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On Properties of Power Matrices over Boolean Algebra

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On Properties of Power Matrices over Boolean algebra

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< 국문초록 >

부울 대수상에서 멱등행렬의 성질 연구

부울 대수는 수학의 기본이 되는 대수적 구조이다. 이 구조상의 행렬도 참고 문헌에서 보듯이 많은 연구자들의 연구 주제가 되어 왔다.

본 논문에서는 부울 대수상의 행렬의 거듭제곱에 대하여 연구하였다. 주어진 부울 행렬을 계속하여 거듭 제곱을 하면, 결국은 똑 같은 행렬이 반복되어 나오게 되는 성질이 있다. 이를 이용하여 [1]에서, 저자들은 반복 되는 행렬의 주기, 순환길이, 주기지수를 정의하였다. 이 정의를 바탕으로 본 논문에서는 부울 행렬의 근본 성질을 연구하고 항등행렬, 또는 특수한 순환행렬을 품는 행렬의 주기, 순환길이, 주기지수를 규명하였다. 또 일반적인 부울 대수로의 확장된 결과를 몇가지 얻어서 증명하였다.

CONTENTS

1. INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Let $\mathbb{B}_k$ be a finite Boolean algebra. We may assume that $\mathbb{B}_k$ consists of the subsets of a k-element set S_k. Union is denoted by $+$, intersection by juxtaposition, and complementation by $*$; 0 denotes the null set and 1 the set S_k. Addition and multiplication of matrices over $\mathbb{B}_k$ are defined as if they were over a field, as are the zero matrix, $\mathbf{0}$, the zero vector, $\mathbf{o}$, and the identity matrix, $\mathbf{I}$. There is a great deal of literature on the study of matrices over a finite Boolean algebra. Yet despite the fact that most Boolean algebras contain zero divisiors, many results in Boolean matrix theory are stated only for binary Boolean matrices, which have no zero divisiors. This is due in part, as Gregory, Kirkland and Pullman points out in their papers of the subject[1],to some properties of the matrices over a binary Boolean algebra $\mathbb{B}_1$.

In this article, we study the properties of period, cycle depth and power index and research the conditions to be a primitive matrix. In Chapter 1, preliminary results and definitions are presented. In Chapter 2, over binary Boolean algebra, we present the relations among period, cycle depth, and power index when a matrix dominates $\mathbf{I}$, and find out the conditions to be a primitive matrix. They are contained in Section 2.1 and 2.2, respectively. In Chapter 3, we generalize the results which are in Chapter 2 to over non-binary Boolean algebra.

1.2 Preliminaries

We will give some definitions for a binary and a general Boolean algebra. We refer to the scalars, vectors, and matrices over $\mathbb{B}_1$ as *binary Boolean.*

A nonempty family $\mathcal{V}$ of n-vectors over $\mathbb{B}_k$, i.e. of members of $\mathbb{B}_k^n$, is a *vector space* in $\mathbb{B}_k^n$ if it is closed under addition and under multiplication by scalars(members of $\mathbb{B}_k$). The notions of a *linear combination* of vectors and a *spanning set* of vectors are the same as for fields. A collection of vectors is said to be *linearly dependent* if one of its members can be written as a linear combination of the others, or if it is $\{\mathbf{o}\}$, the *zero space.* Otherwise the collection is *linearly independent.* In particular, a set consisting of a single nonzero vector is linearly independent set. For matrices $\mathbf{A}$, $\mathbf{B}$, we say $\mathbf{A}$ *dominates* $\mathbf{B}$, written $\mathbf{A} \geq \mathbf{B}$, if $\mathbf{A}$ and $\mathbf{B}$ are compatible for addition and if, for all i and j, $a_{ij} \supseteq b_{ij}$. Then $\mathbf{A} \geq \mathbf{B}$ if, and only if, $\mathbf{A} + \mathbf{B} = \mathbf{A}$. It is easy to verify that whenever $\mathbf{A}$ dominates $\mathbf{B}$, and $\mathbf{C}$ is compatible with $\mathbf{A}$ for right multiplication, then $\mathbf{AC}$ dominates $\mathbf{BC}$. We say that $\mathbf{A}$ and $\mathbf{B}$ are *incomparable* if neither $\mathbf{A} \geq \mathbf{B}$ nor $\mathbf{B} \geq \mathbf{A}$. If $\mathbf{A}$ dominates $\mathbf{B}$, but $\mathbf{A} \neq \mathbf{B}$, we write $\mathbf{A} > \mathbf{B}$.

2. BINARY BOOLEAN MATRIX

In this Chapter, all matrices are binary Boolean matrices.

2.1 Period, Cycle depth and Power index

Before proceding further, we will need some definitions to describe properties of the set of powers of an $n \times n$ matrix $\mathbf{B}$, $\mathcal{P}(\mathbf{B})$. Because the set is finite, there are indices $r \geq 0$ and $s \geq 1$, such that $\mathbf{B}^r = \mathbf{B}^{r+s}$. Let p be the least such index s and let $c = \min\{r : \mathbf{B}^r = \mathbf{B}^{r+p}\}$. The indices p and c are called *the period* and *cycle depth* respectively. Define the *power index* of $\mathbf{B}$, $d = d(\mathbf{B})$, to be the first integer such that it is a linear combination of previous powers of $\mathbf{B}$. Let $\mathbf{B}^0 = \mathbf{I}$. Then we have

$$\mathcal{P}(\mathbf{B}) = \{\ \mathbf{I},\ \mathbf{B},\ \mathbf{B}^2,\ \mathbf{B}^3,\ \cdots,\ \mathbf{B}^{c+p-1}\ \}$$

Proposition 2.1.1([1]). *For any $n \times n$ matrix $\mathbf{B}$, $d \leq c + p$.*

Proof. By definition of the cycle depth and period, we have $\mathbf{B}^c = \mathbf{B}^{c+p}$. Hence, we get $d \leq c + p$.

Example.

Let $\mathbf{B} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$. Then, we have

$$\mathbf{B}^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$\mathbf{B}^3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

$$\mathbf{B}^4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

$$\mathbf{B}^5 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \mathbf{B}^n, \ for \ n \geq 6.$$

Thus we obtain that $p = 1$, $c = 5$ and $d = 3$. □

One of the most impotent properties of Boolean matrices is that they have a period.

Lemma 2.1.2. *For all* $n \geq 1$, *if* $\mathbf{B}$ *is an* $n \times n$ *matrix dominating* $\mathbf{I}$, *then the period* $p = 1$.

Proof. The proof proceeds by contradiction. Let p and c be the period and cycle depth, respectively. Let us assume that $p \neq 1$ (i.e. $p \geq 2$). Then we have

$$\mathbf{I} \leq \mathbf{B} \leq \mathbf{B}^2 \leq \cdots \leq \mathbf{B}^n \leq \cdots.$$

Then we have $\mathbf{B}^c \neq \mathbf{B}^{c+1}$, $\mathbf{B}^{c+(p-1)} \neq \mathbf{B}^{c+p}$, because $p \geq 2$. Hence we have

$$\mathbf{B}^{c-1} \leq \mathbf{B}^c < \mathbf{B}^{c+1} < \cdots < \mathbf{B}^{c+p-1} < \mathbf{B}^{c+p} \leq \mathbf{B}^{c+p+1} \cdots.$$

But $\mathbf{B}^c = \mathbf{B}^{c+p}$, and so $\mathbf{B}^{c-1} \leq \mathbf{B}^c = \mathbf{B}^{c+1} = \cdots = \mathbf{B}^{c+p} = \mathbf{B}^{c+p+1} = \cdots$.

This contradicts to $\mathbf{B}^c \neq \mathbf{B}^{c+1}$ and $\mathbf{B}^{c+(p-1)} \neq \mathbf{B}^{c+p}$.

Therefore, $p = 1$. □

Lemma 2.1.3. *For all $n \geq 1$, if $\mathbf{B}$ is $n \times n$ matrix dominating $\mathbf{I}$, then the power index $d = c + 1$.*

Proof. The proof proceeds by contradiction. By Proposition 2.1.1, 2.1.2, we note that $d \leq c + 1$. Let us assume that $d < c + 1$ (i.e. $d \leq c$).

First, we suppose that $d = c$. Then we have

$$\mathbf{I} \leq \mathbf{B} \leq \cdots \leq \mathbf{B}^i \leq \cdots \leq \mathbf{B}^c = \mathbf{B}^d$$

and there exists i such that $i < c$ and $\mathbf{B}^d = \mathbf{B}^i$. That is, $\mathbf{I} \leq \mathbf{B} \leq \cdots \leq \mathbf{B}^i = \cdots = \mathbf{B}^c = \mathbf{B}^d$.

This contradicts the minimality of c. Next, we suppose that $d < c$.

Let $d = j < c$. Then there exists i such that $i < j < c$ and $\mathbf{B}^i = \cdots = \mathbf{B}^j = \mathbf{B}^d \leq \mathbf{B}^c$. This contradicts the minimality of c. Hence, $d = c + 1$. □

Lemma 2.1.4. *Let $\mathbf{B}$ be an $n \times n$ matrix dominating $\mathbf{I}$. Then $d \leq n$.*

We prove this Lemma at the end of this section 2.2.

2.2 The conditions of primitivity

If $\mathbf{B}$ is primitive, that is, if $\mathbf{B}^c = \mathbf{J}_n$, the $n \times n$ matrix of all ones, then the period of $\mathbf{B}$ is 1.

We denote the $n \times n$ Boolean matrix whose all entries are 1 as $\mathbf{J}_n$. Let $\mathbf{B}$ be an $n \times n$ Boolean matrix. If some power of $\mathbf{B}$ becomes $\mathbf{J}_n$, then we call $\mathbf{B}$ a primitive matrix. Then the period of $\mathbf{B}$ is 1.

Theorem 2.2.1. *Let $\mathbf{C}$ be the $n \times n$ permutation matrix which all diagonal entries are zero with $n \leq 4$. Then*

(1) $\mathbf{C}$, $\mathbf{C}^2$, $\cdots$, *and* $\mathbf{C}^n$ *are distinct permutation matrices.*

(2) $\mathbf{C}^n = \mathbf{I}_n$.

(3) $\mathbf{C} + \mathbf{C}^2 + \cdots + \mathbf{C}^n = \mathbf{J}_n$.

Proof. Let $\mathbf{E}_{ij}$ be a matrix which only (i, j)-entry is 1 over binary Boolean algebra. Mathematical induction provides the best means for confirming this guess. First of all, for $n = 2, 3, 4$, we have the followings;

When $n = 2$,

$$\mathbf{C} = \mathbf{E}_{12} + \mathbf{E}_{21},$$

$$\mathbf{C}^2 = \mathbf{E}_{11} + \mathbf{E}_{22} = \mathbf{I}_2.$$

and so $\mathbf{C} + \mathbf{C}^2 = \mathbf{J}_2$. It holds conditions (1),(2) and (3).

When $n = 3$, let i, j and k be mutually distinct. We have

$$\mathbf{C} = \mathbf{E}_{ij} + \mathbf{E}_{jk} + \mathbf{E}_{ki},$$

$$\mathbf{C}^2 = \mathbf{E}_{ik} + \mathbf{E}_{ji} + \mathbf{E}_{kj},$$

$$\mathbf{C}^3 = \mathbf{C}^2 \cdot \mathbf{C} = \mathbf{E}_{ii} + \mathbf{E}_{jj} + \mathbf{E}_{kk} = \mathbf{I}_3.$$

Thus $\mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 = \mathbf{J}_3$. Also, (1),(2) and (3) hold.

Finally, when $n = 4$, let i, j, k and l be mutually distinct. Then we have

$$\begin{aligned}
\mathbf{C} &= \mathbf{E}_{ij} + \mathbf{E}_{jk} + \mathbf{E}_{kl} + \mathbf{E}_{li}, \\
\mathbf{C}^2 &= \mathbf{E}_{ik} + \mathbf{E}_{jl} + \mathbf{E}_{ki} + \mathbf{E}_{lj}, \\
\mathbf{C}^3 &= \mathbf{C}^2 \cdot \mathbf{C} = \mathbf{E}_{il} + \mathbf{E}_{ji} + \mathbf{E}_{kj} + \mathbf{E}_{lk}, \\
\mathbf{C}^4 &= \mathbf{C}^3 \cdot \mathbf{C} = \mathbf{E}_{ii} + \mathbf{E}_{jj} + \mathbf{E}_{kk} + \mathbf{E}_{ll} = \mathbf{I}_4.
\end{aligned}$$

It is certainly correct. □

Theorem 2.2.2. *Let* $\mathbf{C}$ *be an* $n \times n$ *permutation matrix which all diagonal entries are zero. Let* $\mathbf{B} = \mathbf{I} + \mathbf{C}$. *Then* $\mathbf{I} + \mathbf{C} + \cdots + \mathbf{C}^n = \mathbf{B}^n = \mathbf{B}^{n-1} = \mathbf{J}_n$.

Proof. Since $\mathbf{B} = \mathbf{I}+\mathbf{C}$, we have

$$\begin{aligned}
\mathbf{B}^2 &= (\mathbf{I} + \mathbf{C})^2 \\
&= \mathbf{I}^2 + \mathbf{I} \cdot \mathbf{C} + \mathbf{C} \cdot \mathbf{I} + \mathbf{C}^2 \\
&= \mathbf{I} + \mathbf{C} + \mathbf{C}^2. \\
\mathbf{B}^3 &= (\mathbf{I} + \mathbf{C})^2(\mathbf{I} + \mathbf{C}) \\
&= (\mathbf{I} + \mathbf{C} + \mathbf{C}^2)(\mathbf{I} + \mathbf{C}) \\
&= \mathbf{I}^2 + \mathbf{I} \cdot \mathbf{C} + \mathbf{C} \cdot \mathbf{I} + \mathbf{C}^2 + \mathbf{C}^2 \cdot \mathbf{I} + \mathbf{C}^3 \\
&= \mathbf{I} + \mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3. \\
&\quad \vdots \\
\mathbf{B}^n &= \mathbf{I} + \mathbf{C} + \mathbf{C}^2 + \cdots + \mathbf{C}^{n-1} + \mathbf{C}^n.
\end{aligned}$$

By Theorem 2.2.1.

$$\begin{aligned}
\mathbf{B}^n &= \mathbf{I} + \mathbf{C} + \mathbf{C}^2 + \cdots + \mathbf{C}^{n-1} + \mathbf{C}^n \\
&= \mathbf{I} + \mathbf{C} + \mathbf{C}^2 + \cdots + \mathbf{C}^{n-1} \\
&= \mathbf{B}^{n-1} = \mathbf{J}_n.
\end{aligned}$$

This completes the proof. □

Example.

Let $\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Then we have

$$\mathbf{C}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\begin{aligned}
\mathbf{C}^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \mathbf{I}_3.
\end{aligned}$$

Thus we know that

$$\begin{aligned}
\mathbf{B}^3 &= \mathbf{I} + \mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 \\
&= \mathbf{I} + \mathbf{C} + \mathbf{C}^2 \\
&= \mathbf{B}^2 \\
&= \mathbf{J}_3.
\end{aligned}$$

Hence **B** is primitive. □

Theorem 2.2.3. *Let* $\mathbf{C}$ *be an* $n \times n$ *permutation matrix which all diagonal entries are zero and* $\mathbf{O}$ *a matrix whose nonzero entry is different to any one of the nonzero entries in* $\mathbf{C}$ *with* $n \leq 4$. *Then* $\mathbf{C}+\mathbf{O}$ *is primitive.*

Proof. Let $\mathbf{E}_{ij}$ be matrix which only (i,j)-entry is 1 over binary Boolean algebra. We again argue by induction on n. For $n = 3$,

$$
\begin{aligned}
\mathbf{C}+\mathbf{O} &= \mathbf{E}_{ij} + \mathbf{E}_{jk} + \mathbf{E}_{ki} + \mathbf{E}_{ik}, \\
(\mathbf{C}+\mathbf{O})^2 &= \mathbf{E}_{ik} + \mathbf{E}_{ji} + \mathbf{E}_{kj} + \mathbf{E}_{kk} + \mathbf{E}_{ii}, \\
(\mathbf{C}+\mathbf{O})^3 &= \mathbf{E}_{ii} + \mathbf{E}_{jj} + \mathbf{E}_{kk} + \mathbf{E}_{ij} + \mathbf{E}_{jk} + \mathbf{E}_{ki} + \mathbf{E}_{ik} + \mathbf{E}_{kj}.
\end{aligned}
$$

Thus we obtain $(\mathbf{C}+\mathbf{O})^3 \geq \mathbf{I}+\mathbf{C}$. Therefore $\mathbf{C}+\mathbf{O}$ is a primitive matrix by theorem 2.2.2. For $n = 4$

$$
\begin{aligned}
\mathbf{C}+\mathbf{O} &= \mathbf{E}_{ij} + \mathbf{E}_{jk} + \mathbf{E}_{kl} + \mathbf{E}_{li} + \mathbf{E}_{ik}, \\
(\mathbf{C}+\mathbf{O})^2 &= \mathbf{E}_{ik} + \mathbf{E}_{jl} + \mathbf{E}_{ki} + \mathbf{E}_{lj} + \mathbf{E}_{il} + \mathbf{E}_{lk}, \\
(\mathbf{C}+\mathbf{O})^3 &= \mathbf{E}_{il} + \mathbf{E}_{ji} + \mathbf{E}_{kj} + \mathbf{E}_{kk} + \mathbf{E}_{lk} + \mathbf{E}_{ii} + \mathbf{E}_{ll}, \\
(\mathbf{C}+\mathbf{O})^4 &= \mathbf{E}_{ii} + \mathbf{E}_{jj} + \mathbf{E}_{jk} + \mathbf{E}_{kk} + \mathbf{E}_{kl} + \mathbf{E}_{ll} + \mathbf{E}_{ij} + \mathbf{E}_{ik} + \mathbf{E}_{li} \\
&= \mathbf{E}_{ii} + \mathbf{E}_{jj} + \mathbf{E}_{kk} + \mathbf{E}_{ll} + \mathbf{E}_{ij} + \mathbf{E}_{jk} + \mathbf{E}_{kl} + \mathbf{E}_{li} + \mathbf{E}_{ik}.
\end{aligned}
$$

Thus we have $(\mathbf{C}+\mathbf{O})^4 \geq \mathbf{I}+\mathbf{C}$. Therefore $\mathbf{C}+\mathbf{O}$ is a primitive matrix by theorem 2.2.2. □

Remark. *In the theorems 2.2.1 and 2.2.3, we can show that the results holds for any finite integer* n. *But the proofs are tedious, we omit them.*

Consequently, summing Theorem 2.2.1, Theorem 2.2.2 and Theorem 2.2.3, we have the following Theorem.

Theorem 2.2.4. *Let* $\mathbf{B}$ *be an* $n \times n$ *binary Boolean matrix. Let* $\mathbf{C}$ *be an* $n \times n$ *permutation matrix which all diagonal entries are zero and* $\mathbf{O}$ *be a matrix whose nonzero entry is different to any one of the nonzero entries in* $\mathbf{C}$. *Then*

(1) $\mathbf{B}(= \mathbf{I} + \mathbf{C})$ *is primitive matrix if* $\mathbf{C}$ *dominates a permuatation matrix which is not symmetric in any even* n.

(2) $\mathbf{B}(\ngeq \mathbf{I})$ *is primitive matrix if* $\mathbf{C}$ *satisfies* (1) *condition and* $\mathbf{B} \geq \mathbf{C} + \mathbf{O}$.

Lemma 2.2.5. *Let* $\mathbf{B}$ *be the binary Boolean matrix. Then* $\mathbf{B}$ *and* $\mathbf{B}^T$ *have the same period, cycle depth and power index.*

Proof. To show that $\mathbf{B}$ and $\mathbf{B}^T$ have the same period, cycle depth and power index. Let p, c and d be period, cycle depth and power index of $\mathbf{B}$, respectively. Then we have

$$\mathbf{B}^c = \mathbf{B}^{c+p} = \mathbf{B}^{c+2p} = \cdots.$$

Note that $(\mathbf{B}^c)^T = (\mathbf{B}^T)^c$. We have

$$\mathbf{B}^c = (\overbrace{\mathbf{B}^T \cdot \mathbf{B}^T \cdot \cdots \cdot \mathbf{B}^T}^{c\ times})^T = \{(\mathbf{B}^T)^c\}^T,$$

$$\mathbf{B}^{c+p} = (\overbrace{\mathbf{B}^T \cdot \mathbf{B}^T \cdot \cdots \cdot \mathbf{B}^T}^{c+p\ times})^T = \{(\mathbf{B}^T)^{c+p}\}^T.$$

Since $\mathbf{B}^c = \mathbf{B}^{c+p}$, $\{(\mathbf{B}^T)^c\}^T = \{(\mathbf{B}^T)^{c+p}\}^T$. This shows clearly that $(\mathbf{B}^T)^c = (\mathbf{B}^T)^{c+p}$.

But $\mathbf{B}^d = \sum_{i=1}^{d-1} \alpha_i \mathbf{B}^i$. Consequently we have

$$\begin{aligned}(\mathbf{B}^d)^T &= (\sum_{i=1}^{d-1} \alpha_i \mathbf{B}^i)^T \\ &= \sum_{i=1}^{d-1} \alpha_i (\mathbf{B}^i)^T \\ &= \sum_{i=1}^{d-1} \alpha_i (\mathbf{B}^T)^i \\ &= (\mathbf{B}^T)^d\end{aligned}$$

Thus the period, cycle depth and power index of $\mathbf{B}^T$ is not greater than those of $\mathbf{B}$. If we exchange the role of $\mathbf{B}$ and $\mathbf{B}^T$ in the preceeding proof, we obtain that the period, cycle depth and power index of $\mathbf{B}$ are not greater than those of $\mathbf{B}^T$. Hence they are the same. □

We restate lemma 2.1.4 and give the proof at here.

Lemma 2.1.4. *Let* $\mathbf{B}$ *be an* $n \times n$ *binary Boolean matrix dominating* $\mathbf{I}$. *Then* $d \le n$.

Proof. By theorem 2.2.1 and 2.2.2, we have

$$\mathbf{I} \le \mathbf{B} \le \mathbf{B}^2 \le \cdots \le \mathbf{B}^{n-1} = \mathbf{B}^n = \mathbf{B}^{n+1} = \cdots ,$$

which means that $c \le n-1$ and $c+1 = d \le n$. □

3. GENERAL RESULTS

We continue our program by proving a general result over Boolean algebra $\mathbb{B}_k$ with $k \geq 2$. Let $\sigma_1, \sigma_2, \cdots, \sigma_k$ denote the singleton subsets of S_k. For each $p \times q$ matrix $\mathbf{A}$ over $\mathbb{B}_k$, the lth *constituent* of $\mathbf{A}$,$\mathbf{A}^{(l)}$, is the $p \times q$ $(0,1)$ matrix whose ijth entry is 1 if and only if $a_{ij} \supseteq \sigma_l$.

Evidently $\mathbf{A} = \sum_{l=1}^{k} \sigma_l \mathbf{A}^{(l)}$. The next two propositions follow easily from the definitions and the fact that for any singletons σ and τ, $\sigma\tau = \sigma$ or 0 according as $\tau = \sigma$ or not.

Proposition 3.1([1]). *If* $\mathbf{A} = \sum_{l=1}^{k} \sigma_l \mathbf{C}^{(l)}$ *and the* $\mathbf{C}^{(l)}$ *are all* $(0,1)$ *matrices, then* $\mathbf{C}^{(l)} = \mathbf{A}^{(l)}$ *for all* $1 \leq l \leq k$. *That is, the constituents of* $\mathbf{A}$ *are the unique binary solutions to* $\mathbf{A} = \sum_{l=1}^{k} \sigma_l \mathbf{X}^{(l)}$.

Proof. Let $\mathbf{A} = \sum_{l=1}^{k} \sigma_l \mathbf{C}^{(l)}$. Then

$$\begin{aligned}
\sigma_l \mathbf{A} &= \sigma_l \sum_{l=1}^{k} \sigma_l \mathbf{C}^{(l)} \\
&= \sigma_l(\sigma_1 \mathbf{C}^{(1)} + \sigma_2 \mathbf{C}^{(2)} + \cdots + \sigma_k \mathbf{C}^{(k)}) \\
&= \sigma_l \mathbf{C}^{(l)}, \\
\text{i.e.} \quad \sigma_l \mathbf{A} &= \sigma_l \mathbf{C}^{(l)}.
\end{aligned}$$

Also clearly $\sigma_l \mathbf{A} = \sigma_l \mathbf{A}^{(l)}$ and hence $\sigma_l \mathbf{A}^{(l)} = \sigma_l \mathbf{C}^{(l)}$.

Therefore $\mathbf{A}^{(l)} = \mathbf{C}^{(l)}$, for all $l = 1, 2, \cdots, k$. □

Proposition 3.2([1]). *For all $p \times q$ matrices $\mathbf{A}$, all $q \times r$ matrices $\mathbf{B}$ and $\mathbf{C}$, and all $\alpha \in \mathbb{B}$,*

(a) $(\mathbf{AB})^{(l)} = \mathbf{A}^{(l)}\mathbf{B}^{(l)}$,

(b) $(\mathbf{B}+\mathbf{C})^{(l)} = \mathbf{B}^{(l)} + \mathbf{C}^{(l)}$, *and*

(c) $(\alpha\mathbf{A})^{(l)} = \alpha^{(l)}\mathbf{A}^{(l)}$ *for all* $1 \le l \le k$.

Proof. Let $\mathbf{A} = \sum_{i=1}^{k} \sigma_i \mathbf{A}^{(i)}$, $\mathbf{B} = \sum_{j=1}^{k} \sigma_j \mathbf{B}^{(j)}$.

(a): We have

$$\begin{aligned}
\mathbf{AB} &= \left(\sum_{j=1}^{k} \sigma_j \mathbf{A}^{(j)}\right)\left(\sum_{j=1}^{k} \sigma_j \mathbf{B}^{(j)}\right) \\
&= (\sigma_1 \mathbf{A}^{(1)} + \cdots + \sigma_k \mathbf{A}^{(k)})(\sigma_1 \mathbf{B}^{(1)} + \cdots + \sigma_k \mathbf{B}^{(k)}) \\
&= \sigma_1 \mathbf{A}^{(1)}\mathbf{B}^{(1)} + \cdots + \sigma_k \mathbf{A}^{(k)}\mathbf{B}^{(k)}.
\end{aligned}$$

Thus $\mathbf{AB} = \sum_{l=1}^{k} \sigma_l \mathbf{A}^{(l)}\mathbf{B}^{(l)}$. Note that $\mathbf{AB} = \sum_{l=1}^{k} \sigma_l (\mathbf{AB})^{(l)}$.

Hence $(\mathbf{AB})^{(l)} = \mathbf{A}^{(l)}\mathbf{B}^{(l)}$ by proposition 2.1.1.

(b): We obtain

$$\begin{aligned}
\mathbf{A}+\mathbf{B} &= \left(\sum_{j=1}^{k} \sigma_j \mathbf{A}^{(j)}\right) + \left(\sum_{j=1}^{k} \sigma_j \mathbf{B}^{(j)}\right) \\
&= (\sigma_1 \mathbf{A}^{(1)} + \cdots + \sigma_k \mathbf{A}^{(k)}) + (\sigma_1 \mathbf{B}^{(1)} + \cdots + \sigma_k \mathbf{B}^{(k)}) \\
&= \sigma_1 (\mathbf{A}^{(1)} + \mathbf{B}^{(1)}) + \cdots + \sigma_k (\mathbf{A}^{(k)} + \mathbf{B}^{(k)}).
\end{aligned}$$

Thus $\mathbf{A}+\mathbf{B} = \sum_{l=1}^{k} \sigma_l (\mathbf{A}^{(l)} + \mathbf{B}^{(l)})$. Note that $\mathbf{A}+\mathbf{B} = \sum_{l=1}^{k} \sigma_l (\mathbf{A}+\mathbf{B})^{(l)}$.

Hence, $(\mathbf{A}+\mathbf{B})^{(l)} = \mathbf{A}^{(l)} + \mathbf{B}^{(l)}$, by proposition 2.1.1.

(c): Now we have

$$\begin{aligned}\alpha\mathbf{A} &= \alpha(\sigma_1\mathbf{A}^{(1)} + \cdots + \sigma_k\mathbf{A}^{(k)})\\ &= \alpha\sigma_1\mathbf{A}^{(1)} + \cdots + \alpha\sigma_k\mathbf{A}^{(k)}\\ &= \sigma_1\alpha^{(1)}\mathbf{A}^{(1)} + \cdots + \sigma_k\alpha^{(k)}\mathbf{A}^{(k)}\\ &= \sum_{l=1}^{k}\sigma_l\alpha^{(l)}\mathbf{A}^{(l)}.\end{aligned}$$

Notice that $\alpha\mathbf{A} = \sum_{i=1}^{k}\sigma_i(\alpha\mathbf{A})^{(i)}$.

By Proposition 2.1.1, we have $(\alpha\mathbf{A})^{(l)} = \alpha^{(l)}\mathbf{A}^{(l)}$. Hence the proof is completed. □

Lemma 3.3. *Let* **B** *be the Boolean matrix over* $\mathbb{B}_2$.

Then

$$\begin{aligned}p &= lcm\{p^{(1)},\ p^{(2)}\},\\ c &= max\{c^{(1)},\ c^{(2)}\},\\ d &= max\{d^{(1)},\ d^{(2)}\}.\end{aligned}$$

where, $p^{(1)}$ *and* $p^{(2)}$, $c^{(1)}$ *and* $c^{(2)}$, $d^{(1)}$ *and* $d^{(2)}$ *are periods, cycle depths, power indices of constituents of* **B**, *respectively.*

Proof. It is easy to check them by our new definitions. Let $p^{(1)}$ and $p^{(2)}$, $c^{(1)}$ and $c^{(2)}$, $d^{(1)}$ and $d^{(2)}$ be periods, cycle depths, power indices of constituents of **B**, respectively. Now, consider the following four cases.

Case 1) $c^{(1)} = c^{(2)}$ and $p^{(1)} = p^{(2)}$,

Case 2) $c^{(1)} = c^{(2)}$ and $p^{(1)} \neq p^{(2)}$, Case 3) $c^{(1)} \neq c^{(2)}$ and $p^{(1)} = p^{(2)}$,

Case 4) $c^{(1)} \neq c^{(2)}$ and $p^{(1)} \neq p^{(2)}$.

We prove *Case 4)* only. Because we can also get the same results with a similar way in there proofs.

Case 4): Let $c^{(1)} \neq c^{(2)}$ and $p^{(1)} \neq p^{(2)}$. Put $c^{(1)} < c^{(2)}$ and $p^{(1)} < p^{(2)}$. Then we have

$$(\mathbf{B}^{(1)})^{c^{(1)}+p^{(1)}} = (\mathbf{B}^{(1)})^{c^{(1)}}, \qquad (\mathbf{B}^{(2)})^{c^{(2)}+p^{(2)}} = (\mathbf{B}^{(2)})^{c^{(2)}}.$$

Since $c^{(1)} < c^{(2)}$, and so $(\mathbf{B}^{(1)})^{c^{(2)}+p^{(1)}} = (\mathbf{B}^{(1)})^{c^{(2)}}$. But $(\mathbf{B}^{(2)})^{c^{(1)}+p^{(2)}} \neq (\mathbf{B}^{(2)})^{c^{(1)}}$. Thus, $c = max\{c^{(1)}, c^{(2)}\}$. Hence

$$(\mathbf{B}^{(1)})^{c+p^{(1)}} = (\mathbf{B}^{(1)})^{c}, \qquad (\mathbf{B}^{(2)})^{c+p^{(2)}} = (\mathbf{B}^{(2)})^{c}.$$

But since $p^{(1)} < p^{(2)}$, we get that

$$(\mathbf{B}^{(1)})^{c} = (\mathbf{B}^{(1)})^{c+p^{(1)}} = (\mathbf{B}^{(1)})^{c+2p^{(1)}} = \cdots = (\mathbf{B}^{(1)})^{c+lcm\{p^{(1)},p^{(2)}\}} = \cdots,$$
$$(\mathbf{B}^{(2)})^{c} = (\mathbf{B}^{(2)})^{c+p^{(2)}} = (\mathbf{B}^{(2)})^{c+2p^{(2)}} = \cdots = (\mathbf{B}^{(2)})^{c+lcm\{p^{(1)},p^{(2)}\}} = \cdots,$$

Thus, $p = lcm\{p^{(1)}, p^{(2)}\}$. Similarly, suppose $d^{(1)} \leq d^{(2)}$.

Then we obtain $\mathbf{B}^{d^{(1)}} = \mathbf{B}^{(1)d^{(1)}} + \mathbf{B}^{(2)d^{(1)}}$, $\mathbf{B}^{d^{(2)}} = \mathbf{B}^{(1)d^{(2)}} + \mathbf{B}^{(2)d^{(2)}}$.

But $\mathbf{B}^{(2)d^{(1)}}$ is not linear combinition of previous nonnegative powers, $\mathbf{B}^{d^{(1)}}$ is also not. Consequently, $d = max\{d^{(1)}, d^{(2)}\}$. □

Lemma 3.4. *For all $n \times n$ Boolean matrices* $\mathbf{B}$ *over* $\mathbb{B}_2$, *we have* $d \leq c + p$.

Proof. It is easy to check them by our new definitions. Let $p^{(1)}$ and $p^{(2)}$, $c^{(1)}$ and $c^{(2)}$, $d^{(1)}$ and $d^{(2)}$ be periods, cycle depths, power indices of constituents of $\mathbf{B}$, respectively. Now we have

$$\begin{aligned} d = & \ max\{d^{(1)},\ d^{(2)}\} \\ \leq & \ max\{c^{(1)} + p^{(1)}, c^{(2)} + p^{(2)}\} \\ \leq & \ max\{c^{(1)}, c^{(2)}\} + max\{p^{(1)}, p^{(2)}\} \\ \leq & \ max\{c^{(1)}, c^{(2)}\} + lcm\{p^{(1)}, p^{(2)}\} \\ = & \ c + p. \end{aligned}$$

□

Lemma 3.5. *Let* $\mathbf{B}$ *be a matrix over* $\mathbb{B}_2$ *,such that each constituent of* $\mathbf{B}$ *is idempotent matrix. Then* $\mathbf{B}^T$ *is also idempotent and* $d = c + p$

Proof. It is not hard to see that $\mathbf{B}$ is idempotent if each constituent is idempotent. Since $\mathbf{B} = \sigma_1\mathbf{B}^{(1)} + \sigma_2\mathbf{B}^{(2)}$,

$$\begin{aligned} \mathbf{B}^2 &= \sigma_1\mathbf{B}^{(1)2} + \sigma_2\mathbf{B}^{(2)2} \\ &= \sigma_1\mathbf{B}^{(1)} + \sigma_2\mathbf{B}^{(2)} \\ &= \mathbf{B}. \end{aligned}$$

Thus $\mathbf{B}$ is idempotent matrix. Also the period, the cycle depth and power index of $\mathbf{B}$ are 1, 1 and 2, respectively.

If $\mathbf{B}$ is identity matrix, then $p = 1$, $c = 0$ and $d = 1$. Hence $d = p + c$. □

The next theorem establishes some properties of all $n \times n$ Boolean matrices.

Theorem 3.6. *For all $n \times n$ Boolean matrices $\mathbf{B}$ over $\mathbb{B}_k$, we have the followings;*

(1) $d \leq c + p$.

(2) If $\mathbf{B}$ dominates $\mathbf{I}$, then the period $p = 1$.

(3) If $\mathbf{B}$ dominates $\mathbf{I}$, then the power index $d = c + p$.

(4) If $\mathbf{B}$ dominates $\mathbf{I}$, then $d \leq n$.

(5) If all the constituents of $\mathbf{B}$ satisfy either (1) or (2) condition of Theorem 2.2.4, then $\mathbf{B}$ is primitive matrix.

(6) $\mathbf{B}$ and $\mathbf{B}^T$ have the same period, cycle depth and power index.

(7) If each constituent of $\mathbf{B}$ is idempotent matrix, then $\mathbf{B}^T$ is also idemptent and $d = c + p$.

Proof. It is easy to check them using previous Lemmas.

Let $\mathbf{B} = \sum_{l=1}^{k} \sigma_l \mathbf{B}^{(l)}$.

(1): For all constituents,

$$d^{(l)} \leq c^{(l)} + p^{(l)}, \quad l = 1, 2, \cdots, k.$$

We obtain

$$\begin{aligned}
d &= max\{d^{(1)}, d^{(2)}, \cdots, d^{(k)}\} \\
&\leq max\{c^{(1)} + p^{(1)}, \cdots, c^{(k)} + p^{(k)}\} \\
&\leq max\{c^{(1)}, \cdots, c^{(k)}\} + max\{p^{(1)}, \cdots, p^{(k)}\} \\
&\leq max\{c^{(1)}, \cdots, c^{(k)}\} + lcm\{p^{(1)}, \cdots, p^{(k)}\} \\
&= c + p.
\end{aligned}$$

(2): We note that $\mathbf{I} \le \mathbf{B} = \sum_{l=1}^{k} \sigma_l \mathbf{B}^{(l)}$.

This means that

$$\sigma_l \mathbf{I} = \sigma_l \mathbf{I}^{(l)} \le \sigma_{\mathbf{B}} = \sigma_l \mathbf{B}^{(l)}.$$

Hence $\mathbf{I}^{(l)} \le \mathbf{B}^{(l)}$. Since the period of each constituent is 1,

$p = lcm\{p^{(1)}, p^{(2)}, \cdots, p^{(k)}\} = 1.$

(3): For all the constituents,

$$d^{(l)} = c^{(l)} + 1,\ l = 1, 2, \cdots, k.$$

Thus, there exists i such that $d^{(i)} = c^{(i)} + 1$ and $i = max\{1, 2, \cdots, k\}$.

Therefore,

$$\begin{aligned} d = max\{d^{(1)}, \cdots, d^{(k)}\} = d^{(i)} = c^{(i)} + 1 \\ = max\{c^{(1)}, \cdots, c^{(k)}\} + 1 = c + 1. \end{aligned}$$

(4): Since the cycle depth of all constituents satisfies $c^{(l)} \le n - 1$, for all $l = 1, 2, \cdots, k$. We have

$$c = max\{c^{(1)}, \cdots, c^{(k)}\} \le n - 1, \quad i.e.\ \ d \le n.$$

(5): Since all constituents are primitive, **B** is also primitive.

(6):

$$\begin{aligned} (\mathbf{B}^c)^T &= \left((\sum_{l=1}^{k} \sigma_l \mathbf{B}^{(l)}) \right)^T = \left(\sum_{l=1}^{k} \sigma_l \mathbf{B}^{(l)} \right)^T = \sum_{l=1}^{k} \sigma_l (\mathbf{B}^{(l)})^T \\ &= \sum_{l=1}^{k} \sigma_l ((\mathbf{B}^{(l)})^T)^c = \left(\sum_{l=1}^{k} \sigma_l (\mathbf{B}^{(l)})^T \right)^c = \left(\sum_{l=1}^{k} \sigma_l (\mathbf{B}^T)^{(l)} \right)^c \\ &= (\mathbf{B}^T)^c \end{aligned}$$

and $\mathbf{B}^c = \mathbf{B}^{c+p}$, and so we get $(\mathbf{B}^T)^c = (\mathbf{B}^{c+p})^T = (\mathbf{B}^T)^{c+p}$.

From the definition, $\mathbf{B}^d = \sum_{l=1}^{d-1} \sigma_l(\mathbf{B})^{(l)}$. Consequently

$$(\mathbf{B}^d)^T = \left(\sum_{l=1}^{d-1} \sigma_l \mathbf{B}^{(l)}\right)^T = \sum_{l=1}^{d-1} \sigma_l (\mathbf{B}^{(l)})^T = \sum_{l=1}^{d-1} \sigma_l (\mathbf{B}^T)^{(l)} = (\mathbf{B}^T)^d$$

as desired.

(7): Since $\mathbf{B}^{(l)}$ is idempotent, for all $l = 1, \cdots, k$, we have

$$\mathbf{B}^2 = \left(\sum_{l=1}^{k} \sigma_l \mathbf{B}^{(l)}\right)^2 = \sum_{l=1}^{k} \sigma_l (\mathbf{B}^{(l)})^2 = \sum_{l=1}^{k} \sigma_l \mathbf{B}^{(l)} = \mathbf{B}.$$

□

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< Abstract >

On properties of power matrices over Boolean algebra

We study that the properties of period, cycle depth and power index of power matrices over Boolean algebra and research the conditions for a matrix to be primitive. We also extend some results on these properties over binary Boolean algebra in [1] to the case of non-binary Boolean algebra.

감 사 의 글

먼저 하나님께 감사와 영광을 돌려드립니다. 바쁘신 가운데서도 이 한편의 논문이 완성되기까지 저에게 깊은 관심과 배려로 지도해 주신 송석준 교수님께 무엇보다 깊은 감사를 드립니다. 저가 대학원 공부를 하는동안 끊임없이 연구하시고 훌륭한 강의를 해 주신 고윤희, 양성호, 정승달, 김철수교수님께 감사를 드립니다. 그리고 아낌없이 저를 격려해 주신 방은숙, 윤용식, 양영오 교수님께 감사드립니다. 저를 변함없이 사랑하여 주시고 지금까지 저의 뒷바라지를 아끼지 아니하신 부모님과 언제나 밝은 웃음으로 저에게 힘이되어 준 사랑하는 영임씨와 이 기쁨을 나누고 싶습니다. 지난 4학기 동안 서로에게 격려가 되고 힘이 되어준 대학원 동기생들에게 감사를 드립니다.

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