Measure on the Real Line E^1

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감 사 의 글

이 논문이 완성되기까지 바쁘신 가운데도 지도를 하여 주신 한 철 순 교수님께 심심한 사의를 표합니다.

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김 성 원

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수직선 一차 공간에서의 측도

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이 논문은 수직선 E^1 에서 length를 도입하여 일반적인 measure의 개념을 만족시키는 것을 보이고 length는 σ-algebra상에서 정의된 measure의 성질을 만족시키는 것을 증명하였다.

1. INTRODUCTION

In this paper, We shall study the measure on the real line E^1.

On the other hand, we use the properties of measure to get an exact knowelege of length on the real line.

Finally, we shall show that the length on the real line E^1 is certainly a measure on E^1.

We begin by defining a measure on σ-algebra $\mathbf{X}$ of subsets of X.

2. PRELIMINARY

DEFINITION 2-1

A collection $\mathbf{X}$ of subsets of a set X is a σ-algebra(or a σ-field) if and only if

1) ϕ, $X \in \mathbf{X}$

2) if $A \in \mathbf{X}$, then $A^c = X - A \in \mathbf{X}$

3) if $\{A_n\}$ is a sequence of sets in $\mathbf{X}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathbf{X}$.

An ordered pair $(X, \mathbf{X})$, or simply, X is called a measurable space and any set in $\mathbf{X}$ is a $\mathbf{X}$-measurable set.

It follows from definition 2-1 that E^1, the set of all real numbers, is a measurable space.

PROPOSITION 2-1

Let A be a nonempty collection of subset of X.

Then there exists a smallest σ-algebra of subsets of X containing A.

PROOF Recall that the collection of all subsets of X is a σ-algebra and contains A.

Let $\mathfrak{M} = \cap \{\mathfrak{F} : \mathfrak{F}$ is a σ-algebra containing $A\}$. Then $A \subset \mathfrak{M}$.

It follows from definition 2-1 that $\mathfrak{M}$ is a σ-algebra, and hence $\mathfrak{M}$ is the smallest σ-algebra.

This smallest σ-algebra is often called the σ-algebra generated by A.

DEFINITION 2-2

The Borel algebra is the σ- algebra B generated by all segments in E^1. we say that any set in B is called a Borel set.

DEFINITION 2-3

A measure is an extended real-valued function μ defined on a σ-algebra $\mathbf{X}$ of subsets of X such that

1) $\mu(\phi)=0$

2) $\mu(E)\geqslant 0$ for all $E \in \mathbf{X}$

3) μ is countably additive in the sense that if $\{E_n\}$ is any disjoint sequence of sets in $\mathbf{X}$, then $\mu(\bigcup_{n=1}^{\infty} E_n)=\sum_{n=1}^{\infty}\mu(E_n)$.

It follows from definition 2-3 that the characteristic function of a nonempty set X is a measure on a σ-algebra $\mathbf{X}$ of subsets of X.

PROPOSITION 2-2

If G is open, then there exists a unique countable collection S(G) of segments(a,b) so that the members of S(G) are mutually disjoint and $\bigcup S(G)=G$.

PROOF Let G be an open set in R. Since each point $x\in G$ is contained in a segment which is contained in G,for $x\in G$. Let I_x be the maximal segment containing x which is in G: that is, I_x is the union of all segments which contain x and which are in G.

If x, $x' \in G$ and $x \neq x'$, then either $I_x \cap I_{x'} = \phi$ or $I_x = I_{x'}$.

Since if $I_x \cap I_{x'} \neq \phi$, $I_x \cup I_{x'}$ is a segment containing x and x' and so by the maximality of I_x and $I_{x'}$, $I_x = I_{x'}$.

Clearly $G = \bigcup_{x \in G} I_x$ because if $x \in G$, $I_x \subset G$, then $\bigcup_{x \in G} I_x \subset G$ and if $x \in G$, then $x \in I_x$, $x \in \bigcup_{x \in G} I_x$ and $G \subset \bigcup_{x \in G} I_x$.

Since each I_x contains a rational number, the number of distinct I_x's must be countable, since the set of rationals is countable.

We use proposition 2-2 to define the length of an open set in E^1.

DEFINITION 2-4

If G is open on the real line E^1, then the length of G, denoted by $|G|$, is defined by $|G| = \sum_{S \in S(G)} |S|$, where S(G) is the set of all mutually disjoint segments such that $G = \cup S(G)$.

If $S = (a, b)$, then $|S| = |b - a|$.

If $S = (a, \infty)$ or $(-\infty, a)$, then $|S| = \infty$

If $S = \phi$, then we define $|S| = |\phi| = 0$.

PROPOSITION 2-3

Suppose each of M_1, M_2 ...... is open,

then $\left| \bigcup_{i=1}^{\infty} M_i \right| \leqslant \sum_{i=1}^{\infty} |M_i|$.

PROOF For each M_i, there exists a unique collection $\{s_{ij}\}$ of mutually disjoint segments such that $|M_i| = \left| \sum_{j=1}^{\infty} \right| s_{ij} \Big|$, by proposition 2-2 and definition 2-4.

Let S denote the collection of all segments s_{ij},

then $\sum_{i=1}^{\infty}|M_i| = \sum_{i=1}^{\infty}(\sum_{j=1}^{\infty}|s_{ij}|) = \sum_{i,j=1}^{\infty}|s_{ij}|$.

Since each M_i is open, then $\bigcup_{i=1}^{\infty} M_i$ is open.

Let $T=\{t_1, t_2, \cdots\cdots\}$ denote the collection of mutually disjoint segments such that $|\bigcup_{i=1}^{\infty} M_i| = \sum_{i=1}^{\infty}|t_i|$ and $\bigcup_{i=1}^{\infty} M_i = \bigcup_{i=1}^{\infty} t_i$, then $\bigcup_{i=1}^{\infty} M_i = \bigcup_{i=1}^{\infty} t_i = \bigcup_{i,j=1}^{\infty} s_{ij}$. We claim that for each i, t_i is the union of a subcollection of S.

Suppose there exists a positive integer i such that if S′ is a subcollection of S, then t_i is not the union of segments of S′. Suppose futher there exists no subcollection S′ of S which contains t_i.

Then t_i contains a point of $\bigcup_{i=1}^{\infty} M_i$ but not in $\bigcup_{i,j=1}^{\infty} s_{ij}$, a contradiction.

Hence there exists a subcollection S′ of S with the following properties.

1) every segment in S which intersects t_i is in S′.

2) every segment in S′ contains t_i

3) if $S'=\{s_1', s_2' \cdots\cdots\}$, then $\bigcup_{i=1}^{\infty} s_i'$ contains t_i.

Suppose $\bigcup_{n=1}^{\infty} s_n' - t_i \neq \phi$.

Let p be an endpoint of t_i, then there exists a $j \in N$ such that $p \in t_j$ so t_i intersects t_j, a contradiction.

Therefore, t_i is the union of subcollection of S.

For each $i \in N$, let $V_i = \{v_{i1}, v_{i2} \cdots\cdots\}$ denote the subcollection of S which intersects t_i, then $t_i = \bigcup_{j=1}^{\infty} v_{ij}$.

Also, if V is the set of all v_{ij}, then V=S: that is, V is a rearrangement of S.

Hence $\sum_{i,j=1}^{\infty} |s_{ij}| = \sum_{i,j=1}^{\infty} |v_{ij}|$, and $\bigcup_{i \cdot j=1}^{\infty} v_{ij} = \bigcup_{i \cdot j=1}^{\infty} s_{ij} = \bigcup_{i=1}^{\infty} t_i$.

Thus if V_i contains only one element v_{i1}, then $|t_i| = |v_{i1}|$ and if V_i contains more than one element, then $|t_i| \leq \left|\sum_{j=1}^{\infty} |v_{ij}|\right|$.

Hence $\left|\bigcup_{i=1}^{\infty} M_i\right| = \sum_{i=1}^{\infty} |t_i| \leq \left|\sum_{i,j=1}^{\infty} |v_{ij}|\right| = \sum_{i,j=1}^{\infty} |s_{ij}| = \sum_{i=1}^{\infty} |M_i|$.

3. Properties of measure on E^1

DEFINITION 3-1

Suppose $M \subset E^1$. M has length means for each $\varepsilon > 0$, there are open sets G_1 and G_2 such that $M \subset G_1$ and $G_1 - M \subset G_2$ and $|G_2| < \varepsilon$.

DEFINITION 3-2

If M has length, then we defined the length of M by $|M| = \text{glb}\{|G| : G \text{ is open and } M \subset G\}$.

It follows from definition 2-4 that if G is open, then G has length and the definition 3-1 and 3-2 for $|G|$ coincide.

PROPOSITION 3-1

If each of $M_1, M_2, \cdots\cdots$ has length, then so does $\bigcup_{i=1}^{\infty} M_i$.

PROOF Since each M_i has length, so by definition 3-1 for a given ε, there exists open G_i, H_i such that $M_i \subset G_i$ and $G_i - M_i \subset H_i$ and $|H_i| < \varepsilon/2^i$. Now, $\bigcup_{i=1}^{\infty} M_i \subset \bigcup_{i=1}^{\infty} G_i$ and

$$\bigcup_{i=1}^{\infty} G_i - \bigcup_{i=1}^{\infty} M_i = \bigcup_{i=1}^{\infty} \left(G_i - \bigcup_{i=1}^{\infty} M_i \right) \subset \bigcup_{i=1}^{\infty} (G_i - M_i) \subset \bigcup_{i=1}^{\infty} H_i .$$

It follows from proposition 2-3 that $\left| \bigcup_{i=1}^{\infty} H_i \right| \leq \sum_{i=1}^{\infty} |H_i|$

$$< \sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon .$$

Therefore $\bigcup_{i=1}^{\infty} M_i$ has length.

PROPOSITION 3-2

If G is open and $\varepsilon > 0$, there exists a closed set $F \subset G$ such that $|G - F| < \varepsilon$.

PROOF It follows from definition 2-4 that $|G| = \sum_{i=1}^{\infty} |s_i|$, $s_i \cap s_j \neq \emptyset$, $i \neq j$, and each s_i is a segment.

Let $\varepsilon > 0$ be given, there exists a $m \in N$ such that

$n \geq m \Rightarrow |s_n| < \varepsilon/2^{n+2}$ for each $k \leq m$, choose $F_k = [a_k + \varepsilon/2^{k+2}, b_k - \varepsilon/2^{k+2}]$,

and let $F = \bigcup_{k=1}^{\infty} F_k$, then F is closed and $F \subset G$.

Futhermore, $|G - F| \leq \varepsilon/2^2 + \varepsilon/2^3 + \cdots = \varepsilon/3 < \varepsilon$.

PROPOSITION 3-3

If M has length, then R - M has length.

PROOF Since M has length, so choose open sets G_n such that $M \subset G_n$, $|G_n - M| < \frac{1}{n}$.

It follows from proposition 3-2 that each G_n^c has length.

Let $H = \bigcup_{n=1}^{\infty} G_n^c$, then H has length, and $H \subset M^c$.

Let $M^c = H \cup Z$, where $Z = M^c - H$.

Then $Z = M^c - H = M^c - \bigcup_{n=1}^{\infty} G_n^c = \bigcap_{n=1}^{\infty} (M^c - G_n^c) = \bigcap_{n=1}^{\infty} (M^c \cap G_n)$
$= \bigcap_{n=1}^{\infty} (G_n - M) \subset G_n - M$ so that $|Z| \leq |G_n - M| < \frac{1}{n}$, for every n.

Hence $|Z| = 0$ and Z has length.

Hence $M^c = H \cup Z$ has length by the proposition 3-1.

PROPOSITION 3-4

If M and N are disjoint sets each having length, then $|M \cup N| = |M| + |N|$. More generally, if M_1, M_2 ...... is a sequence of disjoint sets each having length, then $\left|\bigcup_{i=1}^{\infty} M_i\right|$

$$= \sum_{i=1}^{\infty} |M_i| .$$

PROOF It follows from proposition 3-1 that $M_1 \cup M_2$ has length.

For each $\varepsilon > 0$, choose open sets G_1, G_2 such that $M_1 \subset G_1$, $M_2 \subset G_2$ and $|G_1| \leqslant |M_1| + \varepsilon/2$, $|G_2| \leqslant |M_2| + \varepsilon/2$ so

$$|M_1 \cup M_2| \leqslant |G_1 \cup G_2| \leqslant |G_1| + |G_2| \leqslant |M_1| + \varepsilon/2 + |M_2| + \varepsilon/2$$
$$= |M_1| + |M_2| + \varepsilon$$

Hence $|M_1 \cup M_2| \leqslant |M_1| + |M_2|$

Next, we want to show that $|M_1 \cup M_2| \geqslant |M_1| + |M_2|$.

For each $\varepsilon > 0$, choose an open set G such that $M_1 \cup M_2 \subset G$ and $|G| \leqslant |M_1 \cup M_2| + \varepsilon$. Write $G = \bigcup_{k=1}^{\infty} I_k$, where I_k's are muntually disjoint segments.

Then $\sum_{k=1}^{\infty} |I_k| \leqslant |M_1 \cup M_2| + \varepsilon$, let $N_1 = \{n \in N \mid I_n \cap M_1 \neq \phi\}$ and $N_2 = \{m \in N \mid I_m \cap M_2 \neq \phi\}$, then $|M_1| + |M_2| \leqslant \sum_{n \in N_1} |I_n| + \sum_{m \in N_2} |I_m| \leqslant \sum_{k \in N} |I_k| \leqslant |M_1 \cup M_2| + \varepsilon$, so $|M_1| + |M_2| \leqslant |M_1 \cup M_2|$.

Therefore $|M_1 \cup M_2| = |M_1| + |M_2|$, provided that $M_1 \cap M_2 = \phi$ each having length. Proceeding by induction on n, we have $\left|\bigcup_{i=1}^{\infty} M_i\right| = \sum_{i=1}^{\infty} |M_i|$, if each M_i has length and they are mutually disjoint.

Consequently, it follows from defintion 2-1, 2-2 and 2-3 that length is a measure on the Borel algebra E^1.

PROPOSITION 3-5

If M and N have lengths and $M \subset N$ with M finite length, then $|N-M| = |N| - |M|$, in particular $|N| \geq |M|$.

It follows from proposition 3-3 that N-M has length.

PROOF Since $N = M \cup (N-M)$ and $M \cap (N-M) = \phi$, it follows from proposition 3-4 that $|N| = |M \cup (N-M)| = |M| + |N-M|$, so $|N-M| = |N| - |M|$. In particular $|N| \geq |M|$.

PROPOSITION 3-6

If each of M_1, M_2, ... has length, then $\left| \bigcup_{i=1}^{\infty} M_i \right| \leq \sum_{i=1}^{\infty} |M_i|$.

PROOF Let $B_1 = M_1$, $B_2 = M_2 - M_1$, ... $B_n = M_n - (M_1 \cup \cdots \cup M_{n-1})$ for $n \geq 3$, then B_n's are mutually disjoint and each B_n has length.

It follows from proposition 3-4 that

$\left| \bigcup_{i=1}^{\infty} M_i \right| = \left| \bigcup_{i=1}^{\infty} B_i \right| = \sum_{i=1}^{\infty} |B_i|$. But it follows from proposition 3-5 that each $B_i \subset M_i \Rightarrow |B_i| \leq |M_i|$.

Therefore $\left| \bigcup_{i=1}^{\infty} M_i \right| = \sum_{i=1}^{\infty} |B_i| \leq \sum_{i=1}^{\infty} |M_i|$.

PROPOSITION 3 - 7

If $M_1 \subset M_2 \subset \cdots$ and each has length, then $\left| \bigcup_{i=1}^{\infty} M_i \right| = \lim_{n \to \infty} |M_n|$.

If some of M has an infinite length, then we are done.

So we assume each of M_1, M_2 ...... has finite length.

PROOF Consider $\bigcup_{i=1}^{\infty} M_i = M_1 \cup (M_2 - M_1) \cup (M_3 - M_2) \cup \cdots$, then M_1, $M_2 - M_1$, $M_3 - M_2$, ... are mutually disjoint and each has length by proposition 3 - 3

then $\left|\bigcup_{i=1}^{\infty} M_i\right| = |M_1| + |M_2 - M_1| + \cdots\cdots$ by proposition 3-4, and also $\left|\bigcup_{i=1}^{\infty} M_i\right| = |M_1| + |M_2| - |M_1| + \cdots\cdots$ by proposition 3-5.

$\bigcup_{i=1}^{\infty} M_i$ has length by proposition 3 - 1

and hence $\left|\bigcup_{i=1}^{\infty} M_i\right| = \lim_{n\to\infty} |M_n|$

PROPOSITION 3 - 8

If $M_1 \supset M_2 \supset \cdots\cdots$ and each has length, then $\left|\bigcap_{i=1}^{\infty} M_i\right| = \lim_{n\to\infty} |M_n|$.

PROOF Let $B_1 = M_1 - M_2$, $B_2 = M_2 - M_3$, $\cdots\cdots$ and $B_n = M_n - M_{n+1}$, for $n \geqslant 3$.

It follows from proposition 3 - 3 that each B_n has length.

Also, B_n's are mutually disjoint.

We claim that $M_1 - \bigcap_{i=1}^{\infty} M_i = \bigcup_{i=1}^{\infty} B_i$.

In fact, $M_1 - \bigcap_{i=1}^{\infty} M_i = \bigcup_{i=1}^{\infty} (M_1 - M_i) \supset (M_1 - M_2) \cup (M_2 - M_3) \cup \cdots\cdots$ and if $x \in \bigcup_{i=1}^{\infty} B_i$, then $x \in B_n$ for some n,

and $x \in M_n$ but if $x \not\in M_{n+1}$, then $x \in M_1$

but if $x \in \bigcap_{i=1}^{\infty} M_i$, then $x \in M_1 - \bigcap_{i=1}^{\infty} M_i$.

So $\bigcup_{i=1}^{\infty} B_i \subset M_1 - \bigcap_{i=1}^{\infty} M_i$. Therefore $M_1 - \bigcap_{i=1}^{\infty} M_i = \bigcup_{i=1}^{\infty} B_i$.

Note that $\bigcap_{i=1}^{\infty} M_i$ has length, since $\left(\bigcap_{i=1}^{\infty} M_i\right) = \left(\bigcup_{i=1}^{\infty} M_i^c\right)^c$

has length.

It follows from proposition 3-4 that $\left| M_1 - \bigcap_{i=1}^{\infty} M_i \right|$

$$= \left| \bigcup_{i=1}^{\infty} B_i \right| = \sum_{i=1}^{\infty} \left| B_i \right| .$$

Now $\left| M_1 - \bigcap_{i=1}^{\infty} M_i \right| = \left| M_1 \right| - \left| \bigcap_{i=1}^{\infty} M_i \right|$ and $\sum_{i=1}^{\infty} \left| B_i \right| = \left| M_1 \right| - \left| M_2 \right| + \left| M_2 \right| - \left| M_3 \right| + \cdots\cdots + \left| M_n \right| - \left| M_{n-1} \right| + \cdots\cdots$

by proposition 3-5.

Thus, $\left| M_1 \right| - \left| \bigcap_{i=1}^{\infty} M_i \right| = \left| M_1 \right| - [M_2 - (\left| M_2 \right| - \left| M_3 \right|) - \cdots\cdots$

$$- (\left| M_n \right| - \left| M_{n-1} \right|) - \cdots\cdots] = \left| M_1 \right| - \lim_{n \to \infty} \left| M_n \right| .$$

So $\left| \bigcap_{i=1}^{\infty} M_i \right| = \lim_{n \to \infty} \left| M_n \right|$

We conclude that length satisfies the properties of a measure defined on the σ - algebra $\mathbf{X}$ of subsets of x.

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