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Normal fuzzy probability and exponential fuzzy probability for various fuzzy numbers

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宋 宰 忠

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Normal fuzzy probability and exponential fuzzy probability for various fuzzy numbers

Jae Choong Song
(Supervised by professor Yong Sik Yun)

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Department of Mathematics
GRADUATE SCHOOL
CHEJU NATIONAL UNIVERSITY

여러 가지 퍼지수에 대한 정규퍼지확률과 지수퍼지확률

指導敎授　尹　龍　植

宋　宰　忠

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CONTENTS

< Abstract>

Normal fuzzy probability and Exponential fuzzy probability for various fuzzy numbers

Using quadratic curve and trigonometric curve, we define quadratic fuzzy number and trigonometric fuzzy number, then calculate some operations(addition, subtraction, multiplication, division) of two quadratic fuzzy numbers and two trigonometric fuzzy numbers, respectively. The results of addition and subtraction of two quadratic fuzzy numbers and two trigonometric fuzzy numbers become a quadratic fuzzy number and a trigonometric fuzzy numbers possibility, but the results of multiplication and division may not be a quadratic fuzzy number and a trigonometric fuzzy number, respectively. But the results of multiplication and division of two trigonometric fuzzy numbers are expressed by trigonometric function. And we calculate the operations of two fuzzy numbers through an actual examples.

We define the normal fuzzy probability and the exponential fuzzy probability using normal distribution and exponential distribution, and then calculate the normal fuzzy probability and the exponential fuzzy probability of quadratic fuzzy number and trigonometric fuzzy number. Also, we calculate the normal fuzzy probability and the exponential fuzzy probability for the result of operations of two fuzzy numbers and calculate concretely through an actual examples.

1. Introduction

The operations of two fuzzy numbers (A, μ_A) and (B, μ_B) are based on the Zadeh's extension principle([18], [19], [20]). We consider four operations, addition $A(+)B$, subtraction $A(-)B$, multiplication $A(\cdot)B$ and division $A(/)B$ described in section 2.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, where Ω denotes the sample space, $\mathfrak{F}$ the $\sigma-$algebra on Ω, and P a probability measure. A fuzzy set A on Ω is called a fuzzy event. Let $\mu_A(\cdot)$ be the membership function of the fuzzy event A. Then the probability of the fuzzy event A is defined by Zadeh([17]) as

$$\widetilde{P}(A) = \int_{\Omega} \mu_A(\omega)\, dP(\omega), \qquad \mu_A(\omega) : \Omega \to [0,1].$$

In section 3, using quadratic curve and trigonometric curves, we define quadratic fuzzy number and trigonometric fuzzy number. And we calculate some operations of two quadratic fuzzy numbers and two trigonometric fuzzy numbers. The results of multiplication and division of two quadratic fuzzy numbers and two trigonometric fuzzy numbers may not to be a quadratic fuzzy number and a trigonometric fuzzy number, respectively. But the results of multiplication and division of two trigonometric fuzzy numbers are expressed by trigonometric function.

In section 4, we define the normal fuzzy probability and exponential fuzzy probability using the normal distribution and exponential distribution, and we derive the explicit formula for the normal fuzzy probability and exponential fuzzy probability for triangular fuzzy number and fuzzy numbers driven by operations and give some examples.

In section 5, we derive the explicit formula for the normal fuzzy probability and exponential fuzzy probability for quadratic fuzzy number and trigonometric fuzzy number. Also, we calculate the normal fuzzy probability and exponential fuzzy probability of fuzzy numbers driven by operations.

2. Preliminaries

2.1 Fuzzy number

Let X be a set. A classical subset A of X is often viewed as a characteristic function μ_A from X to $\{0, 1\}$ such that $\mu_A(x) = 1$ if $x \in A$, and $\mu_A(x) = 0$ if $x \notin A$. $\{0, 1\}$ is called a valuation set. The following definition is a generalization of this notion.

Definition 2.1. A fuzzy set A on X is a function from X to the interval $[0, 1]$. The function is called the *membership function* of A.

Let A be a fuzzy set on X with a membership function μ_A. Then A is a subset of X that has no sharp boundary. A is completely characterized by the set of pairs

$$A = \{(x, \mu_A(x)), x \in X\}.$$

Elements with a zero degree of membership are normally not listed.

When X is a finite set $\{x_1, \cdots, x_n\}$, a fuzzy set A on X is expressed as

$$A = \mu_A(x_1)/x_1 + \cdots + \mu_A(x_n)/x_n = \sum_{i=1}^{n} \mu_A(x_i)/x_i.$$

When X is not finite, we write

$$A = \int_X \mu_A(x)/x.$$

Two fuzzy sets A and B are said to be equal, denoted by $A = B$, if and only if $\mu_A(x) = \mu_B(x)$, for all $x \in X$.

Example 2.2. Let $X = \{1, 2, 3, 4\}$. Then the function $\mu_A : X \to [0, 1]$ defined by

$$\mu_A(1) = 0, \ \ \mu_A(2) = 1, \ \ \mu_A(3) = 0.5, \ \ \mu_A(4) = 0,$$

can be considered as a membership function for $A = \{\text{two or so}\}$. Thus $A = 0/1 + 1/2 + 0.5/3 + 0/4$.

Example 2.3. Let $X = \mathbb{R}$ and $\mu_A(x) = \dfrac{1}{1+(x-5)^2}$, i.e.,

$$A = \int_{\mathbb{R}} \frac{1}{1+(x-5)^2} / x.$$

Then A is a fuzzy set of real numbers clustered around 5.

Definition 2.4. Let A and B are fuzzy sets. Operations of two fuzzy sets A and B are defined as

1. Union $A \cup B$:

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \quad \forall x \in X.$$

2. Intersection $A \cap B$:

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \quad \forall x \in X.$$

3. Complement A^c :

$$\mu_{A^c}(x) = 1 - \mu_A(x), \quad \forall x \in X.$$

4. Probabilistic sum $A \hat{+} B$:

$$\mu_{A\hat{+}B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x).$$

5. Probabilistic product $A \cdot B$:

$$\mu_{A\cdot B}(x) = \mu_A(x)\cdot\mu_B(x).$$

Example 2.5. Let $X = \{1, 2, 3, 4, \cdots, 10\}$, $A = \{(1, 0.5), (2, 0.9), (3, 1), (4, 0.9), (5, 0.5)\}$, and $B = \{(2, 0.4), (3, 0.8), (4, 1), (5, 1), (6, 0.8)\}$. Then

$A \cup B = \{(1, 0.5), (2, 0.9), (3, 1), (4, 1), (5, 1), (6, 0.8)\}$.

$A \cap B = \{(2, 0.4), (3, 0.8), (4, 0.9), (5, 0.5)\}$.

$A^c = \{(1, 0.5), (2, 0.1), (4, 0.1), (5, 0.5), (6, 1), (7, 1), (8, 1), (9, 1), (10, 1)\}$.

$A \widehat{+} B = \{(1, 0.5), (2, 0.94), (3, 1), (4, 1), (5, 1), (6, 0.8)\}$.

$A \cdot B = \{(2, 0.36), (3, 0.8), (4, 0.9), (5, 0.5)\}$.

Theorem 2.6. Let A, B and C be fuzzy sets on X. Then the followings hold.

1. Commutative law : $A \cup B = B \cup A$, $A \cap B = B \cap A$.
2. Associative law :
   $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$,
   $A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$.
3. Distributive law :
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
4. Involution : $(A^c)^c = A$.
5. Idempotency : $A \cup A = A$, $A \cap A = A$.
6. Absorption : $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$.
7. Identity : $A \cup \phi = A$, $A \cap \phi = \phi$.
8. Absorption by ϕ and X : $A \cap \phi = \phi$, $A \cup X = X$.

9. De Morgan's law :

$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c.$$

10. Excluded-middle law and Contradiction law are not satisfied :

$$A \cup A^c \neq X, \quad A \cap A^c \neq \phi.$$

Definition 2.7. The set $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$ is said to be the *α-cut* of a fuzzy set A.

The membership function of a fuzzy set A can be expressed in terms of the characteristic functions of its α-cuts according to the formula

$$\mu_A(x) = \sup_{\alpha \in (0,1]} \min(\alpha, \mu_{A_\alpha}(x)),$$

where

$$\mu_{A_\alpha}(x) = \begin{cases} 1, & x \in A_\alpha, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily checked that the following properties hold

$$(A \cup B)_\alpha = A_\alpha \cup B_\alpha, \ \ (A \cap B)_\alpha = A_\alpha \cap B_\alpha.$$

Definition 2.8. A fuzzy set A on $\mathbb{R}$ is *convex* if

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)), \quad \forall x_1, x_2 \in \mathbb{R}, \ \forall \lambda \in [0,1].$$

Definition 2.9. A convex fuzzy set A on $\mathbb{R}$ is called a *fuzzy number* if

1. There exists exactly one $x_0 \in \mathbb{R}$ such that $\mu_A(x_0) = 1$.
2. $\mu_A(x)$ is piecewise continuous.

Definition 2.10. A *triangular fuzzy number* is a fuzzy number A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \ \ a_3 \le x, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \le x < a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \le x < a_3. \end{cases}$$

The above triangular fuzzy number is denoted by $A = (a_1, a_2, a_3)$.

Definition 2.11. A fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \ \ a_4 \le x, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \le x < a_2, \\ 1, & a_2 \le x < a_3, \\ \frac{a_4-x}{a_4-a_3}, & a_3 \le x < a_4. \end{cases}$$

is called a *trapezoidal fuzzy set.*

The above trapezoidal fuzzy set is denoted by $A = (a_1, a_2, a_3, a_4)$.

2.2 Operations of two fuzzy numbers

Definition 2.12. The *addition, subtraction, multiplication,* and *division* of two fuzzy numbers are defined as

1. Addition $A(+)B$:

$$\mu_{A(+)B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}, \ x, y \in \mathbb{R}.$$

2. Subtraction $A(-)B$:

$$\mu_{A(-)B}(z) = \sup_{z=x-y} \min\{\mu_A(x), \mu_B(y)\}, \ x, y \in \mathbb{R}.$$

3. Multiplication $A(\cdot)B$:

$$\mu_{A(\cdot)B}(z) = \sup_{z=x\cdot y} \min\{\mu_A(x), \mu_B(y)\},\ x, y \in \mathbb{R}.$$

4. Division $A(/)B$:

$$\mu_{A(/)B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\},\ x, y \in \mathbb{R}.$$

Example 2.13. Let $A = \{(2, 1), (3, 0.5)\}$ and $B = \{(3, 1), (4, 0.5)\}$.

1. Addition :

(i) If $z < 5$, since $x + y \geq 5$ for all $x \in A, y \in B, \mu_{A(+)B}(z) = 0$.

(ii) If $z = 5$, since $\mu_A(2) \wedge \mu_B(3) = 1 \wedge 1 = 1$, $\mu_{A(+)B}(5) = \sup_{2+3}\{1\} = 1$.

(iii) If $z = 6$, since $\mu_A(3) \wedge \mu_B(3) = 0.5 \wedge 1 = 0.5$ and $\mu_A(2) \wedge \mu_B(4) = 1 \wedge 0.5 = 0.5$, we have $\mu_{A(+)B}(6) = \sup_{3+3,2+4}\{0.5, 0.5\} = 0.5$.

(iv) If $z = 7$, since $\mu_A(3) \wedge \mu_B(4) = 0.5 \wedge 0.5 = 0.5$, we have $\mu_{A(+)B}(7) = \sup_{3+4}\{0.5\} = 0.5$.

(v) If $z > 7$, since $x + y \leq 7$ for all $x \in A, y \in B, \mu_{A(+)B}(z) = 0$.

Thus we have $A(+)B = \{(5, 1), (6, 0.5), (7, 0.5)\}$.

By the same way, we have

2. Subtraction : $A(-)B = \{(-2, 0.5), (-1, 1), (0, 0.5)\}$.

3. Multiplication : $A(\cdot)B = \{(6, 1), (8, 0.5), (9, 0.5), (12, 0.5)\}$.

4. Division : $A(/)B = \{(\frac{1}{2}, 0.5), (\frac{2}{3}, 1), (\frac{3}{4}, 0.5), (1, 0.5)\}$.

Theorem 2.14. ([5],[15]) For two triangular fuzzy numbers $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, we have

1. $A(+)B = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3)$.
2. $A(-)B = (a_1 - b_3,\ a_2 - b_2,\ a_3 - b_1)$.
3. $A(\cdot)B$ and $A(/)B$ need not to be triangular fuzzy numbers.

Example 2.15. Let $A = (1,2,4)$ and $B = (2,4,5)$ be triangular fuzzy numbers, i.e.,

$$\mu_A(x) = \begin{cases} 0, & x < 1, \ \ 4 \le x, \\ x - 1, & 1 \le x < 2, \\ -\frac{1}{2}x + 2, & 2 \le x < 4, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < 2, \ \ 5 \le x, \\ \frac{1}{2}x - 1, & 2 \le x < 4, \\ -\ x + 5, & 4 \le x < 5, \end{cases}$$

we calculate exactly the above four operations using $\alpha-$ cuts.

Let A_α and B_α be the α-cuts of A and B, respectively. Let $A_\alpha = [a_1^{(\alpha)},\ a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)},\ b_2^{(\alpha)}]$. Since $\alpha = a_1^{(\alpha)} - 1$ and $\alpha = -\frac{a_2^{(\alpha)}}{2} + 2$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha + 1, -2\alpha + 4]$. Since $\alpha = \frac{b_1^{(\alpha)}}{2} - 1$ and $\alpha = -b_2^{(\alpha)} + 5$, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [2\alpha + 2, -\alpha + 5]$.

1. Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [3\alpha + 3, -3\alpha + 9]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[3,9]^c$ and $\mu_{A(+)B}(x) = 1$ at $x = 6$. By the routine calculation, we have

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < 3, \ \ 9 \le x, \\ \frac{1}{3}x - 1, & 3 \le x < 6, \\ -\frac{1}{3}x + 3, & 6 \le x < 9, \end{cases}$$

i.e., $A(+)B = (3,6,9)$.

2. Subtraction :

Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [2\alpha - 4, -4\alpha + 2]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-4,2]^c$ and $\mu_{A(-)B}(x) = 1$ at $x = -2$. By the routine calculation, we have

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -4, \ \ 2 \le x, \\ \frac{1}{2}x + 2, & -4 \le x < -2, \\ -\frac{1}{4}x + \frac{1}{2}, & -2 \le x < 2, \end{cases}$$

i.e., $A(-)B = (-4, -2, 2)$.

3. Multiplication :

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_1^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] = [2\alpha^2 + 4\alpha + 2, 2\alpha^2 - 14\alpha + 20]$, $\mu_{A(\cdot)B}(x) = 0$ on the interval $[2, 20]^c$ and $\mu_{A(\cdot)B}(x) = 1$ at $x = 8$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < 2, \ \ 20 \le x, \\ \frac{-2+\sqrt{2x}}{2}, & 2 \le x < 8, \\ \frac{7-\sqrt{9+2x}}{2}, & 8 \le x < 20. \end{cases}$$

Thus $A(\cdot)B$ is not a triangular fuzzy number.

4. Division :

Since $A_\alpha(/)B_\alpha = [\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}] = [\frac{\alpha+1}{-\alpha+5}, \frac{-\alpha+2}{\alpha+1}]$, $\mu_{A(/)B}(x) = 0$ on the interval $[\frac{1}{5}, 2]^c$ and $\mu_{A(/)B}(x) = 1$ at $x = \frac{1}{2}$. By the routine calculation, we have

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{1}{5}, \ \ 2 \le x, \\ \frac{5x-1}{x+1}, & \frac{1}{5} \le x < \frac{1}{2}, \\ \frac{-x+2}{x+1}, & \frac{1}{2} \le x < 2. \end{cases}$$

Thus $A(/)B$ is not a triangular fuzzy number.

Theorem 2.16. ([5],[15]) For two trapezoidal fuzzy sets $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$, we have

1. $A(+)B = (a_1 + b_1, \ a_2 + b_2, \ a_3 + b_3, \ a_4 + b_4)$.
2. $A(-)B = (a_1 - b_4, \ a_2 - b_3, \ a_3 - b_2, \ a_4 - b_1)$.
3. $A(\cdot)B$ and $A(/)B$ need not to be trapezoidal fuzzy sets.

Example 2.17. Let $A = (1, 5, 6, 9)$ and $B = (2, 3, 5, 8)$ be trapezoidal fuzzy sets, i.e.,

$$\mu_A(x) = \begin{cases} 0, & x < 1, \ \ 9 \le x, \\ \frac{x-1}{4}, & 1 \le x < 5, \\ 1, & 5 \le x < 6, \\ \frac{-x+9}{3}, & 6 \le x < 9, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < 2, \ \ 8 \le x, \\ x-2, & 2 \le x < 3, \\ 1, & 3 \le x < 5, \\ \frac{-x+8}{3}, & 5 \le x < 8, \end{cases}$$

we calculate exactly the above four operations using $\alpha-$ cuts.

Let A_α and B_α be the α-cuts of A and B, respectively. Put $A_\alpha = [a_1^{(\alpha)},\ a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)},\ b_2^{(\alpha)}]$. Since $\alpha = \frac{a_1^{(\alpha)}-1}{4}$ and $\alpha = \frac{-a_2^{(\alpha)}+9}{3}$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [4\alpha+1, -3\alpha+9]$. Similarly, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha+2, -3\alpha+8]$.

1. Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)}+b_1^{(\alpha)}, a_2^{(\alpha)}+b_2^{(\alpha)}] = [5\alpha+3, -6\alpha+17]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[3, 17]^c$ and $\mu_{A(+)B}(x) = 1$ on the interval $[8, 11]$. By the routine calculation, we have

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < 3, \ \ 17 \le x, \\ \frac{x-3}{5}, & 3 \le x < 8, \\ 1, & 8 \le x < 11, \\ \frac{-x+17}{6}, & 11 \le x < 17, \end{cases}$$

i.e., $A(+)B = (3, 8, 11, 17)$.

2. Subtraction :

Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [7\alpha - 7, -4\alpha + 7]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-7, 7]^c$ and $\mu_{A(-)B}(x) = 1$ on the interval $[0, 3]$. By the routine calculation, we have

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -7,\ \ 7 \le x, \\ \frac{x+7}{7}, & -7 \le x < 0, \\ 1, & 0 \le x < 3, \\ \frac{-x+7}{4}, & 3 \le x < 7, \end{cases}$$

i.e., $A(-)B = (-7, 0, 3, 7)$.

3. Multiplication :

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_1^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] = [4\alpha^2 + 9\alpha + 2, 9\alpha^2 - 51\alpha + 72]$, $\mu_{A(\cdot)B}(x) = 0$ on the interval $[2, 72]^c$ and $\mu_{A(\cdot)B}(x) = 1$ on the interval $[15, 30]$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < 2,\ \ 72 \le x, \\ \frac{-9+\sqrt{49+16x}}{8}, & 2 \le x < 15, \\ 1, & 15 \le x < 30, \\ \frac{17-\sqrt{1+4x}}{6}, & 30 \le x < 72. \end{cases}$$

Thus $A(\cdot)B$ is not a trapezoidal fuzzy set.

4. Division :

Since $A_\alpha(/)B_\alpha = [\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}] = [\frac{4\alpha+1}{-3\alpha+8}, \frac{-3\alpha+9}{\alpha+2}]$, $\mu_{A(/)B}(x) = 0$ on the interval $[\frac{1}{8}, \frac{9}{2}]^c$ and $\mu_{A(/)B}(x) = 1$ on the interval $[1, 2]$. By the routine calculation, we have

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{1}{8},\ \ \frac{9}{2} \le x, \\ \frac{8x-1}{3x+4}, & \frac{1}{8} \le x < 1, \\ 1, & 1 \le x < 2, \\ \frac{-2x+9}{x+3}, & 2 \le x < \frac{9}{2}. \end{cases}$$

Thus $A(/)B$ is not a trapezoidal fuzzy set.

3. Quadratic fuzzy number and trigonometric fuzzy number

3.1 Quadratic fuzzy number

Similar to triangular fuzzy number, the quadratic fuzzy number is defined by quadratic curve.

Definition 3.1. A *quadratic fuzzy number* is a fuzzy number A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < \alpha, \ \ \beta \leq x, \\ -a(x-\alpha)(x-\beta) = -a(x-k)^2 + 1, & \alpha \leq x < \beta, \end{cases}$$

where $a > 0$.

The above quadratic fuzzy number is denoted by $A = [\alpha, k, \beta]$.

Theorem 3.2. For two quadratic fuzzy numbers $A = [x_1, k, x_2]$ and $B = [x_3, m, x_4]$, we have

1. $A(+)B = [x_1 + x_3, k + m, x_2 + x_4]$.
2. $A(-)B = [x_1 - x_4, k - m, x_2 - x_3]$.
3. $\mu_{A(\cdot)B}(x) = 0$ on the interval $[x_1x_3, x_2x_4]^c$ and $\mu_{A(\cdot)B}(x) = 1$ at $x = km$. Note that $A(\cdot)B$ needs not to be a quadratic fuzzy number.
4. $\mu_{A(/)B}(x) = 0$ on the interval $[\frac{x_1}{x_4}, \frac{x_2}{x_3}]^c$ and $\mu_{A(/)B}(x) = 1$ at $x = \frac{k}{m}$. Note that $A(/)B$ needs not to be a quadratic fuzzy number.

Proof. Note that

$$\mu_A(x) = \begin{cases} 0, & x < x_1, \ \ x_2 \leq x, \\ -a(x-k)^2 + 1 = -a(x-x_1)(x-x_2), & x_1 \leq x < x_2, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < x_3, \quad x_4 \le x, \\ -b(x-m)^2+1 = -b(x-x_3)(x-x_4), & x_3 \le x < x_4. \end{cases}$$

We calculate exactly four operations using α-cuts. Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ are the α-cuts of A and B, respectively. Since $\alpha = -a(a_1^{(\alpha)} - k)^2 + 1$ and $\alpha = -a(a_2^{(\alpha)} - k)^2 + 1$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = \left[k - \sqrt{\frac{1-\alpha}{a}},\ \ k + \sqrt{\frac{1-\alpha}{a}}\ \right].$$

Similarly, we have

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = \left[m - \sqrt{\frac{1-\alpha}{b}},\ \ m + \sqrt{\frac{1-\alpha}{b}}\ \right].$$

1. Addition : By the above facts,

$$\begin{aligned} A_\alpha(+)B_\alpha &= [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] \\ &= \Big[k + m - \sqrt{\frac{1-\alpha}{a}} - \sqrt{\frac{1-\alpha}{b}}, \\ &\qquad\qquad k + m + \sqrt{\frac{1-\alpha}{a}} + \sqrt{\frac{1-\alpha}{b}}\ \Big]. \end{aligned}$$

Thus $\mu_{A(+)B}(x) = 0$ on the interval $[k+m-\frac{1}{\sqrt{a}}-\frac{1}{\sqrt{b}}, k+m+\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}]^c$ $= [x_1+x_3, x_2+x_4]^c$ and $\mu_{A(+)B}(x) = 1$ at $x = k+m$. Therefore

$\mu_{A(+)B}(x)$

$$= \begin{cases} 0, & x < x_1+x_3, \quad x_2+x_4 \le x, \\ -\frac{ab}{(\sqrt{a}+\sqrt{b})^2}\{x-(k+m)\}^2+1, & x_1+x_3 \le x < x_2+x_4, \end{cases}$$

i.e., $A(+)B = [x_1+x_3, k+m, x_2+x_4]$.

2. Subtraction : Since

$$A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}]$$
$$= \Big[k - m - \sqrt{\frac{1-\alpha}{a}} - \sqrt{\frac{1-\alpha}{b}},$$
$$k - m + \sqrt{\frac{1-\alpha}{a}} + \sqrt{\frac{1-\alpha}{b}}\ \Big],$$

we have $\mu_{A(-)B}(x) = 0$ on the interval $[k-m-(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}), k-m+(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}})]^c = [x_1-x_4, x_2-x_3]$ and $\mu_{A(-)B}(x) = 1$ at $x = k-m$. Therefore

$\mu_{A(-)B}(x)$

$$= \begin{cases} 0, & x < x_1 - x_4,\ \ x_2 - x_3 \le x, \\ -\frac{ab}{(\sqrt{a}+\sqrt{b})^2}\{x-(k-m)\}^2 + 1, & x_1 - x_4 \le x < x_2 - x_3, \end{cases}$$

i.e., $A(-)B = [x_1 - x_4, k-m, x_2 - x_3]$.

3. Multiplication : Since

$$A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} b_1^{(\alpha)}, a_2^{(\alpha)} b_2^{(\alpha)}]$$
$$= \Big[\Big(k - \sqrt{\frac{1-\alpha}{a}}\Big)\Big(m - \sqrt{\frac{1-\alpha}{b}}\Big),$$
$$\Big(k + \sqrt{\frac{1-\alpha}{a}}\Big)\Big(m + \sqrt{\frac{1-\alpha}{b}}\Big)\Big],$$

$\mu_{A(\cdot)B}(x) = 0$ on the interval

$$\Big[\Big(k - \frac{1}{\sqrt{a}}\Big)\Big(m - \frac{1}{\sqrt{b}}\Big), \Big(k + \frac{1}{\sqrt{a}}\Big)\Big(m + \frac{1}{\sqrt{b}}\Big)\Big]^c = [x_1x_3, x_2x_4]^c$$

and $\mu_{A(\cdot)B}(x) = 1$ at $x = km$. Therefore

$\mu_{A(\cdot)B}(x)$

$$= \begin{cases} 0, & x < x_1x_3,\ \ x_2x_4 \le x, \\ \frac{1}{4}\Big(4 - 2k^2a - 2m^2b - 4\sqrt{ab}x \\ \quad +2(k\sqrt{a}+m\sqrt{b})\sqrt{(k\sqrt{a}-m\sqrt{b})^2 + 4\sqrt{ab}x}\Big), \\ & x_1x_3 \le x < x_2x_4. \end{cases}$$

4. Division : Since

$$A_\alpha(/)B_\alpha = \left[\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\right] = \left[\frac{k-\sqrt{\frac{1-\alpha}{a}}}{m+\sqrt{\frac{1-\alpha}{b}}}, \frac{k+\sqrt{\frac{1-\alpha}{a}}}{m-\sqrt{\frac{1-\alpha}{b}}}\right],$$

$\mu_{A(/)B}(x) = 0$ on the interval

$$\Big[\frac{\sqrt{b}(k\sqrt{a}-1)}{\sqrt{a}(m\sqrt{b}+1)}, \frac{\sqrt{b}(k\sqrt{a}+1)}{\sqrt{a}(m\sqrt{b}-1)}\Big]^c = \Big[\frac{x_1}{x_4}, \frac{x_2}{x_3}\Big]^c$$

and $\mu_{A(/)B}(x) = 1$ at $x = \frac{k}{m}$. Therefore

$$\mu_{A(/)B}(x)$$

$$= \begin{cases} 0, & x < \frac{x_1}{x_4}, \ \ \frac{x_2}{x_3} \le x, \\ \frac{a(1-bm^2)x^2+2\sqrt{ab}(1+\sqrt{ab}km)x+b(1-ak^2)}{(\sqrt{a}x+\sqrt{b})^2}, & \frac{x_1}{x_4} \le x < \frac{x_2}{x_3}. \end{cases} \qquad \square$$

Example 3.3. For two quadratic fuzzy numbers $A = [1,2,3]$ and $B = [2,5,8]$, we calculate exactly the above four operations using α-cuts. Note that

$$\mu_A(x) = \begin{cases} 0, & x < 1, \ \ 3 \le x, \\ -(x-2)^2+1, & 1 \le x < 3, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < 2, \ \ 8 \le x, \\ -\frac{1}{9}(x-5)^2+1, & 2 \le x < 8. \end{cases}$$

Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$. Since $\alpha = -(a_1^{(\alpha)}-2)^2+1$ and $\alpha = -(a_2^{(\alpha)}-2)^2+1$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [2-\sqrt{1-\alpha}, 2+\sqrt{1-\alpha}]$. Similarly, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [5-3\sqrt{1-\alpha}, 5+3\sqrt{1-\alpha}]$.

1. Addition : By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [7 - 4\sqrt{1-\alpha}, 7 + 4\sqrt{1-\alpha}]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[3, 11]^c$ and $\mu_{A(+)B}(x) = 1$ at $x = 7$. By the routine calculation, we have

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < 3, \ \ 11 \le x, \\ -\frac{1}{16}(x-7)^2 + 1, & 3 \le x < 11, \end{cases}$$

i.e., $\mu_{A(+)B}(x) = [3, 7, 11]$.

2. Subtraction : Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [-3 - 4\sqrt{1-\alpha}, -3 + 4\sqrt{1-\alpha}]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-7, 1]^c$ and $\mu_{A(-)B}(x) = 1$ at $x = -3$. By the routine calculation, we have

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -7, \ \ 1 \le x, \\ -\frac{1}{16}(x+3)^2 + 1, & -7 \le x < 1, \end{cases}$$

i.e., $\mu_{A(-)B}(x) = [-7, -3, 1]$.

3. Multiplication : Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_1^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] = [13 - 3\alpha - 11\sqrt{1-\alpha}, 13 - 3\alpha + 11\sqrt{1-\alpha}\,]$, $\mu_{A(\cdot)B}(x) = 0$ on the interval $[2, 24]^c$ and $\mu_{A(\cdot)B}(x) = 1$ at $x = 10$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < 2, \ \ 24 \le x, \\ -\frac{1}{18}(6x + 43 - 11\sqrt{12x+1}), & 2 \le x < 24. \end{cases}$$

Thus $A(\cdot)B$ is not a quadratic fuzzy number.

4. Division : Since

$$A_\alpha(/)B_\alpha = \left[\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\right] = \left[\frac{2 - \sqrt{1-\alpha}}{5 + 3\sqrt{1-\alpha}}, \frac{2 + \sqrt{1-\alpha}}{5 - 3\sqrt{1-\alpha}}\right],$$

$\mu_{A(/)B}(x) = 0$ on the interval $[\frac{1}{8}, \frac{3}{2}]^c$ and $\mu_{A(/)B}(x) = 1$ at $x = \frac{2}{5}$. By the routine calculation, we have

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{1}{8}, \ \ \frac{3}{2} \le x, \\ \dfrac{-(8x-1)(2x-3)}{(3x+1)^2}, & \frac{1}{8} \le x < \frac{3}{2}. \end{cases}$$

Thus $A(\cdot)B$ is not a quadratic fuzzy number.

3.2 Trigonometric fuzzy number

Similar to triangular fuzzy number, the trigonometric fuzzy number is defined by trigonometric curve.

Definition 3.4. A *trigonometric fuzzy number* is a fuzzy number A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < \theta_1, \ \ \theta_3 \le x, \\ \sin(x-\theta_1), & \theta_1 \le x < \theta_3, \end{cases}$$

where $\theta_3 - \theta_1 = \pi$.

The above trigonometric fuzzy number is denoted by $A =< \theta_1, \theta_2, \theta_3 >$, where $\theta_2 = \theta_1 + \frac{\pi}{2}$. And for all trigonometric fuzzy number $A =< \theta_1, \theta_2, \theta_3 >$, we define $\sin^{-1}(\cdot)$ as an inverse of $\sin(\cdot) : [0, \theta_2 - \theta_1] \to [0, 1]$.

Theorem 3.5. For two trigonometric fuzzy numbers $A =< c_1, c_2, c_3 >$ and $B =< d_1, d_2, d_3 >$, we have

1. $A(+)B =< c_1 + d_1, c_2 + d_2, c_3 + d_3 >$.
2. $A(-)B =< c_1 - d_3, c_2 - d_2, c_3 - d_1 >$.
3. $A(\cdot)B$ and $A(/)B$ are expressed by trigonometric function.

Proof. Note that

$$\mu_A(x) = \begin{cases} 0, & x < c_1, \ \ \pi + c_1 \le x, \\ \sin(x - c_1), & c_1 \le x < \pi + c_1, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < d_1, \ \ \pi + d_1 \le x, \\ \sin(x - d_1), & d_1 \le x < \pi + d_1. \end{cases}$$

We calculate exactly four operations using α-cuts. Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. Since $\alpha = \sin(a_1^{(\alpha)} - c_1)$ and $a_2^{(\alpha)} = \pi + 2c_1 - a_1^{(\alpha)}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = \Big[\sin^{-1}\alpha + c_1, \ \pi + c_1 - \sin^{-1}\alpha\Big],$$

where $0 \le \sin^{-1}\alpha \le c_2 - c_1$. Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = \Big[\sin^{-1}\alpha + d_1, \ \pi + d_1 - \sin^{-1}\alpha\Big].$$

1. Addition : By the above facts,

$$\begin{aligned} A_\alpha(+)B_\alpha &= [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] \\ &= \Big[2\sin^{-1}\alpha + c_1 + d_1, \ 2(\pi - \sin^{-1}\alpha) + c_1 + d_1\Big]. \end{aligned}$$

Thus $\mu_{A(+)B}(x) = 0$ on the interval $[c_1+d_1, 2\pi+c_1+d_1]^c = [c_1+d_1, c_3+d_3]^c$ and $\mu_{A(+)B}(x) = 1$ at $x = \pi + c_1 + d_1 = c_2 + d_2$. Therefore

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < c_1 + d_1, \ \ c_3 + d_3 \le x, \\ \sin\frac{1}{2}(x - c_1 - d_1), & c_1 + d_1 \le x < c_3 + d_3, \end{cases}$$

i.e., $A(+)B = < c_1 + d_1, c_2 + d_2, c_3 + d_3 >$.

2. Subtraction : Since

$$\begin{aligned} A_\alpha(-)B_\alpha &= [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] \\ &= \Big[2\sin^{-1}\alpha + c_1 - d_1 - \pi, \ \pi + c_1 - d_1 - 2\sin^{-1}\alpha\Big], \end{aligned}$$

we have $\mu_{A(-)B}(x) = 0$ on the interval $[c_1 - d_1 - \pi, \pi + c_1 - d_1]^c = [c_1 - d_3, c_3 - d_1]^c$ and $\mu_{A(-)B}(x) = 1$ at $x = c_1 - d_1$. Therefore

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < c_1 - d_3,\ \ c_3 - d_1 \le x, \\ \sin\frac{1}{2}(x + \pi + d_1 - c_1), & c_1 - d_3 \le x < c_3 - d_1, \end{cases}$$

i.e., $A(-)B =< c_1 - d_3, c_1 - d_1, c_3 - d_1 >$.

3. Multiplication : Since

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_1^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] \\ &= [c_1 d_1 + (d_1 + c_1)\sin^{-1}\alpha + (\sin^{-1}\alpha)^2,\ \pi^2 + (d_1 + c_1)\pi \\ &\qquad + c_1 d_1 - (2\pi + d_1 + c_1)\sin^{-1}\alpha + (\sin^{-1}\alpha)^2], \end{aligned}$$

we have $\mu_{A(\cdot)B}(x) = 0$ on the interval $[c_1 d_1, \pi^2 + (d_1 + c_1)\pi + c_1 d_1]^c = [c_1 d_1, c_3 d_3]^c$ and $\mu_{A(\cdot)B} = 1$ at $x = \frac{\pi^2}{4} + (d_1 + c_1)\frac{\pi}{2} + c_1 d_1 = c_2 d_2$. Therefore

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < c_1 d_1,\ \ c_3 d_3 \le x, \\ \sin\frac{1}{2}\Big(-(c_1 + d_1) + \sqrt{(d_1 + c_1)^2 - 4(c_1 d_1 - x)}\Big), \\ & c_1 d_1 \le x < c_3 d_3. \end{cases}$$

Note that $A(\cdot)B$ is not a trigonometric fuzzy number, but the membership function of $A(\cdot)B$ is expressed by trigonometric function.

4. Division : Since

$$\begin{aligned} A_\alpha(/)B_\alpha &= \left[\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\right] \\ &= \left[\frac{\sin^{-1}\alpha + c_1}{-\sin^{-1}\alpha + \pi + d_1}, \frac{-\sin^{-1}\alpha + c_1 + \pi}{\sin^{-1}\alpha + d_1}\right], \end{aligned}$$

we have $\mu_{A(/)B}(x) = 0$ on the interval $[\frac{c_1}{\pi + d_1}, \frac{\pi + c_1}{d_1}]^c = [\frac{c_1}{d_3}, \frac{c_3}{d_1}]^c$

and $\mu_{A(/)B}(x)=1$ at $x=\frac{\frac{\pi}{2}+c_1}{\frac{\pi}{2}+d_1}=\frac{c_2}{d_2}$. Therefore

$$\mu_{A(/)B}(x)=\begin{cases} 0, & x<\frac{c_1}{d_3},\ \ \frac{c_3}{d_1}\le x, \\ \sin\frac{(\pi+d_1)x-c_1}{x+1}, & \frac{c_1}{d_3}\le x<\frac{c_3}{d_1}. \end{cases}$$

Note that $A(/)B$ is not a trigonometric fuzzy number, but the membership function of $A(/)B$ is expressed by trigonometric function. □

Example 3.6. For two trigonometric fuzzy numbers A=$<\frac{\pi}{2},\pi,\frac{3\pi}{2}>$ and B=$<\frac{\pi}{3},\frac{5\pi}{6},\frac{4\pi}{3}>$, we calculate exactly the above four operations using α-cuts. Note that

$$\mu_A(x)=\begin{cases} 0, & x<\frac{\pi}{2},\ \ \frac{3}{2}\pi\le x, \\ \sin(x-\frac{\pi}{2}), & \frac{\pi}{2}\le x<\frac{3}{2}\pi, \end{cases}$$

and

$$\mu_B(x)=\begin{cases} 0, & x<\frac{\pi}{3},\ \ \frac{4}{3}\pi\le x, \\ \sin(x-\frac{\pi}{3}), & \frac{\pi}{3}\le x<\frac{4}{3}\pi. \end{cases}$$

Put $A_\alpha=[a_1^{(\alpha)},a_2^{(\alpha)}]$ and $B_\alpha=[b_1^{(\alpha)},b_2^{(\alpha)}]$. Since $\alpha=\sin(a_1^{(\alpha)}-\frac{\pi}{2})$ and $a_2^{(\alpha)}=2\pi-a_1^{(\alpha)}$, we have $A_\alpha=[a_1^{(\alpha)},a_2^{(\alpha)}]=[\frac{\pi}{2}+\sin^{-1}\alpha,\frac{3}{2}\pi-\sin^{-1}\alpha]$. Similarly, $B_\alpha=[b_1^{(\alpha)},b_2^{(\alpha)}]=[\frac{\pi}{3}+\sin^{-1}\alpha,\frac{4}{3}\pi-\sin^{-1}\alpha]$.

1. Addition : By the above facts, $A_\alpha(+)B_\alpha=[a_1^{(\alpha)}+b_1^{(\alpha)},a_2^{(\alpha)}+b_2^{(\alpha)}]=[\frac{5}{6}\pi+2\sin^{-1}\alpha,\frac{17}{6}\pi-2\sin^{-1}\alpha]$. Thus $\mu_{A(+)B}(x)=0$ on the interval $[\frac{5}{6}\pi,\frac{17}{6}\pi]^c$ and $\mu_{A(+)B}(x)=1$ at $x=\frac{11}{6}\pi$. Therefore

$$\mu_{A(+)B}(x)=\begin{cases} 0, & x<\frac{5}{6}\pi,\ \ \frac{17}{6}\pi\le x, \\ \sin\frac{1}{2}(x-\frac{5}{6}\pi), & \frac{5}{6}\pi\le x<\frac{17}{6}\pi, \end{cases}$$

i.e., $A(+)B=<\frac{5}{6}\pi,\frac{11}{6}\pi,\frac{17}{6}\pi>$.

2. Subtraction : Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [2\sin^{-1}\alpha - \frac{5}{6}\pi, \frac{7}{6}\pi - 2\sin^{-1}\alpha]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-\frac{5}{6}\pi, \frac{7}{6}\pi]^c$ and $\mu_{A(-)B}(x) = 1$ at $x = \frac{\pi}{6}$. Therefore

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -\frac{5}{6}\pi, \quad \frac{7}{6}\pi \le x, \\ \sin\frac{1}{2}(x + \frac{5}{6}\pi), & -\frac{5}{6}\pi \le x < \frac{7}{6}\pi, \end{cases}$$

i.e., $A(-)B = < -\frac{5}{6}\pi, \frac{1}{6}\pi, \frac{7}{6}\pi >$.

3. Multiplication : Since

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_1^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] \\ &= [\frac{\pi^2}{6} + \frac{5}{6}\pi\sin^{-1}\alpha + (\sin^{-1}\alpha)^2, \\ &\qquad 2\pi^2 - \frac{17}{6}\pi\sin^{-1}\alpha + (\sin^{-1}\alpha)^2], \end{aligned}$$

we have $\mu_{A(\cdot)B}(x) = 0$ on the interval $[\frac{\pi^2}{6}, 2\pi^2]^c$ and $\mu_{A(\cdot)B} = 1$ at $x = \frac{5}{6}\pi^2$. Therefore

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < \frac{\pi^2}{6}, \quad 2\pi^2 \le x, \\ \sin\left(-\frac{5}{12}\pi + \sqrt{\frac{\pi^2}{144} + x}\right), & \frac{\pi^2}{6} \le x < 2\pi^2. \end{cases}$$

4. Division : Since

$$A_\alpha(/)B_\alpha = \left[\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\right] = \left[\frac{2\sin^{-1}\alpha + 3\pi}{8\pi - 6\sin^{-1}\alpha}, \frac{9\pi - 6\sin^{-1}\alpha}{6\sin^{-1}\alpha + 2\pi}\right],$$

we have $\mu_{A(/)B}(x) = 0$ on the interval $[\frac{3}{8}, \frac{9}{2}]^c$ and $\mu_{A(/)B}(x) = 1$ at $x = \frac{4}{5}$. Therefore

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{3}{8}, \quad \frac{9}{2} \le x, \\ \sin\frac{8\pi x - 3\pi}{2(1+3x)}, & \frac{3}{8} \le x < \frac{9}{2}. \end{cases}$$

4. Fuzzy probability

4.1 Probability space and fuzzy probability

Let Ω be a nonempty set. Let $\mathfrak{F}$ be a σ*-field* of subsets of Ω, that is, a nonempty class of subsets of Ω which is closed under countable union and complement.

Let P be a measure defined on $\mathfrak{F}$ satisfying $P(\Omega) = 1$, i.e., P satisfies

(1) $P(A) \geq 0$ for all $A \in \mathfrak{F}$.

(2) $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$, for all disjoint subsets A_n.

(3) $P(\Omega) = 1$.

Then the triple $(\Omega, \mathfrak{F}, P)$ is called a *probability space*, and P, a probability measure. The set Ω is the sure event, and elements of $\mathfrak{F}$ are called *events*.

We note that, if $A_n \in \mathfrak{F}, n = 1, 2, \cdots,$ then $A_n^c, \cup_{n=1}^{\infty} A_n, \cap_{n=1}^{\infty} A_n$, $\liminf_{n\to\infty} A_n, \limsup_{n\to\infty} A_n$, and $\lim_{n\to\infty} A_n$, if it exists, are events. Moreover,

$$P(\liminf_{n\to\infty} A_n) \leq \liminf_{n\to\infty} P(A_n) \leq \limsup_{n\to\infty} P(A_n) \leq P(\limsup_{n\to\infty} A_n),$$

and, if $\lim_{n\to\infty} A_n$ exists, then

$$P(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n).$$

This is known as the continuity property of probability measures.

Definition 4.1. Let $(\Omega, \mathfrak{F}, P)$ be a probability space. A real-valued function X defined on Ω is said to be a *random variable* if

$$X^{-1}(E) = \{\omega \in \Omega : X(\omega) \in E\} \in \mathfrak{F} \quad \text{for all} \quad E \in \mathcal{B},$$

where $\mathcal{B}$ is the σ-field of Borel sets in $\mathbb{R} = (-\infty, \infty)$; that is, a random variable X is a measurable transformation of $(\Omega, \mathfrak{F}, P)$ into $(\mathbb{R}, \mathcal{B})$.

Remark. X is a random variable if and only if $X^{-1}(I) \in \mathfrak{F}$ for all intervals $I = (a, b],\ a, b \in \mathbb{R}$.

We note that a random variable X defined on $(\Omega, \mathfrak{F}, P)$ induces a measure P_X on $\mathcal{B}$ defined by the relation

$$P_X(E) = P\{X^{-1}(E)\} \quad (E \in \mathcal{B}).$$

Clearly, P_X is a probability measure on $\mathcal{B}$ and is called the probability distribution or the distribution of X. We note that P_X is a Lebesgue-Stieltjes measure on $\mathcal{B}$.

Definition 4.2. Let X be a random variable. The function $F_X : \mathbb{R} \to \mathbb{R}$ defined by

$$F_X(x) = P_X(-\infty, x] = P\{\omega \in \Omega : X(\omega) \le x\}.$$

is called the *distribution* of X.

Theorem 4.3. The distribution function F of a random variable X is a nondecreasing, right-continuous function on $\mathbb{R}$ which satisfies

$$F(-\infty) = \lim_{x \to -\infty} F(x) = 0$$

and

$$F(\infty) = \lim_{x \to \infty} F(x) = 1.$$

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, and X be a random variable defined on it. Let g be a real-valued Borel-measurable function on $\mathbb{R}$. Then $g(X)$ is also a random variable.

Definition 4.4. We say that the *mathematical expectation* of $g(X)$ exists if

$$E[g(X)] = \int_{\Omega} g(X(\omega))\, dP(\omega) = \int_{\Omega} g(X)\, dP$$

is finite.

It is known that if $E[g(X)]$ exists, then g is also integrable over $\mathbb{R}$ with respect to P_X. Moreover, the relation

$$\int_{\Omega} g(X)\, dP = \int_{\mathbb{R}} g(t)\, dP_X(t) \tag{4.1}$$

holds. We note that the integral on the right-hand side of (4.1) is the Lebesgue-Stieltjes integral of g with respect to P_X.

In particular, if g is continuous on $\mathbb{R}$ and $E[g(X)]$ exists, we can rewrite (4.1) as follows

$$\int_{\Omega} g(X)\, dP = \int_{\mathbb{R}} g\, dP_X = \int_{-\infty}^{\infty} g(x)\, dF(x),$$

where F is the distribution function corresponding to P_X, and the last integral is a Riemann-Stieltjes integral.

Let F be absolutely continuous on $\mathbb{R}$ with probability density function $f(x) = F'(x)$. Then $E[g(X)]$ exists if and only if $\int_{-\infty}^{\infty} |g(x)| f(x) dx$ is finite and in that case we have

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx.$$

We note some elementary properties of random variables with finite expectations which follow as immediate consequences of the properties of integrable functions. Let $\mathfrak{F}_1 = \mathfrak{F}_1(\Omega, \mathfrak{F}, P)$ be the set of all random variables with finite expectations. In the followings, we write a.s. to abbreviate "almost surely with respect to the probability distribution of X on $(\mathbb{R}, \mathfrak{B})$".

1. $X, Y \in \mathfrak{F}_1$ and $\alpha, \beta \in \mathbb{R} \Rightarrow \alpha X + \beta Y \in \mathfrak{F}_1$ and $E(\alpha X + \beta Y) = \alpha E[X] + \beta E[Y]$.
2. $X \in \mathfrak{F}_1 \Rightarrow |E[X]| \le E[|X|]$.
3. $X \in \mathfrak{F}_1, X \ge 0$ a.s. $\Rightarrow E[X] \ge 0$.
4. Let $X \in \mathfrak{F}_1$. Then $E[|X|] = 0 \Leftrightarrow X = 0$ a.s..
5. For $A \in \mathfrak{F}$, write χ_A for the indicator function of the set A, that is, $\chi_A = 1$ on A and $\chi_A = 0$ otherwise. Then $X \in \mathfrak{F}_1 \Rightarrow X \cdot \chi_A \in \mathfrak{F}_1$, and we write
$$\int_A X dP = E[X \cdot \chi_A].$$
Also, $E[|X| \cdot \chi_A] = 0 \Leftrightarrow$ either $P(A) = 0$ or $X = 0$ a.s. on A.
6. If $X \in \mathfrak{F}_1$, then $X = 0$ a.s. $\Leftrightarrow E[X \cdot \chi_A] = 0$ for all $A \in \mathfrak{F}$.
7. Let $X \in \mathfrak{F}_1$ and $A \in \mathfrak{F}$. If $\alpha \le X \le \beta$ a.s. on A for $\alpha, \beta \in \mathbb{R}$, then
$$\alpha P(A) \le \int_A X dP \le \beta P(A).$$
8. Let $Y \in \mathfrak{F}_1$, and X be a random variable such that $|X| \le |Y|$ a.s.. Then $X \in \mathfrak{F}_1$ and $E[|X|] \le E[|Y|]$. In particular, if X is bounded a.s., then $X \in \mathfrak{F}_1$.

Example 4.5. Let the random variable X have the normal distribution, denoted by $X \sim N(m, \sigma^2)$, with the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-m)^2}{2\sigma^2}}, \quad x \in \mathbb{R},$$

where $\sigma^2 > 0$ and $m \in \mathbb{R}$. Then $E[|X|^\gamma] < \infty$ for every $\gamma > 0$, and we have

$$E[X] = m \quad \text{and} \quad E[(X-m)^2] = \sigma^2.$$

The induced measure P_X is called the *normal distribution*.

A fuzzy set A on Ω is called a *fuzzy event*. Let $\mu_A(\cdot)$ be the membership function of the fuzzy event A. Then the probability of the fuzzy event A is defined by Zadeh([17]) as

$$\widetilde{P}(A) = \int_\Omega \mu_A(\omega)\, dP(\omega), \quad \mu_A(\omega) : \Omega \to [0,1].$$

Theorem 4.6. The probability of the fuzzy event satisfies the following properties.

1. $\widetilde{P}(A \cup B) = \widetilde{P}(A) + \widetilde{P}(B) - \widetilde{P}(A \cap B)$.
2. $\widetilde{P}(A^c) = 1 - \widetilde{P}(A)$.
3. $\widetilde{P}(A \hat{+} B) = \widetilde{P}(A) + \widetilde{P}(B) - \widetilde{P}(A \cdot B)$.

Proof.

1. $\widetilde{P}(A \cup B) = \int_\Omega \mu_{A\cup B}(\omega) dP(\omega)$
$= \int_\Omega (\mu_A(\omega) + \mu_B(\omega) - \mu_{A\cap B}(\omega)) dP(\omega)$
$= \int_\Omega \mu_A(\omega) dP(\omega) + \int_\Omega \mu_B(\omega) dP(\omega) - \int_\Omega \mu_{A\cap B}(\omega) dP(\omega)$
$= \widetilde{P}(A) + \widetilde{P}(B) - \widetilde{P}(A \cap B)$
2. $\widetilde{P}(A^c) = \int_\Omega \mu_{A^c}(\omega) dP(\omega)$

$$= \int_\Omega \{1 - \mu_A(\omega)\} dP(\omega)$$
$$= 1 - \widetilde{P}(A)$$

3. $\widetilde{P}(A \widehat{+} B) = \int_\Omega \mu_{A \widehat{+} B}(\omega) dP(\omega)$

$$\begin{aligned}
&= \int_\Omega (\mu_A(\omega) + \mu_B(\omega) - \mu_A(\omega) \cdot \mu_B(\omega)) dP(\omega) \\
&= \int_\Omega \mu_A(\omega) dP(\omega) + \int_\Omega \mu_B(\omega) dP(\omega) \\
&\qquad - \int_\Omega \mu_A(\omega) \cdot \mu_B(\omega) dP(\omega) \\
&= \widetilde{P}(A) + \widetilde{P}(B) - \widetilde{P}(A \cdot B)
\end{aligned}$$

□

4.2 Normal fuzzy probability

Definition 4.7. The *normal fuzzy probability* $\widetilde{P}(A)$ of a fuzzy set A on $\mathbb{R}$ is defined by

$$\widetilde{P}(A) = \int_{\mathbb{R}} \mu_A(x)\, dP_X,$$

where P_X is the normal distribution.

Theorem 4.8. The normal fuzzy probability $\widetilde{P}(A)$ of a triangular fuzzy number $A = (a_1, a_2, a_3)$ is

$$\begin{aligned}
\widetilde{P}(A) = &\frac{m - a_1}{a_2 - a_1} \Big(N(\frac{a_2 - m}{\sigma}) - N(\frac{a_1 - m}{\sigma}) \Big) \\
&+ \frac{\sigma}{\sqrt{2\pi}(a_2 - a_1)} \Big(e^{-\frac{(a_1 - m)^2}{2\sigma^2}} - e^{-\frac{(a_2 - m)^2}{2\sigma^2}} \Big) \\
&+ \frac{m - a_3}{a_2 - a_3} \Big(N(\frac{a_3 - m}{\sigma}) - N(\frac{a_2 - m}{\sigma}) \Big) \\
&+ \frac{\sigma}{\sqrt{2\pi}(a_2 - a_3)} \Big(e^{-\frac{(a_2 - m)^2}{2\sigma^2}} - e^{-\frac{(a_3 - m)^2}{2\sigma^2}} \Big),
\end{aligned}$$

where $N(a)$ is the *standard normal distribution*, that is,

$$N(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{t^2}{2}}\, dt.$$

Proof. Since

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \quad a_3 \le x, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \le x < a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \le x < a_3, \end{cases}$$

we have

$$\begin{aligned} \widetilde{P}(A) &= \int_{\mathbb{R}} \mu_A(x)\, dP_X \\ &= \int_{a_1}^{a_2} g_1(x) f(x)\, dx + \int_{a_2}^{a_3} g_2(x) f(x)\, dx, \end{aligned}$$

where $g_1(x) = \dfrac{x-a_1}{a_2-a_1}$, $g_2(x) = \dfrac{a_3-x}{a_3-a_2}$ and $f(x) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$.
Put $\frac{x-m}{\sigma} = t$, then

$$\begin{aligned} \widetilde{P}(A) &= \int_{a_1}^{a_2} \frac{x-a_1}{a_2-a_1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} dx + \int_{a_2}^{a_3} \frac{a_3-x}{a_3-a_2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}(a_2-a_1)} \int_{\frac{a_1-m}{\sigma}}^{\frac{a_2-m}{\sigma}} (m+\sigma t) e^{-\frac{t^2}{2}}\, dt \\ &\quad - \frac{a_1}{\sqrt{2\pi}(a_2-a_1)} \int_{\frac{a_1-m}{\sigma}}^{\frac{a_2-m}{\sigma}} e^{-\frac{t^2}{2}}\, dt \\ &\quad + \frac{1}{\sqrt{2\pi}(a_2-a_3)} \int_{\frac{a_2-m}{\sigma}}^{\frac{a_3-m}{\sigma}} (m+\sigma t) e^{-\frac{t^2}{2}}\, dt \\ &\quad - \frac{a_3}{\sqrt{2\pi}(a_2-a_3)} \int_{\frac{a_2-m}{\sigma}}^{\frac{a_3-m}{\sigma}} e^{-\frac{t^2}{2}}\, dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{m}{\sqrt{2\pi}(a_2-a_1)} \int_{\frac{a_1-m}{\sigma}}^{\frac{a_2-m}{\sigma}} e^{-\frac{t^2}{2}} dt + \frac{\sigma}{\sqrt{2\pi}(a_2-a_1)} \int_{\frac{a_1-m}{\sigma}}^{\frac{a_2-m}{\sigma}} te^{-\frac{t^2}{2}} dt \\
&\quad - \frac{a_1}{\sqrt{2\pi}(a_2-a_1)} \int_{\frac{a_1-m}{\sigma}}^{\frac{a_2-m}{\sigma}} e^{-\frac{t^2}{2}} dt + \frac{m}{\sqrt{2\pi}(a_2-a_3)} \int_{\frac{a_2-m}{\sigma}}^{\frac{a_3-m}{\sigma}} e^{-\frac{t^2}{2}} dt \\
&\quad + \frac{\sigma}{\sqrt{2\pi}(a_2-a_3)} \int_{\frac{a_2-m}{\sigma}}^{\frac{a_3-m}{\sigma}} te^{-\frac{t^2}{2}} dt - \frac{a_3}{\sqrt{2\pi}(a_2-a_3)} \int_{\frac{a_2-m}{\sigma}}^{\frac{a_3-m}{\sigma}} e^{-\frac{t^2}{2}} dt \\
&= \frac{m-a_1}{a_2-a_1}\Big(N(\frac{a_2-m}{\sigma}) - N(\frac{a_1-m}{\sigma})\Big) \\
&\quad + \frac{\sigma}{\sqrt{2\pi}(a_2-a_1)}\Big(e^{-\frac{(a_1-m)^2}{2\sigma^2}} - e^{-\frac{(a_2-m)^2}{2\sigma^2}}\Big) \\
&\quad + \frac{m-a_3}{a_2-a_3}\Big(N(\frac{a_3-m}{\sigma}) - N(\frac{a_2-m}{\sigma})\Big) \\
&\quad + \frac{\sigma}{\sqrt{2\pi}(a_2-a_3)}\Big(e^{-\frac{(a_2-m)^2}{2\sigma^2}} - e^{-\frac{(a_3-m)^2}{2\sigma^2}}\Big).
\end{aligned}
$$

□

Example 4.9. 1. Let $A = (1, 3, 5)$ be a triangular fuzzy number. Then the normal fuzzy probability of A with respect to $X \sim N(3, 2^2)$ is 0.3687. In fact,

putting $\frac{x-3}{2} = t$,

$$
\begin{aligned}
\widetilde{P}(A) &= \int_1^3 \frac{x-1}{2} \cdot \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-3)^2}{8}} dx + \int_3^5 \frac{5-x}{2} \cdot \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-3)^2}{8}} dx \\
&= \frac{1}{2\sqrt{2\pi}} \int_{-1}^0 (2t+2) e^{-\frac{t^2}{2}} dt + \frac{1}{2\sqrt{2\pi}} \int_0^1 (2-2t) e^{-\frac{t^2}{2}} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^0 te^{-\frac{t^2}{2}} dt + \frac{1}{\sqrt{2\pi}} \int_{-1}^0 e^{-\frac{t^2}{2}} dt + \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-\frac{t^2}{2}} dt \\
&\quad - \frac{1}{\sqrt{2\pi}} \int_0^1 te^{-\frac{t^2}{2}} dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}(e^{-\frac{1}{2}} - 1) + \Big(N(0) - N(-1)\Big) + \Big(N(1) - N(0)\Big) \\
&\quad - \frac{1}{\sqrt{2\pi}}(1 - e^{-\frac{1}{2}}) \\
&= 0.3687.
\end{aligned}
$$

2. Let $B = (2, 4, 6)$ be a triangular fuzzy number. Then the normal fuzzy probability of A with respect to $X \sim N(3, 2^2)$ is 0.3315. In fact, putting $\frac{x-3}{2} = t$,

$$
\begin{aligned}
\widetilde{P}(A) &= \int_2^4 \frac{x-2}{2} \cdot \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-3)^2}{8}} dx + \int_4^6 \frac{6-x}{2} \cdot \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-3)^2}{8}} dx \\
&= \frac{1}{2\sqrt{2\pi}} \int_{\frac{-1}{2}}^{\frac{1}{2}} (2t+1) e^{-\frac{t^2}{2}} dt + \frac{1}{2\sqrt{2\pi}} \int_{\frac{1}{2}}^{\frac{3}{2}} (3-2t) e^{-\frac{t^2}{2}} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{\frac{-1}{2}}^{\frac{1}{2}} t e^{-\frac{t^2}{2}} dt + \frac{1}{2\sqrt{2\pi}} \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-\frac{t^2}{2}} dt + \frac{3}{2\sqrt{2\pi}} \int_{\frac{1}{2}}^{\frac{3}{2}} e^{-\frac{t^2}{2}} dt \\
&\quad - \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{2}}^{\frac{3}{2}} t e^{-\frac{t^2}{2}} dt \\
&= \frac{1}{2}\Big(N(\frac{1}{2}) - N(\frac{-1}{2})\Big) + \frac{3}{2}\Big(N(\frac{3}{2}) - N(\frac{1}{2})\Big) \\
&\quad - \frac{1}{\sqrt{2\pi}}(e^{-\frac{1}{8}} - e^{-\frac{9}{8}}) \\
&= 0.3315.
\end{aligned}
$$

3. Let $A = (1, 3, 5)$ be a triangular fuzzy number. Then the normal fuzzy probability of A with respect to $X \sim N(3, 3^2)$ is 0.2565. In fact, putting $\frac{x-3}{3} = t$,

$$
\begin{aligned}
\widetilde{P}(A) &= \int_1^3 \frac{x-1}{2} \cdot \frac{1}{3\sqrt{2\pi}} e^{-\frac{(x-3)^2}{18}} dx + \int_3^5 \frac{5-x}{2} \cdot \frac{1}{3\sqrt{2\pi}} e^{-\frac{(x-3)^2}{18}} dx \\
&= \frac{1}{2\sqrt{2\pi}} \int_{\frac{-2}{3}}^{0} (3t+2) e^{-\frac{t^2}{2}} dt + \frac{1}{2\sqrt{2\pi}} \int_0^{\frac{2}{3}} (2-3t) e^{-\frac{t^2}{2}} dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{3}{2\sqrt{2\pi}} \int_{\frac{-2}{3}}^{0} te^{-\frac{t^2}{2}} dt + \frac{1}{\sqrt{2\pi}} \int_{\frac{-2}{3}}^{0} e^{-\frac{t^2}{2}} dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{\frac{2}{3}} e^{-\frac{t^2}{2}} dt \\
&\quad - \frac{3}{2\sqrt{2\pi}} \int_{0}^{\frac{2}{3}} te^{-\frac{t^2}{2}} dt \\
&= \frac{3}{2\sqrt{2\pi}} (e^{-\frac{2}{9}} - 1) + \Big(N(0) - N(\frac{-2}{3}) \Big) + \Big(N(\frac{2}{3}) - N(0) \Big) \\
&\quad - \frac{3}{2\sqrt{2\pi}} (1 - e^{-\frac{2}{9}}) \\
&= 0.2565.
\end{aligned}
$$

By Example 4.9, we have the following property of normal fuzzy probability for triangular fuzzy numbers.

Remark. 1. The normal fuzzy probability of the triangular fuzzy number $A = (b-a, b, b+a)$ with respect to fixed $X \sim N(m, \sigma^2)$ takes its maximum at $b = m$.

2. For fixed triangular fuzzy number $A = (m-a, m, m+a)$, let $\widetilde{P}_1$ and $\widetilde{P}_2$ be the normal fuzzy probabilities of A with respect to $X \sim N(m, \sigma_1^2)$ and $X \sim N(m, \sigma_2^2)$, respectively. Then $\widetilde{P}_1 \leq \widetilde{P}_2$ if $\sigma_1 \geq \sigma_2$.

Now, we calculate the normal fuzzy probability for the four operations of two triangular fuzzy numbers. Since addition and subtraction of two triangular fuzzy numbers are triangular fuzzy numbers, the normal fuzzy probabilities of addition and subtraction can be calculated by the same way. But, in calculating of the normal fuzzy probabilities for multiplication and division of two triangular fuzzy numbers, we have to calculate the integrals of the following two forms ;

Form 1. $\mu_A(x) = \sqrt{ax+b}$

$$\begin{aligned}\widetilde{P} &= \int \sqrt{ax+b}\ \frac{1}{\sqrt{2\pi}\sigma}\ e^{-\frac{(x-m)^2}{2\sigma^2}}dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int \sqrt{ax+b}\ e^{-\frac{(x-m)^2}{2\sigma^2}}dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\Big(-\frac{\sqrt{ax+b}\sqrt{\frac{\pi}{2}}\ \mathrm{Erf}(\frac{\sigma(m-x)}{\sqrt{2}})}{\sigma}\Big),\end{aligned}$$

where

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz.$$

Form 2. $\mu_A(x) = \dfrac{cx+d}{ax+b} = \dfrac{p}{ax+b} + q$

$$\begin{aligned}\widetilde{P} &= \int\Big(\frac{p}{ax+b}+q\Big)\frac{1}{\sqrt{2\pi}\sigma}\ e^{-\frac{(x-m)^2}{2\sigma^2}}dx \\ &= \frac{p}{\sqrt{2\pi}\sigma}\int\frac{e^{-\frac{(x-m)^2}{2\sigma^2}}}{ax+b}\ dx + \frac{q}{\sqrt{2\pi}\sigma}\int e^{-\frac{(x-m)^2}{2\sigma^2}}\ dx \\ &= \frac{p}{\sqrt{2\pi}\sigma}\Big(\frac{\sqrt{\frac{\pi}{2}}\ \mathrm{Erf}(\frac{\sigma(m-x)}{\sqrt{2}})}{(ax+b)\sigma}\Big) + \frac{q}{\sqrt{2\pi}\sigma}\Big(-\frac{\sqrt{\frac{\pi}{2}}\ \mathrm{Erf}(\frac{\sigma(m-x)}{\sqrt{2}})}{\sigma}\Big).\end{aligned}$$

Example 4.10. We calculate the normal fuzzy probabilities for multiplication and division of two triangular fuzzy numbers in Example 2.15 with respect to $X \sim N(5,4)$.

1. Multiplication

$$\begin{aligned}\widetilde{P} &= \int \mu_{A(\cdot)B}(x)dP_X \\ &= \int_2^8 \frac{-2+\sqrt{2x}}{2}\frac{1}{2\sqrt{2\pi}}\ e^{-\frac{(x-5)^2}{2\cdot 4}}dx \\ &\quad + \int_8^{20}\frac{7-\sqrt{9+2x}}{2}\cdot\frac{1}{2\sqrt{2\pi}}\ e^{-\frac{(x-5)^2}{2\cdot 4}}dx\end{aligned}$$

$$= \frac{-2}{4\sqrt{2\pi}} \int_2^8 e^{-\frac{(x-5)^2}{8}} dx + \frac{1}{4\sqrt{2\pi}} \int_2^8 \sqrt{2x}\, e^{-\frac{(x-5)^2}{8}} dx$$
$$+ \frac{7}{4\sqrt{2\pi}} \int_8^{20} e^{-\frac{(x-5)^2}{8}} dx - \frac{1}{4\sqrt{2\pi}} \int_8^{20} \sqrt{9+2x}\, e^{-\frac{(x-5)^2}{8}} dx$$
$$= -\frac{1}{\sqrt{2\pi}} \int_{-\frac{3}{2}}^{\frac{3}{2}} e^{-\frac{t^2}{2}} dt + \frac{1}{4\sqrt{2\pi}} \int_2^8 \sqrt{2x}\, e^{-\frac{(x-5)^2}{8}} dx$$
$$+ \frac{7}{2\sqrt{2\pi}} \int_{\frac{3}{2}}^{\frac{15}{2}} e^{-\frac{t^2}{2}} dt - \frac{1}{4\sqrt{2\pi}} \int_8^{20} \sqrt{9+2x}\, e^{-\frac{(x-5)^2}{8}} dx$$
$$= N(-\frac{3}{2}) - N(\frac{3}{2}) + \frac{1}{4\sqrt{2\pi}} \int_2^8 \sqrt{2x}\, e^{-\frac{(x-5)^2}{8}} dx$$
$$+ \frac{7}{2}\Big((N(\frac{15}{2}) - N(\frac{3}{2})\Big) - \frac{1}{4\sqrt{2\pi}} \int_8^{20} \sqrt{9+2x}\, e^{-\frac{(x-5)^2}{8}} dx$$
$$= 0.5485.$$

2. Division

$$\widetilde{P} = \int \mu_{A(/)B}(x) dP_X$$

$$= \int_{\frac{1}{5}}^{\frac{1}{2}} \frac{5x-1}{x+1} \cdot \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-5)^2}{2\cdot 4}} dx + \int_{\frac{1}{2}}^{2} \frac{-x+2}{x+1} \cdot \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-5)^2}{2\cdot 4}} dx$$
$$= \frac{1}{2\sqrt{2\pi}} \int_{\frac{1}{5}}^{\frac{1}{2}} \Big(-\frac{6}{x+1} + 5\Big) e^{-\frac{(x-5)^2}{8}} dx$$
$$+ \frac{1}{2\sqrt{2\pi}} \int_{\frac{1}{2}}^{2} \Big(\frac{3}{x+1} - 1\Big) e^{-\frac{(x-5)^2}{8}} dx$$
$$= \frac{-3}{\sqrt{2\pi}} \int_{\frac{1}{5}}^{\frac{1}{2}} \frac{e^{-\frac{(x-5)^2}{8}}}{x+1} dx + \frac{5}{2\sqrt{2\pi}} \int_{\frac{1}{5}}^{\frac{1}{2}} e^{-\frac{(x-5)^2}{8}} dx$$
$$+ \frac{3}{2\sqrt{2\pi}} \int_{\frac{1}{2}}^{2} \frac{e^{-\frac{(x-5)^2}{8}}}{x+1} dx - \frac{1}{2\sqrt{2\pi}} \int_{\frac{1}{2}}^{2} e^{-\frac{(x-5)^2}{8}} dx$$
$$= \frac{-3}{\sqrt{2\pi}} \int_{\frac{1}{5}}^{\frac{1}{2}} \frac{e^{-\frac{(x-5)^2}{8}}}{x+1} dx + \frac{5}{\sqrt{2\pi}} \int_{-\frac{12}{5}}^{-\frac{9}{4}} e^{-\frac{t^2}{2}} dt$$

$$+\frac{3}{2\sqrt{2\pi}}\int_{\frac{1}{2}}^{2}\frac{e^{-\frac{(x-5)^2}{8}}}{x+1}\,dx-\frac{1}{\sqrt{2\pi}}\int_{-\frac{9}{4}}^{-\frac{3}{2}}e^{-\frac{t^2}{2}}dt$$
$$=\frac{-3}{\sqrt{2\pi}}\int_{\frac{1}{5}}^{\frac{1}{2}}\frac{e^{-\frac{(x-5)^2}{8}}}{x+1}\,dx+5\Big(N(-\frac{9}{4})-N(-\frac{12}{5})\Big)$$
$$+\frac{3}{2\sqrt{2\pi}}\int_{\frac{1}{2}}^{2}\frac{e^{-\frac{(x-5)^2}{8}}}{x+1}\,dx-\Big(N(-\frac{3}{2})-N(-\frac{9}{4})\Big)$$
$$=0.0177.$$

4.3 Exponential fuzzy probability

In this section, we define the exponential fuzzy probability using the exponential distribution.

Example 4.11. Let the random variable X have the exponential distribution with the probability density function

$$f(x)=\lambda e^{-\lambda x},$$

for $x \geq 0$ and $\lambda > 0$. Then we have

$$E[X]=\frac{1}{\lambda}\quad\text{and}\quad E[(X-\frac{1}{\lambda})^2]=\frac{1}{\lambda^2}.$$

The induced measure P_X is called the *exponential distribution.*

Definition 4.12. The *exponential fuzzy probability* $\widetilde{P}(A)$ of a fuzzy set A on $\mathbb{R}$ is defined by

$$\widetilde{P}(A)=\int_{\mathbb{R}}\mu_A(x)\,dP_X,$$

where P_X is exponential distribution.

Theorem 4.13. The exponential fuzzy probability $\widetilde{P}(A)$ of a triangular fuzzy number $A = (a_1, a_2, a_3)$ is

$$\widetilde{P}(A) = \frac{\lambda a_2}{(a_2 - a_1)}\Big[e^{-\lambda a_2}(-a_1 a_2 + \frac{a_2^2}{2}) - e^{-\lambda a_1}(-\frac{a_1^2}{2})\Big] + \frac{\lambda a_3}{a_3 - a_2}\Big[e^{-\lambda a_3}(\frac{a_3^2}{2}) - e^{-\lambda a_1}(a_3 a_2 - \frac{a_2^2}{2})\Big].$$

Proof. Since

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \quad a_3 \le x, \\ \frac{x - a_1}{a_2 - a_1}, & a_1 \le x < a_2, \\ \frac{a_3 - x}{a_3 - a_2}, & a_2 \le x < a_3, \end{cases}$$

we have

$$\begin{aligned} \widetilde{P}(A) &= \int_{\mathbb{R}} \mu_A(x)\, dP_X \\ &= \int_{a_1}^{a_2} g_1(x) f(x)\, dx + \int_{a_2}^{a_3} g_2(x) f(x)\, dx, \end{aligned}$$

where $g_1(x) = \dfrac{x - a_1}{a_2 - a_1}$, $g_2(x) = \dfrac{a_3 - x}{a_3 - a_2}$ and $f(x) = \lambda e^{-\lambda x} (x \ge 0)$.

$$\begin{aligned} \widetilde{P}(A) &= \int_{a_1}^{a_2} \frac{x - a_1}{a_2 - a_1} \lambda e^{-\lambda x} dx + \int_{a_2}^{a_3} \frac{a_3 - x}{a_3 - a_2} \lambda e^{-\lambda x} dx \\ &= \frac{\lambda}{(a_2 - a_1)} \int_{a_1}^{a_2} (x - a_1) e^{-\lambda x} dx + \frac{\lambda}{(a_3 - a_2)} \int_{a_2}^{a_3} (a_3 - x) e^{-\lambda x} dx \\ &= \frac{\lambda a_2}{(a_2 - a_1)}\Big[e^{-\lambda a_2}(-a_1 a_2 + \frac{a_2^2}{2}) - e^{-\lambda a_1}(-\frac{a_1^2}{2})\Big] \\ &\quad + \frac{\lambda a_3}{a_3 - a_2}\Big[e^{-\lambda a_3}(\frac{a_3^2}{2}) - e^{-\lambda a_1}(a_3 a_2 - \frac{a_2^2}{2})\Big]. \end{aligned}$$

□

Example 4.14. Let $A = (1, 4, 6)$ be a triangular fuzzy number. Then the exponential fuzzy probability with respect to $\lambda = 2$ is

$$\begin{aligned}\widetilde{P}(A) &= \int_1^4 \frac{x-1}{3} \cdot 2e^{-2x} dx + \int_4^6 \frac{6-x}{2} \cdot 2e^{-2x} dx \\ &= \Big[\frac{2}{3} e^{-2x} (\frac{1}{4} - \frac{x}{2})\Big]_1^4 + \Big[e^{-2x}(-\frac{11}{4} + \frac{x}{2})\Big]_4^6 \\ &= 0.0225.\end{aligned}$$

Example 4.15. We calculate the exponential fuzzy probabilities for multiplication and division of two triangular fuzzy numbers in Example 2.15 with respect to $\lambda = 2$.

1. Multiplication

$$\begin{aligned}\widetilde{P} &= \int \mu_{A(\cdot)B}(x) dP_X \\ &= \int_2^8 \frac{-2+\sqrt{2x}}{2} \cdot 2e^{-2x} dx + \int_8^{20} \frac{7-\sqrt{9+2x}}{2} \cdot 2e^{-2x} dx \\ &= -2 \int_2^8 e^{-2x} dx + \int_2^8 \sqrt{2x}\, e^{-2x} dx + 7 \int_8^{20} e^{-2x} dx \\ &\quad - \int_8^{20} \sqrt{9+2x}\, e^{-2x} dx \\ &= 0.0021.\end{aligned}$$

2. Division

$$\begin{aligned}\widetilde{P} &= \int \mu_{A(/)B}(x) dP_X \\ &= \int_{\frac{1}{5}}^{\frac{1}{2}} \frac{5x-1}{x+1} \cdot 2\, e^{-2x} dx + \int_{\frac{1}{2}}^{2} \frac{-x+2}{x+1} \cdot 2\, e^{-2x} dx \\ &= 0.3604.\end{aligned}$$

5. Fuzzy probability for fuzzy numbers driven by operations

We derive the explicit formula for the normal and the exponential fuzzy probability for quadratic fuzzy number and trigonometric fuzzy number and give some examples.

5.1 Normal fuzzy probability for quadratic fuzzy number and trigonometric fuzzy number

Theorem 5.1. Let $X \sim N(m,\sigma^2)$ and $A=[\alpha,k,\beta]$ be a quadratic fuzzy number. Then the normal fuzzy probability of A is

$$\begin{aligned}\widetilde{P}(A) =& \frac{\sigma}{\sqrt{2\pi}}\Big((a\alpha+am+b)e^{-\frac{(\alpha-m)^2}{2\sigma^2}}-(a\beta+am+b)e^{-\frac{(\beta-m)^2}{2\sigma^2}}\Big)\\ &+(a\sigma^2+am^2+bm+c)\Big[N\big(\frac{\beta-m}{\sigma}\big)-N\big(\frac{\alpha-m}{\sigma}\big)\Big].\end{aligned}$$

Proof. Note that

$$\mu_A(x)=\begin{cases}0, & x<\alpha,\ \ \beta\le x,\\ ax^2+bx+c=a(x-\alpha)(x-\beta), & \alpha<x<\beta.\end{cases}$$

Putting $\frac{x-m}{\sigma}=t$,

$$\begin{aligned}\widetilde{P}(A)&=\int_\alpha^\beta (ax^2+bx+c)\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}dx\\ &=\frac{1}{\sqrt{2\pi}}\int_{\frac{\alpha-m}{\sigma}}^{\frac{\beta-m}{\sigma}}\{a(\sigma t+m)^2+b(\sigma t+m)+c\}e^{-\frac{t^2}{2}}dt\\ &=\frac{a\sigma^2}{\sqrt{2\pi}}\int_{\frac{\alpha-m}{\sigma}}^{\frac{\beta-m}{\sigma}}t^2e^{-\frac{t^2}{2}}dt+\frac{2a\sigma m+b\sigma}{\sqrt{2\pi}}\int_{\frac{\alpha-m}{\sigma}}^{\frac{\beta-m}{\sigma}}te^{-\frac{t^2}{2}}dt\end{aligned}$$

$$
\begin{aligned}
&+ \frac{(am^2+bm+c)}{\sqrt{2\pi}} \int_{\frac{\alpha-m}{\sigma}}^{\frac{\beta-m}{\sigma}} e^{-\frac{t^2}{2}} dt \\
&= \frac{\sigma}{\sqrt{2\pi}}\Big((a\alpha+am+b)e^{-\frac{(\alpha-m)^2}{2\sigma^2}} - (a\beta+am+b)e^{-\frac{(\beta-m)^2}{2\sigma^2}}\Big) \\
&\quad + (a\sigma^2+am^2+bm+c)\Big[N\big(\frac{\beta-m}{\sigma}\big) - N\big(\frac{\alpha-m}{\sigma}\big)\Big]. \qquad \square
\end{aligned}
$$

Corollary 5.2. If a quadratic fuzzy number $A = [\alpha, k, \beta]$ is represented by

$$
\mu_A(x) = \begin{cases} 0, & x < \alpha, \ \ \beta \le x, \\ a(x-k)^2+1, & \alpha \le x < \beta, \end{cases}
$$

then

$$
\begin{aligned}
\tilde{P}(A) = &\frac{\sigma}{\sqrt{2\pi}}\Big(a(\alpha+m-2k)e^{-\frac{(\alpha-m)^2}{2\sigma^2}} - a(\beta+m-2k)e^{-\frac{(\beta-m)^2}{2\sigma^2}}\Big) \\
&+ \big(a(\sigma^2+(m-k)^2)+1\big) \cdot \Big[N\big(\frac{\beta-m}{\sigma}\big) - N\big(\frac{\alpha-m}{\sigma}\big)\Big].
\end{aligned}
$$

Example 5.3. Let $A = [1, 2, 3]$ be a quadratic fuzzy number. Then the normal fuzzy probability of A with respect to $X \sim N(3, 2^2)$ is

$$
\begin{aligned}
\widetilde{P}(A) &= \int_{-\infty}^{\infty} (-x^2+4x-3)\frac{1}{2\sqrt{2\pi}}\, e^{-\frac{(x-3)^2}{8}} dx \\
&= \frac{-4}{\sqrt{2\pi}}\int_{-1}^{0} t^2 e^{-\frac{t^2}{2}} dt - \frac{4}{\sqrt{2\pi}}\int_{-1}^{0} te^{-\frac{t^2}{2}} dt \\
&= \frac{4}{\sqrt{2\pi}}(0+e^{-\frac{1}{2}}) - 4(N(0)-N(-1)) + \frac{4}{\sqrt{2\pi}}(1-e^{-\frac{1}{2}}) \\
&= 0.2304.
\end{aligned}
$$

Since the results of addition and subtraction of two quadratic fuzzy numbers are quadratic fuzzy numbers, the normal fuzzy probability can be calculated by Theorem 5.1. But multiplication and division of two quadratic fuzzy number may not to be quadratic fuzzy numbers. So we have to calculate by definition.

Example 5.4. Let $X \sim N(5, 2^2)$ and consider the fuzzy numbers in Example 3.3.

1. Multiplication

$$
\begin{aligned}
\widetilde{P} &= \int_{2}^{24} \frac{-(6x+43)+11\sqrt{12x+1}}{18} \, \frac{1}{2\sqrt{2\pi}} \, e^{-\frac{(x-5)^2}{8}} dx \\
&= -\frac{1}{6\sqrt{2\pi}} \int_{2}^{24} x \, e^{-\frac{(x-5)^2}{8}} dx + \frac{43}{36\sqrt{2\pi}} \int_{2}^{24} e^{-\frac{(x-5)^2}{8}} dx \\
&\quad + \frac{1}{2\sqrt{2\pi}} \int_{2}^{24} 11\sqrt{12x+1} \, e^{-\frac{(x-5)^2}{8}} dx \\
&= 0.6402.
\end{aligned}
$$

2. Division

$$
\begin{aligned}
\widetilde{P} &= \int_{\frac{1}{8}}^{\frac{3}{2}} \frac{-(8x-1)(2x-3)}{(3x+1)^2} \, \frac{1}{2\sqrt{2\pi}} \, e^{-\frac{(x-5)^2}{8}} dx \\
&= \frac{-1}{2\sqrt{2\pi}} \int_{\frac{1}{8}}^{\frac{3}{2}} \frac{16x^2-26x+3}{9x^2+6x+1} \, e^{-\frac{(x-5)^2}{8}} dx \\
&= 0.0145.
\end{aligned}
$$

Theorem 5.5. Let $X \sim N(m, \sigma^2)$ and $A =< \theta_1, \theta_2, \theta_3 >$ be a trigonometric fuzzy number. Then the normal fuzzy probability of A is

$$
\widetilde{P}(A) = \int_{\theta_1}^{\theta_3} \sin(x-\theta_1) \, \frac{1}{\sqrt{2\pi}\sigma} \, e^{-\frac{(x-m)^2}{2\sigma^2}} dx
$$

$$= -\frac{1}{4}\, e^{-\frac{m^2}{2\sigma^2}} \Big[e^{\frac{m^2-2im\sigma^2-\sigma^4}{2\sigma^2}} \cos\theta_1 \,\mathrm{Erfi}\big(\frac{\sigma^2+im-ix}{\sqrt{2}\sigma}\big)$$

$$+ e^{\frac{m^2+2im\sigma^2-\sigma^4}{2\sigma^2}} \cos\theta_1 \,\mathrm{Erfi}\big(\frac{\sigma^2-im+ix}{\sqrt{2}\sigma}\big)$$

$$+ ie^{\frac{m^2-2im\sigma^2-\sigma^4}{2\sigma^2}} \sin\theta_1 \,\mathrm{Erfi}\big(\frac{\sigma^2+im-ix}{\sqrt{2}\sigma}\big)$$

$$- ie^{\frac{m^2+2im\sigma^2-\sigma^4}{2\sigma^2}} \sin\theta_1 \,\mathrm{Erfi}\big(\frac{\sigma^2-im+ix}{\sqrt{2}\sigma}\big)\Big]_{\theta_1}^{\theta_3}$$

where

$$\mathrm{Erfi}(x) = \frac{2}{i\sqrt{\pi}} \int_0^{ix} e^{-z^2} dz.$$

Example 5.6. Let $A =< 0, \frac{\pi}{2}, \pi >$ be a trigonometric fuzzy number. Then the normal fuzzy probability of A with respect to $X \sim N(3, 2^2)$ is

$$\widetilde{P}(A) = \int_0^{\pi} \sin x \,\frac{1}{2\sqrt{2\pi}}\, e^{-\frac{(x-3)^2}{8}} dx$$

$$= -\frac{1}{4} e^{-\frac{9}{8}} \Big[e^{-\frac{7}{8}-3i} \,\mathrm{Erfi}\Big(\frac{4+3i-ix}{2\sqrt{2}}\Big)$$

$$+ e^{-\frac{7}{8}+3i} \,\mathrm{Erfi}\Big(\frac{4-3i+ix}{2\sqrt{2}}\Big)\Big]_0^{\pi}$$

$$= 0.3003.$$

Since the results of addition and subtraction of two trigonometric fuzzy numbers are trigonometric fuzzy numbers, the normal fuzzy probability can be calculated by Theorem 5.5. But multiplication and division of two trigonometric fuzzy number may not to be trigonometric fuzzy numbers. So we have to calculate by definition.

Example 5.7. Let $X \sim N(3, 2^2)$ and consider the fuzzy numbers in Example 3.6.

1. Multiplication

$$
\begin{aligned}
\widetilde{P} &= \int_{\frac{\pi^2}{6}}^{2\pi^2} \sin\Big(-\frac{5}{12}\pi + \sqrt{\frac{\pi^2}{144} + x}\,\Big)\ \frac{1}{2\sqrt{2\pi}}\ e^{-\frac{(x-3)^2}{8}} dx \\
&= 0.0001.
\end{aligned}
$$

2. Division

$$
\begin{aligned}
\widetilde{P} &= \int_{\frac{3}{8}}^{\frac{9}{2}} \sin\Big(\frac{8\pi x - 3\pi}{2(1+3x)}\Big)\ \frac{1}{2\sqrt{2\pi}}\ e^{-\frac{(x-3)^2}{8}} dx \\
&= 0.0045.
\end{aligned}
$$

5.2 Exponential fuzzy probability for quadratic fuzzy number and trigonometric fuzzy number

Theorem 5.8. The exponential fuzzy probability $\widetilde{P}(A)$ for quadratic fuzzy number $A = [\alpha, k, \beta]$ is

$$
\begin{aligned}
\widetilde{P}(A) = &-e^{-\lambda\beta}\Big(\frac{2a + b\lambda + c\lambda^2}{\lambda^2} + \frac{2a + b\lambda}{\lambda}\beta + a\beta^2\Big) \\
&+ e^{-\lambda\alpha}\Big(\frac{2a + b\lambda + c\lambda^2}{\lambda^2} + \frac{2a + b\lambda}{\lambda}\ \alpha + a\alpha^2\Big),
\end{aligned}
$$

where the membership function of A is $-d(x-\alpha)(a-\beta) = ax^2 + bx + c$.

Proof. Since

$$
\mu_A(x) = \begin{cases} 0, & x < \alpha, \quad \beta \le x, \\ ax^2 + bx + c = -d(x-k)^2 + 1, & \alpha \le x < \beta, \end{cases}
$$

we have

$$
\begin{aligned}
\widetilde{P}(A) &= \int_{\alpha}^{\beta} (ax^2 + bx + c)\lambda e^{-\lambda x} dx \\
&= \Big[-e^{-\lambda x} \big(\frac{2a + b\lambda + c\lambda^2}{\lambda^2} + \frac{2a + b\lambda}{\lambda} x + ax^2 \big) \Big]_{\alpha}^{\beta} \\
&= -e^{-\lambda\beta} \Big(\frac{2a + b\lambda + c\lambda^2}{\lambda^2} + \frac{2a + b\lambda}{\lambda} \beta + a\beta^2 \Big) \\
&\quad + e^{-\lambda\alpha} \Big(\frac{2a + b\lambda + c\lambda^2}{\lambda^2} + \frac{2a + b\lambda}{\lambda} \alpha + a\alpha^2 \Big).
\end{aligned}
$$

□

Example 5.9. Let $A = [1, 2, 3]$ be a quadratic number. Then the exponential fuzzy probability of A with respect to $\lambda = 2$ is

$$
\begin{aligned}
\widetilde{P}(A) &= \int_{-1}^{3} (-x^2 + 4x - 3) 2 \cdot e^{-2x} dx \\
&= \frac{3}{2} \cdot e^{-6} + \frac{1}{2} \cdot e^{-2} \\
&= 0.0714.
\end{aligned}
$$

Similar to Example 5.4, we can calculate the exponential fuzzy probability for the results of multiplication and division of two quadratic fuzzy numbers by definition.

Example 5.10. The exponential fuzzy probabilities of the fuzzy numbers in Example 3.3 with respect to $\lambda = 2$ are

1. Multiplication

$$
\widetilde{P} = \int_{2}^{24} \frac{-(6x + 43) + 11\sqrt{12x + 1}}{18} \, 2 \, e^{-2x} dx
$$

$$= -\frac{1}{9}\int_2^{24}(6x-43)\ e^{-2x}dx + \frac{11}{9}\int_2^{24}\sqrt{12x+1}\ e^{-2x}dx$$
$$= 0.0906.$$

2. Division

$$\widetilde{P} = \int_{\frac{1}{8}}^{\frac{3}{2}} \frac{-(8x-1)(2x-3)}{(3x+1)^2}\ 2\ e^{-2x}dx$$
$$= -2\int_{\frac{1}{8}}^{\frac{3}{2}} \frac{16x^2-26x+3}{9x^2+6x+1}\ e^{-2x}dx$$
$$= 0.5062.$$

Theorem 5.11. The exponential fuzzy probability $\widetilde{P}(A)$ of a trigonometric fuzzy number $A =< \theta_1, \theta_2, \theta_3 >$ is

$$\widetilde{P}(A) = -\frac{\lambda}{\lambda^2+1}(e^{-\lambda\theta_3}\cos(\theta_3-\theta_1) + \lambda e^{-\lambda\theta_3}\sin(\theta_3-\theta_1) - e^{-\lambda\theta_1}).$$

Proof.

$$\widetilde{P}(A) = \int_{\theta_1}^{\theta_3}\sin(x-\theta_1)\ \lambda\ e^{-\lambda x}dx$$
$$= -\lambda\Big[\frac{e^{-\lambda x}\cos(x-\theta_1)}{\lambda^2+1} + \frac{\lambda e^{-\lambda x}\sin(x-\theta_1)}{\lambda^2+1}\Big]_{\theta_1}^{\theta_3}$$
$$= -\frac{\lambda}{\lambda^2+1}(e^{-\lambda\theta_3}\cos(\theta_3-\theta_1) + \lambda e^{-\lambda\theta_3}\sin(\theta_3-\theta_1) - e^{-\lambda\theta_1}). \quad \square$$

Example 5.12. Let $A =< 0, \frac{\pi}{2}, \pi >$ be a trigonometric fuzzy number. Then the exponential fuzzy probability of A with respect to $\lambda = 2$ is

$$
\begin{aligned}
\widetilde{P}(A) &= \int_0^{\pi} \sin x \; 2\, e^{-2x} dx \\
&= -\frac{2}{5}(e^{-2\pi} \cos \pi + 2\, e^{-2\pi} \sin \pi - e^0) \\
&= 0.4007.
\end{aligned}
$$

Example 5.13. The exponential fuzzy probabilities of the fuzzy numbers in Example 3.6 with respect to $\lambda = 2$ are

1. Multiplication

$$
\begin{aligned}
\widetilde{P} &= \int_{\frac{\pi^2}{6}}^{2\pi^2} \sin\Big(-\frac{5}{12}\pi + \sqrt{\frac{\pi^2}{144} + x}\Big) \;\; 2\, e^{-2x} dx \\
&= 9.5105 \times 10^{-10}.
\end{aligned}
$$

2. Division

$$
\begin{aligned}
\widetilde{P} &= \int_{\frac{3}{8}}^{\frac{9}{2}} \sin\Big(\frac{8\pi x - 3\pi}{2(1+3x)}\Big) \; 2\, e^{-2x} dx \\
&= 0.3184.
\end{aligned}
$$

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<국문초록>

여러가지 퍼지수에 대한
정규퍼지확률과 지수퍼지확률

이차곡선과 삼각곡선을 이용하여 이차퍼지수와 삼각퍼지수를 정의하였고 두 이차퍼지수와 두 삼각퍼지수에 대해 각각의 사칙연산을 알아보았다. 두 이차퍼지수와 두 삼각퍼지수의 덧셈과 뺄셈의 연산의 결과는 이차퍼지수와 삼각퍼지수가 되었고 곱셈, 나눗셈의 연산의 결과는 각각 이차퍼지수와 삼각퍼지수는 되지 않았으나 두 삼각퍼지수의 곱셈, 나눗셈의 결과는 삼각함수로 표현이 되었다. 그리고 실제 예 들을 들어 두 퍼지수의 연산의 결과를 구체적으로 계산해 보았다.

정규분포와 지수분포를 이용하여 정규 퍼지확률과 지수 퍼지확률을 정의하였고 이차퍼지수와 삼각퍼지수에 대한 정규 퍼지확률과 지수 퍼지확률을 계산하였다. 또한 두 퍼지수의 연산의 결과에 대해서도 정규 퍼지확률과 지수 퍼지확률을 계산하였고 실제 예 들을 들어서도 구체적으로 계산해 보았다.

감사의 글

박사과정 동안의 시간을 정리해야 할 때가 왔습니다. 이제 그 시간동안 이루어낸 결실을 이 논문에 담았습니다. 하나의 논문으로 담아내면서 너무나 많은 것을 배우고 체험했던 소중한 시간이었습니다. 이 작은 결과물을 남기고 아쉬운 마음과 함께 새로운 삶의 무대로 나아가고자 합니다.

지난 박사과정 동안 너무나도 많은 사랑으로 바쁘신 가운데도 많은 시간을 할애하여 친절하고 세심하게 지도해주신 윤용식 교수님께 깊은 감사를 드립니다. 또한 논문심사에서 세심히 지적해주시고 지도해주신 양영오 교수님, 방은숙 교수님, 송석준 교수님, 정승달 교수님과 꼼꼼한 교정과 조언을 해주신 박진원 교수님, 그 외 정보수학과 교수님, 수학교육과 교수님들께도 감사드립니다. 그리고 함께 공부하며 힘이 되어 주셨던 선배 원생들을 비롯한 후배 원생들께도 고마운 마음을 전하고 싶습니다.

끝으로 이 한 편의 논문으로 항상 아들이 건강하고 잘되기를 바라면서 한 평생을 살아오신 부모님과 지금은 고인이 되어 멀리서나마 지켜보고 계실 장인·장모, 어려움을 참으면서 끊임없는 인내와 사랑으로 내조해 준 아내 繽과 항상 열심히 공부하고 건강하며 밝게 자라는 용택, 민아, 용준에게 아빠로서 최선을 다하는 모습을 조금이나마 보여주었다는 사실이 자랑스럽고, 나를 아는 주위의 모든 분들과 이 조그마한 기쁨을 함께 나누고 싶습니다.

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