# On Function Space of Topological Semigroups

이를 敎育學碩士學位 論文으로 提出함



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# 감 사 의 글

이 논문이 완성되기 까지 바쁘신 가운데도 자상하게 지도해 주신 현진오 교수님께 감사를 드리며, 아울러 지도와 편달을 아끼지 않으신 송석준 교수님, 김창식 선생님과 수학과 여러교수님들께 감사드립니다.

그리고 그동안 저에게 사랑과 격려를 주신 가족, 친지 및 주위의 여러분들께 또한 감사를 드립니다.



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김 정 림

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## 0. Introduction and preliminaries

The study of semigroup and topological semigroup is very important in algebra, because the structure of semigroup is basic of all the other algebraic structure. So many mathematicians have studied on semigroup theory.

In this paper, we study the function space of topological semigroup. The motivation of this study is the theorems with respect to adjoint associativity on algebra as follows ;

**Lemma 1**. Let R and S be rings and $A_R$, ${}_RB_S$, $C_S$ modules. Then there is an isomorphism of abelian groups $\alpha : \mathrm{Hom}_S(A_R \otimes B, C) \to \mathrm{Hom}_R(A, \mathrm{Hom}(B, C))$, defined for each $f : A \otimes_R B \to C$ by $[(\alpha f)(a)](b) = f(a \otimes b)$.

([3] Chapter Ⅳ − 5)



**Lemma 2**. Let X, Y, Z be three topolgical space such that X is Hausdorff and Y is locally compact.

Then the restriction to $C(X \times Y, Z)$ of canonical bijection $F(X \times Y, Z) \to F(X, F(Y, Z)$ is a homeomorphism of $C_c(X \times Y, Z)$ onto $C_c(X, C(Y, Z))$.

([1] Chapter X − 3.4)

which are due to [3] and [1], respectively.

We apply these two lemmas to the topological semigroup on a family of homomorphisms between semigroups.

This paper is divided into three sections. The first section includes basic concepts which will pave the way for the further development.

In particular, we show that for any semigroup X and Y, H(X, Y) is a mono-source if and only if X is a subsemigroup of H(H(X,Y) Y), ie, X is considered as a subsemigroup of a power of Y. Using the concept that a map f between semigroups is a homomorphism if and only if we have a commute diagram, we show that $H^m_\tau$ (X, Y) is a topological semigroup which compatible with the initial topology H(X, Y) with respect to $(P_x : H(X,Y) \rightarrow Y)_{x \in X}$ and pointwise multiplication for any semigroup X and Y a topological commutative semigroup.

In the final section, using the tensor product on semigroups, we try to have a corresponding result on topological semigroups to adjoint associativity as follows ;

**Theorem** Let X and Y be semigroups and Z a commutative topological semigroup, then $H^m_\tau$( X$\otimes$Y, Z) is a topological isomorphic with $H^m_\tau$(X, $H^m_\tau$(Y, Z)).

## 1. Definitions and basic properties

In this section, we collect a list of some definiticons, notations and basic results which will be used throughout this thesis.

By a *semigroup* $(X, \cdot)$ we shall mean a non-empty set X whic has an *associative* binary operation. In the following, we shall write $\cdot(x, y)$ simply as $xy$. The associative condition on S states that $x(yz) = (xy)z$ for each $x, y$ $Z \in S$. If S has a Hausdorff topology such that its binary operation is continous on $S \times S$ to S where $S \times S$ is the product topological space, than S is called *a topological semigroup*. A semigroup S is *commutative* if $xy = yx$ for any $x, y \in S$. An element $a \in S$ is said to be *idempotent* if $a^2 = aa = a$ and we denote $E(S) = \{a \in S \mid a^2 = a\}$, the set of all idempotent elements of a semigroup S. A Commutative semigroup S is a *semilattice* if $E(S) = S$. A semigroup S which has an identity 1, i·e, $1x = x = x1$ for any $x \in X$, is called a *monoid*. A subset A of a semigroup is a *subsemigroup* if $AA = [ab \mid a, b \in A] \subset A$. A subsemigroup of a semigroup is a semigroup.



**Notation.** For any sets X and Y, let F(X, Y) be the family of all functions from X into Y, and denote F(X, X) by F(X). For any semigroups X and Y, let H(X, Y) be the set of all homomorphisms from X and Y, and denote End(X), the set of endomorphisms on X into itself.

In the above notation, if Y has an idempotent element, then H(X, Y) is a non-empty. In particular, if $y$ is an idempotent element on Y, then the constant map $\underline{y}$ on X, $\underline{y}(x) = y$ for each $x \in X$, is a homomorphism.

We will assume that $H(X, Y)$ is a non-empty family for any semigroups $X$ and $Y$ throughout this thesis.

A family of maps $(f_i)_{i \in I}$ is called a *source* if dom $f_i$ = dom $f_j$ for any $i, j \in I$, Where dom $f_i$ and dam $f_j$ are the domain of map $f_i$ and $f_j$, resp.[2]

**Definition 1.1** A source $(f_i : X \to X_i)_{i \in I}$ is called a mono-source if $f_i(x) = f_i(y)$, for any $i \in I$, implies $x = y$. [2]

**Remark 1.2** (1) If a source $(f : X \to Y)$, the singleton source, is a mono-source if and only if $f$ is one-to-one.

(2) If $(f_i : X \to X_i)_{i \in I}$ is a mono-source and $I \subset J$, $(f_j : X \to X_j)_{j \in J}$ is a mono-source.



The following concept is a generatization of the topological properties, i·e, productive and hereditary.

**Definition 1.3** Let $(f_i : X \to X_i)_{i \in I}$ be a source, where $(X, \mathcal{T})$ and $(X_i, \mathcal{T}_i)$, for all $i \in I$, are topological space. Then $(f_i)_{i \in I}$ is called a *initial source*. If it satisfies the followings;

i) For each $i \in I$, $f_i : X \to X_i$ is continous,

ii) Any map $h : Y \to X$ is continous, Where $(Y, \mathcal{T}_Y)$ is a topological space if and only if for each $i \in I$, $f_i \cdot h$ is continous. [2]

**Remark 1.4** (1) It is well known that if a source $(f_i : X \to X_i)_{i \in I}$ is an initial mono-source, then $X$ is homeomorphic with a subspace of $\pi_{i \in I} X_i$ and vice versa.

(2) If A is a subspace of a topological space X, then the inclusion map $(i : A \to X)$ is an initial source.

**Proposition 1.5** Let $(f_i : X \to X_i)_{i \in I}$ be an initial source and for each $i \in I$, $(g_{i\lambda} : X_i \to Y_{i\lambda})_{\lambda \in \Lambda}$ is an initial source. Then $(g_{i\lambda} \cdot f_i : X \to Y_{i\lambda})_{i \in I,\ \lambda \in \Lambda}$ is an initial source.

**Proposition 1.6** Let $(f_i : X \to X_i)_{i \in I}$ and $(g_{i\lambda} : X_i \to Y_{i\lambda})_{\lambda \in \Lambda}$ for each $i \in I$ be sources of continous maps.

Then $(f_i)_{i \in I}$ is an initial source if $(g_{i\lambda} \cdot f_i)_{i \in I,\ \lambda \in \Lambda}$ is an initial source.

**Theorem 1.7** Let $(f_i : X \to X_i)_{i \in I}$ be a source, where X is a set and each $X_i$ is a topological space. Then there exists unique initial topological structure $\mathcal{T}$ on X which $(f_i : (X, \mathcal{T}) \to X_i)_{i \in I}$ is an initial source.



<u>Proof</u>. Clearly $\{f_i^{-1}(G_i) \mid G_i$ is open in $X_i$, $i \in I\}$ is a subbase for the initial topolgy on X with respect to $(f_i : X \to X_i)_{i \in I}$.

**Theorem 1.8** Let $(f : X \to Y)$ be an initial source. Then f is a homeomorphism if and only if f is bijective.

<u>Proof</u>. If f is bijective, then the inverse map $f^{-1}$ of f exists and satisfy $f \cdot f^{-1} = 1_Y$. Since (f) is an initial source and $1_Y$ is continous, $f^{-1}$ is continous.

**Definition 1.9** Let X and Y be topological semigroups.

(1) A map f on X into Y is called <u>*a continous homomorphism*</u>, if it is continous and homomorphism.[4]

(2) A map f on X onto Y is called a <u>*topological isomo-*</u>

*rphism* if f is a homeomorphism and isomorphism.[4]

(3) X is said to be *topolgical isomorphic* with Y if there is a topological isomorphism on X onto Y.[4]

## 2. Topological semigroups of homomorphisms.

In this section, we will contruct a topological semigroup on a family of homomorphisms for a semigroup, and topological commutative semigroup using the initial topology and pointwise multiplication.

**Proposition 2.1** For any set X, F(X) is a monoid under composition of maps. Moreover, for any semigroup X, End(X) is a subsemigroup of F(X).



**Proposition 2.2** Define a binary operation m: $F(X, Y)^2 \rightarrow F(X, Y)$ by $m(fg)(x) = f(x)g(x)$, in general $fg(x) = f(x)g(x)$, for all $f, g \in F(X, Y)$ and $x \in X$. Then (F(X,Y), m) is a [commutative] semigroup if Y is a [commutative] semigroup and if Y is a semilattice, then (F(X, Y), m) is a semilattice.

**Proposition 2.3** If Y is a commutative semigroup, then (H(X, Y), m) is a commutative semigroup of (F(X,Y), m).

Proof. Take any f,g in H(X, Y).

Then $(fg)(xy) = f(xy)\,g(xy) = f(x)f(y)g(x)g(y) = f(x)g(x)f(y)g(y)$
$= f(xy)g(xy)$, i·e, $fg \in H(X, Y)$;

(H(X, Y), m) is a subsemigroup of (F(X, Y), m).

Hence (H(X, Y), m) is a commutative subsemigroup of (F(X, Y), m).

In the above proposition 2.3, if Y is not commutative, then (H(X, Y), m) need not be a subsemigroup of (F(X, Y), m).

In the above proposition 2.2, proposition 2.3, we denote (F(X, Y), m) by $F^m(X, Y)$ and (H(X, Y), m) by $H^m(X, Y)$.

**Example 2.4** Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

Then we know that $Q_8$ is a non abelian group.

Moreover, End$(Q_8)$ is not a semigroup.

Clearly $1_{Q_8} \in \mathrm{End}(Q_8)$, but

$1_{Q_8}1_{Q_8}(ij) = 1_{Q_8}1_{Q_8}(k) = 1_{Q_8}(k)1_{Q_8}(k) = k^2 = -1$.

$[1_{Q_8}1_{Q_8}(i)][1_{Q_8}1_{Q_8}(j)] = [1_{Q_8}(i)1_{Q_8}(i)][1_{Q_8}(j)1_{Q_8}(j)] = i^2 \cdot j^2$

$= (-1)\cdot(-1) = 1$, and so $1_{Q_8}1_{Q_8} \notin \mathrm{End}(Q_8)$.



**Lemma 2.5** For any $x \in X$, define a map

$P_x: F(X, Y) \to Y$ by $P_x(f) = f(x)$ for all $f \in F(X, Y)$, and we will say th-at this map is x-th projection for all $x \in X$. Then

(1) each x-th projections is a homomorphisms,

(2) the collection of all projections is a mono source.

Proof. (1) Let $x \in X$ be taken.

Then for any $f, g \in F(X, Y)$, $P_x(fg) = f(x)g(x) = P_x(f)\, P_x(g)$, and hence $P_x$ is a homomorphism.

(2) Take any $f, g \in F(X, Y)$ With $P_x(f) = P_x(g)$, i·e, $f(x) = g(x)$ for any $x \in X$. so $f = g$.

Hence $(P_x)_{x\in X}$ is a mono-source.

**Proposition 2.6** Let X and Y be semigroups. Then H(X, Y) is a mono-source if and only if the map $\pi$ : X→H(H(X, Y), Y) defined by $\pi(x) = \overline{P_x}$, where $\overline{P_x}$ is the restriction to H(X, Y) for $P_x$, and we simply denote $P_x$ for $\overline{P_x}$, is one-to-one, and $\pi$ is a homomorphism.

Proof. Let H(X, Y) be a mono-source and suppose $\pi(x) = \pi(y)$.

Then $P_x(f) = P_y(f)$ for any $f \in H(X, Y)$, and so $f(x) = f(y)$ for any $f \in H(X, Y)$.

Hence $x = y$. Thus $\pi$ is one-to-one.

Take any $x, y \in X$ and $f \in H(X, Y)$.

Then $P_{xy}(f) = f(xy) = f(x)f(y) = P_x(f)P_y(f) = (P_xP_y)(f)$, i·e, $P_{xy} = P_xP_y$.

In all $\pi(xy) = P_{xy} = P_xP_y = \pi(x)\pi(y)$, i·e. $\pi$ is a homomorphism.

Conversely, Let $\pi$ be one-to-one. Take any $x, y \in X$ with $f(x) = f(y)$ for any $f \in H(X, Y)$.

Then $P_x(f) = f(x) = f(y) = P_y(f)$ for any $f \in H(X, Y)$, i·e, $P_x = P_y$.

Since $\pi$ is one-to-one and $\pi(x) = P_x = P_y = \pi(y)$; $x = y$

Thus H(X, Y) is a mono-source.

**Theorem 2.7** Let X and Y be semigroups. Then X is isomorphic with a subsemigroup of H(H(X, Y), Y) if H(X, Y) is a mono-source.

Proof. Clearly $P_x \in H(H(X, Y), Y)$ for any $x \in X$, and H(X, Y) is a mono-source, and $\pi$: X→H(H(X, Y), Y) is one-to-one and a homomorphism.

Hence X is a isomorphic with $\pi(x)$.

**Corollary 2.8** Let X and Y be semigroups, then X is isomorphic with a subsemigroup of $H(H(H(X,Y),Y)Y)$.

<u>Proof</u>. Clearly the family of x−th projections is a mono−source. And by theorem 2.7, X is isomorphic with a subsemigroup of $H(H(H(X,Y),Y),Y)$.

From the above theorem 2.7, if $H(X, Y)$ is a mono−source, we can consider that X is a subsemigroup of $F(H(X, Y), Y)$, i·e, a subsemigroup of $Y^{H(X,Y)}$

**Lemma 2.9** A function f on X to Y is a homomorphism if and only if the following diagram commutes, i·e $f\cdot m_X = m_Y\cdot f\times f$,

$$\begin{array}{ccc} X\times X & \xrightarrow{\ m_X\ } & X \\ \big\downarrow{\scriptstyle f\times f} & & \big\downarrow{\scriptstyle f} \\ Y\times Y & \xrightarrow{\ m_Y\ } & Y \end{array}$$



Where X, Y are semigroups and $m_x$, $m_Y$ are the binary operation on X and Y, resp.

<u>Proof</u>. Let f be a homomorphism.

Then $(f\cdot m_x)(a,b) = f(m_x(a,b)) = f(ab) = f(a)f(b) = m_Y(f(a),f(b))$

$= m_Y(f\times f(a,b)) = (m_Y\cdot f\times f)(a,b)$ for any $a,b\in X$. Thus the diagram commutes.

Conversely, if the diagram commutes, i·e, $f\cdot m_x = m_Y\cdot f\times f$, then for any $a,b\in X$, $f(ab) = f(m_x(a,b)) = f\cdot m_x(a,b) = (m_Y\cdot f\times f)(a,b) = m_Y(f\times f(a,b))$
$= m_Y(f(a), f(b)) = f(a)f(b)$.

Hence f is a on X into Y.

Using proposition 2.3, Lemma 2.5, Lemma 2.9, one has the following:

**Theorem 2.10** Let Y be a commutative topological semigroup and let X be a semigroup and $\tau$ be the initial topology on $F(X, Y)$ with respect to $(P_x : F(X, Y) \to Y)$ and denote $F_i(X, Y) = (F(X, Y), Z)$. Then $H^m_i(X, Y)$ is a commutative topological semigroup.

<u>Proof</u>. Since Y is a commutative semigroup, by proposition 2.3, $H^m_i(X,Y)$ is a commutative semigroup

In order to show that $H^m_i(X, Y)$ is a topological semigroup, it suffices to show that the binary operation on $H^m_i(X, Y)$ is continous. By Lemma 2.5, each $P_x$ is a homomorphism. So, by Lemma 2.9, we have a diagram.

$$\begin{array}{ccc} H^m_i(X, Y) \times H^m_i(X, Y) & \xrightarrow{\quad m \quad} & H^m_i(X, Y) \\ \downarrow P_x \times P_x & & \downarrow P_x \qquad (x \in X) \\ Y \times Y & \xrightarrow{\quad m_Y \quad} & Y \end{array}$$



Thus $P_x \cdot m = m_Y \cdot P_x \times P_x$. Since Y is a topological semigroup, $m_Y$ is continous and $(P_x)_{x \in X}$ is continous, m is continous. Therefore $H^m_i(X, Y)$ is a commutative topological semigroup.

**Corollary 2.11** Let X and Y be semigroups and let Z be a commutative topological semigroup [semilattice], then $H^m_i(Y, Z)$ and $H^m_i(Y, Z))$ are topological semigroup [semilattice].

<u>Proof.</u> Clearly $H^m_i(Y, Z)$ is a commutative topological semigroup[semilattice], since Z is a commutative topological semigroup[semilattice]. Hence $H^m_i(X, H^m_i(Y, Z))$ is also commutative topological semigroup[semilattice].

**Theorem 2.12** Let X be a semingroup and Y a compact semigroup, then $H^m_?(X, Y)$ is a compact semigroup.

Proof. It suffices to show that $H^m_?(X, Y)$ is a closed subspace of $F^m_?(X, Y)$.

For any $x$, $y \in X$, $K(x, y) = (P_x \times [m_Y \cdot (P_x \sqcap P_y)])^{-1}(\triangle_Y)$

$$= Equ(P_{xy}, m_Y \cdot (P_x \sqcap P_y))$$

$$= \{f \in F(X, Y) \mid P_{xy}(f) = [m_Y \cdot (P_x \sqcap P_y)](f)\}$$

$$= \{f \in F(X, Y) \mid f(xy) = f(x)f(y)\}.$$

Where $m_Y$ is a map on $Y \times Y$ to Y and $\triangle_Y = \{(t, t) \mid t \in Y\}$ and $(P_x \sqcap P_y)(f) = (f(x), f(y))$, $x \in X$, $y \in Y$.

Since Y is Hausdorff, $\triangle_Y$ is closed and hence $\{k(x, y) \mid x \in X, y \in Y\}$ is closed.



Now let's show that $H(X, Y) = \cap \{k(x, y) \mid x, y \in X\}$.

Take any $f \in H(X, Y)$, then $f(xy) = f(x)f(y)$ for any $x$, $y \in X$, i·e, $f \in k(x, y)$ for all $x, y \in X$. Hance $f \in \cap \{k(x, y) \mid x, y \in X\}$. Moreover; for any $f \in \cap \{k(x, y) \mid x, y \in X\}$, $f(xy) = f(x)f(y)$ for any $x$, $y \in X$, i·e, $f \in H(X, Y)$. This completes the proof.

Thus let Iu be the unit interval [0, 1] with the usual topology and usual multiplication, then Iu is a compact semigroup. Thus by above theorem 2.11 we have

**Corollary 2.13** For any semigroup X, $H^m_?(X, I_u)$ is compact semi - group.

## 3. Adjoint associativity (functor)

Let X and Y be semigroups and denote $X \otimes Y = \{ \overset{n}{\underset{i=1}{\pi}} (x_i \otimes y_i) | x \in X,$ $y_i \in Y,\ i = 1, \cdots, n,\ n \in N\}$,

where $\overset{n}{\underset{i=1}{\pi}} (x_i \otimes y_i) = (x_1 \otimes y_1)(x_2 \otimes y_2) \cdots (X_n \otimes y_n)$ of which elements satisfy the following relation.

for all $x, x_i \in X$, $y, y_i \in Y$, $i = 1, 2$.

i) $(x_1\ x_2) \otimes y = (x_1 \otimes y)(x_2 \otimes y)$

ii) $x \otimes (y_1\ y_2) = (x \otimes y_1)(x \otimes y_2)$.

Now, let's define a binary operation

$$m: X \otimes Y \times X \otimes Y \rightarrow X \otimes Y$$

by $m( \overset{m}{\underset{i=1}{\pi}} (x_i \otimes y_i), \overset{m+n}{\underset{i=m+1}{\pi}} (x_i \otimes y_i)) = \overset{m+n}{\underset{i=1}{\pi}} (x_i \otimes y_i)$ for any $x_i \in X$, $y_i \in Y$, $i = 1, \cdots, m+n$.

Then $m(m ( \overset{k}{\underset{i=1}{\pi}} (x_i \otimes y_i), \overset{k+m}{\underset{i=k+1}{\pi}} (x_i \otimes y_i), \overset{k+m+n}{\underset{i=k+m+1}{\pi}} (x_i \otimes y_i)))$

$$= m ( \overset{k+m}{\underset{i=1}{\pi}} (x_i \otimes y_i), \overset{k+m+n}{\underset{i=k+m+1}{\pi}} (x_i \otimes y_i))$$

$$= \overset{k+m+n}{\underset{i=1}{\pi}} (x_i \otimes y_i), \text{ and}$$

$$m ( \overset{k}{\underset{i=1}{\pi}} (x_i \otimes y_i), m( \overset{k+m}{\underset{i=k+1}{\pi}} (x_i \otimes y_i), \overset{k+m+n}{\underset{i=k+m+1}{\pi}} (x_i \otimes y_i)))$$

$$= m ( \overset{k}{\underset{i=1}{\pi}} (x_i \otimes y_i), \overset{k+m+n}{\underset{i=k+1}{\pi}} (x_i \otimes y_i))$$

$$= \overset{k+m+n}{\underset{i=1}{\pi}} (x_i \otimes y_i)$$

Hence $( X \otimes Y, m)$ is a semigroup

**Definition 3.1** For any semigroups X and Y, $(X \otimes Y, m)$ is called the *tensor product* of X and Y. [3]

**Lemma 3.2** Let X, Y and Z be semigroups, if for any two homomorphisms f and g on $X\otimes Y$ into Z, $f(x\otimes y) = g(x\otimes y)$ for any $(x, y) \in X\times Y$, then $f = g$.

Proof. Take any $(x_1\otimes y_1), \cdots, (x_n\otimes y_n) \in X\times Y$. Then

$$f((x_1\otimes y_1)\cdots(x_n\otimes y_n))$$
$$= f(x_1\otimes y_1)\cdots f(x_n\otimes y_n)$$
$$= g(x_1\otimes y_1)\cdots g(x_n\otimes y_n);\ f = g.$$

**Theorem 3.3 (Adjoint associativity)** Let X and Y be semigroups and Z a commutative topological semigroup, then $H_i^m(X\otimes Y, Z)$ is a topolgical isomorphic with $H_i^m(X, H_i^m(Y, Z))$.

**Step 1.** The evalution map $ev: H_i^m(Y, Z)\otimes Y \to Z$



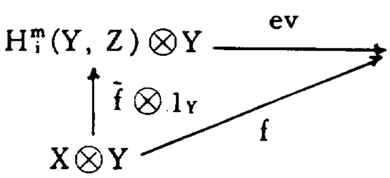
defined by $ev\left(\pi_{i=1}^{n}(f_i\otimes x_i)\right) = \pi_{i=1}^{n}(f_i(x_i))$ is a homomorphism.

Proof. Take any $\pi_{i=1}^{m}(f_i\otimes x_i),\ \pi_{i=m+1}^{m+n} f_i\otimes x_i \in H_i^m(Y, Z)\otimes Y$.

Then $ev\left(m\left(\pi_{i=1}^{m}(f_i\otimes x_i),\ \pi_{i=m+1}^{m+n}(f_i\otimes x_i)\right)\right)$

$$= ev\left(\pi_{i=1}^{m+n}(f_i\otimes x_i)\right)$$
$$= \pi_{i=1}^{m+n} f_i(x_i)$$
$$= \pi_{i=1}^{m} f_i(x_i)\ \pi_{i=m+1}^{m+n} f_i(x_i)$$
$$= ev\left(\pi_{i=1}^{m}(f_i\otimes x_i)\right)\ ev\left(\pi_{i=m+1}^{m+n}(f_i\otimes x_i)\right).$$

**Step 2.** For any homomorphism $f: X\otimes Y \to Z$, there is a homomorphism $\tilde{f}: X \to H_i^m(Y, Z)$ with a diagram;




, i.e, $f = ev\cdot(\tilde{f}\otimes 1_Y)$

Proof. Define $\tilde{f} : X \to H^m_1(Y, Z)$ by $\tilde{f}(x)(y) = f(x \otimes y)$ for any $x \in X$, $y \in Y$. Let $x \in X$ be given.

Then for any $y_1, y_2 \in Y$,

$$\begin{aligned} \tilde{f}(x)(y_1 y_2) &= f(x \otimes (y_1 y_2)) \\ &= f((x \otimes y_1)(x \otimes y_2)) \\ &= f(x \otimes y_1)\ f(x \otimes y_2) \\ &= \tilde{f}(x)(y_1)\ \tilde{f}(x)(y_2)\ ;\ \tilde{f} \in H(Y, Z) \end{aligned}$$

Let $y \in Y$ and $x_1, x_2 \in X$, Then

$$\begin{aligned} \tilde{f}(x_1\ x_2)(y) &= f((x_1\ x_2) \otimes y) \\ &= f((x_1 \otimes y)(x_2 \otimes y)) \\ &= \tilde{f}(x_1)(y)\ \tilde{f}(x_2)(y) \\ &= [\tilde{f}(x_1)\ \tilde{f}(x_2)](y)\ ;\ \tilde{f}(x_1\ x_2) = \tilde{f}(x_1)\ \tilde{f}(x_2) \end{aligned}$$

Hence $\tilde{f}$ is a homomorphism.



**Step 3.** Let X, Y, A and B be semigroups and $f \in H(X, A)$ and $g \in H(Y, B)$. Then a map $f \otimes g$ on $X \otimes Y$ into $A \otimes B$ defined by

$$(f \otimes g)\left(\prod_{i=1}^{n} (x_i \otimes y_i)\right) = \prod_{i=1}^{n} (f(x_i) \otimes f(y_i)) \quad \text{for any } x_i \in X \ \ y_i \in Y,$$

$i = 1, \cdots, n$ is a homomorphism.

Proof. Take any $\prod_{i=1}^{m} (x_i \otimes y_i)$, $\prod_{i=m+1}^{m+n} (x_i \otimes y_i) \in X \otimes Y$. Then

$$\begin{aligned} f \otimes g\left(m\left(\prod_{i=1}^{m} (x_i \otimes y_i), \prod_{i=m+1}^{m+n} (x_i \otimes y_i)\right)\right) &= f \otimes g\left(\prod_{i=1}^{m+n} (x_i \otimes y_i)\right) \\ &= f \otimes g\left(\prod_{i=1}^{m+n} (x_i \otimes y_i)\right) \\ &= \prod_{i=1}^{m+n} (f(x_i) \times g(y_i)) \\ &= \left(\prod_{i=1}^{m} (f(x_i) \otimes g(y_i))\right)\left(\prod_{i=1}^{m+n} (f(x_i) \otimes g(y_i))\right). \end{aligned}$$

Hence $f \otimes g$ is a homomorphism.

**Step 4.** Define $T: H_1^m(X \otimes Y, Z) \rightarrow H_1^m(X, H_1^m(Y, Z))$ by $T(f) = \tilde{f}$ and $G: H_1^m(X, H_1^m(Y, Z)) \rightarrow H_1^m(X \otimes Y, Z)$ by $G(g) = ev \cdot (g \otimes 1_Y)$ for any $f \in H_1^m(X \otimes Y, Z)$ and $g \in H_1^m(X, H_1^m(Y, Z))$. Then $G \cdot T = 1_{H_1^m(X \otimes Y, Z)}$ and $T \cdot G = 1_{H_1^m(X, H_1^m(Y, Z))}$.

<u>Proof</u>. Let $f \in H_1^m(X \otimes Y, Z)$. Then by step2 and step3, $ev \cdot (\tilde{f} \otimes 1_Y)$ is a homomorphism, and

$(G \cdot T)(f) = G(\tilde{f}) = ev \cdot (\tilde{f} \otimes 1_Y) = f$. Hence $G \cdot T = 1_{H_1^m(X \otimes Y, Z)}$.

Let $g \in H_1^m(X, H_1^m(Y, Z))$.

$$\begin{aligned} \text{Then} \quad (T \cdot G)(g) &= T(ev \cdot (g \otimes 1_Y)) \\ &= \overline{ev \cdot (g \otimes 1_Y)} \end{aligned}$$

$$\begin{aligned} \text{Hence for any } x \in X,\ y \in Y, \quad \overline{ev \cdot (g \otimes 1_Y)}(x)(y) &= [ev \cdot (g \otimes 1_Y)](x \otimes y) \\ &= ev(g(x) \otimes y) \\ &= g(x)(y), \text{ i.e., } \overline{ev \cdot (g \otimes 1_Y)} = g. \end{aligned}$$

Hence $T \cdot G = 1_{H_1^m(X, H_1^m(Y, Z))}$.

**Step 5.** In step4, $T$ is a homomorphism, i.e, $H_1^m(X \otimes Y, Z)$ is isomorphic with $H_1^m(X, H_1^m(Y, Z))$.

<u>Proof</u>. Let $f, g \in H_1^m(X \otimes Y, Z)$ and $x \in X$, $y \in Y$. Then

$$\begin{aligned} \overline{fg}(x)(y) &= (fg)(x \otimes y) \\ &= f(x \otimes y)\ g(x \otimes y) \\ &= \bar{f}(x)(y)\ \bar{g}(x)(y) \\ &= (\bar{f}\bar{g})(xy)\ ;\ \overline{fg} = \bar{f} \cdot \bar{g} \end{aligned}$$

Hence $T(fg) = T(f)\ T(g)$.

**Step 6**. In step 4, T is a homeomorphism, i·e, $H^m_1(X\otimes Y, Z)$ is homeomorphic with $H^m_1(X, H^m_1(Y, Z))$.

Proof. Consider a diagram;

$$\begin{array}{ccccc} H^m_1(X, H^m_1(Y, Z)) & \xrightarrow{P_x} & H^m_1(Y, Z) & \xrightarrow{P_y} & Z \\ \uparrow T & & & \nearrow_{P_x \otimes y} & \\ H^m_1(X\otimes Y, Z) & & & & \end{array} \qquad (x\in X,\ y\in Y)$$

By proposition 1.5, $(P_y \cdot P_x)_{(x\in X, y\in Y)}$ is an initial source.

And since each $P_{x\otimes y}$ is continous, T is continous.

Moreover, $(P_{x\otimes y})_{(x\in X,\ y\in Y)}$ is an initial source and $P_y \cdot P_x \cdot T = P_{x\otimes y}$ for each $x\in X$, $y\in Y$. By proposition 1.5, (T) is an initial source. By theorem 1.8, T is a homeomorphism.

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(國文抄錄)

# topological semigroup들의 함수공간에 관하여

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본 논문은 Semigroup 들 사이에 Homomorphism 공간에 대하여 알아보고 더나아가 topological semigroup 들의 Homomorphiom 공간을 연구하였다.

특히, 이 위상 공간들은 compact semigroup 이 됨을 보였으며, 또 semigroup 상에 tensor 를 도입하여 adjoimt associativity 가 성립함을 증명하였다.