)\| \leq a \cdot \|x\| \cdot \|y\|$ with $a = 1/r^2$. If $x = 0$ or $y = 0$, $u(x,y) = 0$, hence the preceding inequality still holds.

Theorem 2-3. Let E, F, G be three Banach spaces, $(x, y) \mapsto x \ast y$ a continuous bilinear mapping of $E \times F$ and the derivative is the linear mapping

$$(s, t) \mapsto x \ast t + s \ast y.$$

Proof. For we have

$$(x+s) \ast (y+t) - x \ast y - x \ast t - s \ast y = s \ast t$$

and by assumption, there is a constant $c > 0$ that $\|s \ast t\| \leq c \cdot \|s\| \cdot \|t\|$ (By Lemma 2-1) For any $\varepsilon > 0$, the relation $\sup(\|s\|, \|t\|) = \|(s, t)\| \leq \frac{\varepsilon}{c}$ implies therefore

$$(x+s) \ast (y+t) - x \ast y - x \ast t - s \ast y / \|(s, t)\| \leq \varepsilon,$$

which proves our assertion.

Remark. Let E, F be two normed spaces; the set $L(E:F)$ of all continuous linear functions of E into F is a vector space.

For any $u \in L(E:F)$, let $\|u\|$ be the g.l.b. of all constants $a > 0$ which satisfy the relation $\|u(x)\| \leq a \cdot \|x\|$ for all x. We can also write

$$\|u\| = \sup_{\|x\| \leq 1} \|u(x)\|$$

and we can show that $\|u\|$ is a norm on the vector space $L(E:F)$.

Theorem 2-4. Suppose E, F_1, F_2 are three Banach spaces and $f = (f_1, f_2)$ a continuous function of an open subset A of E into $F_1 \times F_2$. In order that f be differentiable at x_0, a necessary and sufficient condition that each $f_i (i = 1, 2)$ be differentiable at x_0, and when $f'(x_0) = (f_1'(x_0), f_2'(x_0))$ (when $L(E:F_1 \times F_2)$ is identified with the product of the spaces $L(E:F_1)$, $L(E:F_2)$.

Proof. Indeed, any linear function u of E into $F_1 \times F_2$ can be written in a unique way $u = (u_1, u_2)$, where u_i is a linear function of E into F_i $(i = 1, 2)$, and we have by definition $\| u(x) \| = \sup(\| u_1(x) \|, \| u_2(x) \|)$, whence it follows that $\| u \| = \sup(\| u_1 \|, \| u_2 \|)$, which allows the identification of $L(E:F_1 \times F_2)$ with the product $L(E:F_1) \times L(E:F_2)$. From the definition, it follows at once that u is the derivative of f at x_0 if and only if u_i is the derivative of f_i at x_0 for $i = 1, 2$.

Remark. Let E, F be complex Banach spaces, and E_0, F_0 the underlying real Banach spaces. Then if a function f of an open subset A of E into F is differentiable at a point x_0, it is also differentiable with the same derivative, when considered as a function of A into F_0 (a linear function of E into F being also as a function of E_0 into F_0). But the converse is not true, as the example of the function $z \mapsto \bar{z}$ (complex conjugate) of C into itself or as a function of R^2 into itself, $u : z \to \bar{z}$ (which can be written $(x, y) \mapsto (x, -y)$) is differentiable and has at each point a derivative equal to u, by Corollary : but u is not a complex linear function.

3. Formal rules of derivation.

Theorem 3-1. Let E, F, G be three Banach spaces. A an open neighborhood of $x_0 \in E$, f a continuous function of A into F, $y_0 = f(x_0)$, B an open neighborhood of y_0 in F, g a continuous function of B into G. Then if f is differentiable at x_0 and g differentiable at y_0, the function $h = g \circ f$ (which is defined and continuous in a neighborhood of x_0) is differentiable at x_0, and we have

$$h'(x_0) = g'(y_0) \circ f'(x_0).$$

Proof. By assumption, given $\varepsilon > 0$ such that $0 < \varepsilon < 1$, there is an $r > 0$ such that, for $\| s \| \leq r$ and $\| t \| \leq r$, we can write

$$f(x_0 + s) = f(x_0) + f'(x_0)(s) + 0_1(s)$$

$$g(y_0 + t) = g(y_0) + g'(y_0)(t) + 0_2(t)$$

with $\| 0_1(s) \| \leq \varepsilon \| s \|$ and $\| 0_2(t) \| \leq \varepsilon \| t \|$. On the other hand, by the linearity of $f'(x_0)$ and $g'(y_0)$, there are constants a, b such that, for any s and t,

$$\| f'(x_0)(s) \| \leq a \cdot \| s \| \text{ and } \| g'(y_0)(t) \| \leq b \| t \|$$

hence

$$\| f'(x_0)(s) + 0_1(s) \| \leq (a+1) \| s \|$$

for $\| s \| \leq r$. Therefore, for $\| s \| \leq r/(a+1)$, we have

$$\| 0_2(f'(x_0)(s) + 0_1(s)) \| \leq (a+1) \varepsilon \| s \|$$

and

$$\| g'(y_0)(0_1(s)) \| \leq b \ \varepsilon \| s \|$$

hence we can write

$$h(x_0 + s) = g(y_0 + f'(x_0)(s) + 0_1(s))$$

$$= g(y_0) + g'(y_0)(f'(x_0)(s)) + 0_3(s)$$

with $\| 0_3(s) \| \leqq (a+b+1)\varepsilon \| s \|$, which proves the theorem.

Theorem 3-1 has innumerable applications, of which we mention only the following one ;

Corollary. Let f, g be two continuous functions of the open subset A of E into F. If f and g are differentiable at x_0, so are $f+g$ and αf (α scalar) and we have $(f+g)'(x_0) = f'(x_0) + g'(x_0)$ and $(\alpha f)'(x_0) = \alpha f'(x_0)$.

Proof. The function $f+g$ is composed of $(u,v) \mapsto u+v$, function of F x F into F, and of $x \mapsto (f(x), g(x))$, function F x F ; both are differentiable by theorem 2-3, 2-4, and the result follow (for $f+g$) from theorem 3-1. For αf the argument is still simpler, using the fact that the function $u \mapsto \alpha u$ of F into itself is differentiable by Corollary in section 2.

Remark. Let E, F be two Banach spaces, A an open subset of E, B an open subset of F. If A and B are homeomorphic, and there exists a differentiable homeomorphism f of A onto B, it does not follow that, for each $x_0 \in A$, $f'(x_0)$ is linear homeomorphism of E onto F.

Theorem 3-2. Let f be a homeomorphism of an open subset A of a Banach space E onto an open subset B of a Banach space F, g the inverse homeomorphism. Suppose f is differentiable at the point x_0, and $f'(x_0)$ is a linear homeomorphism of E onto F; then g is differentiable at $y_0 = f(x_0)$ and $g'(y_0)$ is the inverse function to $f'(x_0)$.

Proof. By assumption, the function $s \mapsto f(x_0+s) - f(x_0)$ is a homeomorphism of a neighborhood V of 0 in E onto a neighborhood W of 0 in F, and the inverse homeomorphism is $t \mapsto g(y_0+t)-g(y_0)$. By assumption, the linear function $f'(x_0)$ of E onto F has an inverse u which is continuous, hence (Lemma) there is $c>0$ such that $\|u(t)\| \leq c \cdot \|t\|$ for any $t \in F$. Given any $\varepsilon > 0$ such that $0 < \varepsilon \leq \frac{1}{2c}$, there is an $r>0$ such that, if we write.

$$f(x_0+s)-f(x_0) = f'(x_0)s + 0_1(s),$$

the relation $\|s\| \leq r$ implies $\|0_1(s)\| \leq \varepsilon \|s\|$. Let now r_0 be a number such that the ball $\|t\| \leq r_0$ is contained in W and that its image by the function $t \mapsto g(y_0+t)-g(y_0)$ is contained in the ball $\|s\| \leq r$. Let

$$z = g(y_0+t) - g(y_0);$$

By definition, for $\|t\| \leq r_0$, this equation implies $t = f(x_0+z)-f(x_0)$ and as $\|z\| \leq r$, we can write $t = f'(x_0)z + 0_1(z)$, with $\|0_1(z)\| \leq \varepsilon \|z\|$. From that relation we deduce

$$u(t) = u(f'(x_0)z) + u(0_1(z)) = z + u(0_1(z))$$

by definition of u, and moreover

$$\|u(0_1(z))\| \leq c\|0_1(z)\| \leq c\varepsilon\|z\| \leq \frac{1}{2}\|z\|,$$

hence $\|u(t)\| \geq \|z\| - \frac{1}{2}\|z\| = \frac{1}{2}\|z\|$; therefore $\|z\| \leq 2\|u(t)\| \leq 2c\|t\|$, and finally $\|u(0_1(z))\| \leq c\varepsilon\|z\| \leq 2c^2\varepsilon\|t\|$. We have therefore proved that the relation $\|t\| \leq r_0$ implies $\|g(y_0+t)-g(y_0)-u(t)\| \leq 2c^2\varepsilon\|t\|$, and as ε is arbitrary, this completes the proof.

Theorem 3-3. Let f be a differentiable real valued function defined in an open subset A of a Banach space E.

(a). If at a point $x_0 \in A$, f reaches a relative maximum, then $f'(x_0) = 0$.

(b). Suppose E is finite dimensional, A is relatively compact, f is defined and continuous on $\bar{A}$ and equal to 0 on the boundary of A. Then there exists a point $x_0 \in A$ where $f'(x_0) = 0$.

Proof. Part(a). Since f is differentiable at $x_0 \in A$,

$$f(x_0 + s) = f(x_0) + f'(x_0)(s) + 0_1(s)$$

and

$$f(x_0 - s) = f(x_0) - f'(x_0)(s) + 0_2(-s),$$

where $\lim_{s \to 0} \frac{0_1(s)}{\|s\|} = 0$ and $\lim_{s \to 0} \frac{0_2(-s)}{\|s\|} = 0$. Since f reaches a relative maximum at $x_0 \in A$,

$$\frac{f(x_0 + s) - f(x_0)}{\|s\|} = \frac{f'(x_0)(s)}{\|s\|} + \frac{0_1(s)}{\|s\|} \leq 0$$

and

$$\frac{f(x_0 - s) - f(x_0)}{\|s\|} = \frac{-f'(x_0)(s)}{\|s\|} + \frac{0_2(-s)}{\|s\|} \leq 0$$

for all s in some neighborhood V of 0. Therefore

$$\lim_{s \to 0} \frac{f'(x_0)(s)}{\|s\|} \leq 0 \text{ and } \lim_{s \to 0} \frac{f'(x_0)(s)}{\|s\|} \geq 0,$$

and hence

$$\lim_{s \to 0} \frac{f'(x_0)(s)}{\|s\|} = 0.$$

By the method of proof in theorem 2-1, $f'(x_0)(s) = 0$ for all s.

Part(b). If f is a constant function, then $f(x) = 0$ for all $x \in \bar{A}$, and then the proof is clear. Suppose f is not constant function. By assumption, there exists two points $a, b \in \bar{A}$ such that $f(a) = \inf_{x \in \bar{A}} f(x)$, $f(b) = \sup_{x \in \bar{A}} f(x)$. Since $f(x) = 0$ on the boundary of A, f reaches a relative maximum(or minimum) on the interior of A. By part (a) of this theorem, the part(b) holds.

4. Derivatives in Spaces of Linear Functions.

Lemma 4-1. Let u be a continuous linear function of a normed space E into a normed space F and v a continuous linear function of F into a normed space G. Then

$$\| v \cdot u \| \leq \| v \| \circ \| u \| ,$$

where the symbol ∘ denotes the composition among functions.

Proof. For if $\| x \| \leq 1$, then from Remark 2

$$\| v(u(x)) \| \leq \| v \| \circ \| u(x) \| \leq \| v \| \circ \| u \| ,$$

and the result follows from $\| v \cdot u \| = \sup_{\|x\| \leq 1} \| v(u(x)) \|$.

Theorem 4-2. Let E, F, G be three Banach spaces. Then the function$(u . v) \rightarrow v \cdot u$ (also written vu) of $L(E:F) \times L(F:G)$ into $L(E:G)$ is differentiable, and the derivative at the point (u_0 , v_0) is the mapping $(s, t) \rightarrow v \cdot s + t \cdot u_0$

Proof. If we observe that, by Lemma (4-1), the function$(u, v) \rightarrow v \cdot u$ is bilinear and continuous, the result is a special case of Theorem (2-3).

Remark. We note that the normed space $L(E:F)$ is complete whenever F is.

Theorem 4-3. Let E be a Banach space. If $\| \omega \| \angle 1$ in $L(E:E)$, the linear function $1 + \omega$ (where 1 is identity funtion) is a homeomorphism, its inverse$(1 + \omega)^{-1}$ is equal to the sum of the absolutely convergent series $\sum_{n=0}^{\infty} (-1)^n \omega^n$, and we have.

$$\| (1 + \omega)^{-1} - 1 + \omega \| \leq \frac{\| \omega \|^2}{(1 - \| \omega \|)}$$

Proof. We have.

$$\sum_{n=0}^{n} \|\omega\|^n = \frac{1-\|\omega\|^{N+1}}{1-\|\omega\|} \leqq \frac{1}{1-\|\omega\|},$$

hence, by an absolutely convergent series is convergent in a Banach space, Lemma (4-1), and Remark, the series $\sum_{n=0}^{\infty}(-1)^n \omega^n$ is absolutely convergent in L(E:E).

Moreover, we have

$$(1+\omega)(1-\omega+\omega^2+\cdots+(-1)^N \omega^N)$$

$$=(1-\omega+\omega^2+\cdots+(-1)^N \omega^N)(1+\omega) = 1-\omega^{N+1}$$

and as ω^{N+1} tends to 0, we have by Lemma (4-1), for the element $\nu = \sum_{n=0}^{\infty}(-1)^n \omega^n$ of L(E:E), $(1+\omega)v = \nu(1+\omega) = 1$ which proves the first two statements; the inequality follows from the relation

$$(1+\omega)^{-1} - 1 + \omega = \omega^2(1-\omega+\omega^2+\cdots),$$

and from Lema (4-1) and $\sum_{n=0}^{\infty}(-1)^n \omega^n$ is absolutely convergent.

Theorem 4-4. Let E, F be two Banach spaces, such that there exists at least a linear homeomorphism of E onto F. Then the functions $u \to u^{-1}$ of H (of linear homeomorphisms of E onto F) onto H^{-1} (of linear homeomorphisms of F onto E) is continuous and differentiable, and the derivative of $u \to u^{-1}$ at the point u_0 is the linear function (of L(E:F) into L(F:E)) $s \to -u_0^{-1} \circ s \circ u_0^{-1}$

Proof. Suppose $s \in L(E:F)$ is such that $\|s\| \circ \|u_0^{-1}\| \angle 1$; then the element $1+u_0^{-1}s$, which belongs to L(E:E), has an inverse, due to Lemma (4-1) and Theorem (4-3); as we can write $u_0 + s = n_0(1+u_0^{-1}s)$, the same is true for u_0+s, the inverse being $(1+u_0^{-1}s)$, the same is true for u_0+s, the inverse being $(1+u_0^{-1}s)^{-1}u_0^{-1}$; hence we have

$$(u_0+s)^{-1}-u_0^{-1} = ((1+u_0^{-1}s)^{-1}-1)u_0^{-1}.$$

Applying Theorem (4-3) to $\omega = u_0^{-1}s$, we obtain, for $\|s\| < \frac{1}{\|u_0^{-1}\|}$,

$$\|(u_0+s)^{-1}-u_0^{-1}+u_0^{-1}su_0^{-1}\| \leq \frac{\|u_0^{-1}\|^3\ \|s\|^2}{(1-\|u_0^{-1}\|\ \|s\|)}$$

Therefore, if we take $\|s\| \leq \frac{1}{n}\|u_0^{-1}\|$, we have

$$\frac{\|(u_0+s)^{-1}-u_0^{-1}+u_0^{-1}su_0^{-1}\|}{\|s\|} \leq \frac{\|u_0^{-1}\|^3\ \|s\|}{\frac{n-1}{n}},$$

and this ends the proof.

5. Applications.

Remark. A Hermitian form on a real vector space E is a mapping f of E x E into R which has the following properties ;

(1). $f(x+x', y) = f(x, y) + f(x', y)$

(2). $f(x, y+y') = f(x, y) + f(x, y')$

(3). $f(\lambda x, y) = \lambda f(x, y)$

(4). $f(x, \lambda y) = \lambda f(x, y)$

(5). $f(y, x) = f(x, y)$

A pair of vectors x, y of a vector space E is orthogonal with respect to a hermitian form f on E if $f(x, y) = 0$. For any subset M of E, the set of vectors y which are orthogonal to all vectors $x \in M$ is a vector subspace of E, which is said to be orthogonal to M(with respect to f). It may happen that there exists a vector $a \neq 0$ which is orthogonal to the whole space E, in which case we say the form f is degenerate. And we say a hermitian form f on a vector space E is positive if $f(x, x) \geq 0$ for any $x \in E$.

The function $x \mapsto \sqrt{f(x, x)}$ satisfies the properties of a norm if the form f is nondegenerate, ie. when f is a nondegenerate positive hermitian form, $\sqrt{f(x, x)}$ is a norm on E. A prehilbert space is a vector space E with a given nondegenerate positive hermitian form on E ; when no confusion arises, that form is written $(x \mid y)$ and its value is called the scalar product of x and y ; we always consider a prehilbert space E as a normed space, with the norm $\|x\| = \sqrt{(x \mid x)}$; and of course, such a space is considered as a metric space for the corresponding distance $\| x-y \|$.

Application 1. Let E be a real prehilbert space. In E the mapping $x \mapsto \|x\|$ of E into R is differentiable at every point $x \neq 0$ and its derivative at such a point is the linear mapping $x \mapsto (s \mid x)/\|x\|$.

Solution. To find a linear function for the mapping $x \mapsto \|x\|$, we must solve the following equation:

$$\frac{\left| \sqrt{f(x+s,\ x+s)} - \sqrt{f(x,x)} - g(s) \right|}{\sqrt{f(s,s)}} \to 0$$

as $s \to 0$ for some linear function g on E. Then if we solve the left of the above statement, it becomes

$$\left| \frac{2f\left(x, \dfrac{s}{\sqrt{f(x,x)}}\right)}{\sqrt{f(x+s,\ x+s)} + \sqrt{f(x,x)}} + \frac{f(s,s)}{\sqrt{f(x+s,\ x+s)} + \sqrt{f(s,s)}} + g\left(\frac{s}{\sqrt{f(s,s)}}\right) \right|.$$

Let $g(t) = f(x,t)/\sqrt{f(x,x)} = (x \mid t)/\|x\|$. The equation then tends to 0 as $t \to 0$.

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〈國 文 抄 錄〉

선형함수들에 의한 함수들의 국지적 접근

夫 光 薰

濟州大學校 教育大學院 數學教育專攻
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본 논문의 주요한 목적은 미분적분학에서 나타나는 기본정리들을 연구하는 데 있다. 이 논문이 전개되는 과정에서 그러한 정리들은 새롭게 표현된다. 그러나 그 표현에서는 미분적분학의 기본개념(즉, 선형함수들에 의한 함수들의 국지적 접근)을 가능한 한 유지하려고 한다.