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ON THE BISHOP CURVATURES OF CURVES IN EUCLIDEAN SPACES

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CONTENTS

⟨**Abstract**⟩

ON THE BISHOP CURVATURES OF CURVES IN EUCLIDEAN SPACES

In this thesis, we study the properties of curves in Euclidean space. Firstly we review the Frenet formulas in Euclidean space and Lorentz space, respectively. And we study the several curves with the Frenet theory. Also, we introduce the Bishop theory and study the relations between Frenet theory and Bishop theory.

1 Introduction

Curves arise in many different ways. Specially, curves arise naturally in the motion of a particle. This is the most appropriate for a formal mathematical research of curves. The history of the curve theory were initiated by G. Monge and his school (Meusnier, Lancret, and Dupin). One of the study of curves is to use the moving frames. Classically, the basic tool in the study of curves is the Frenet-Serret theory. In 3-dimensional Euclidean space, it consists of three vector fields along the given curve and two scalar functions, so called the curvature and torsion. In $\mathbb{R}^n$, the Frenet frame can be defined, but the curves need to be of class C^{n-1} and the scalar functions should not be zeros. In this thesis, we introduce an another moving frame, so called the Bishop frame. In the Bishop theory, it is enough to satisfy that the curves are C^2-class. So, in this thesis we summarize the well-known facts about the Frenet theory. And then we study the Bishop theory and give the relationships between them. This thesis is organized a follows. In Chapter 2, we give the well-known facts and review the isometries. In Chapter 3, the Frenet formulas are studied in $\mathbb{R}^3$, $\mathbb{R}^3_1$ and $\mathbb{R}^n$, respectively. Moreover, many interesting curves, including sphere curves, Bertrand curves, involutes and evolutes, are studied. In Chapter 4, we give the Bishop formula in $\mathbb{R}^3$ and study the properties of curves with Bishop curvatures. Lastly, we study the relationships between Frenet theory and Bishop theory.

2 Curves in $\mathbb{R}^n$

2.1 Curves in $\mathbb{R}^n$

Let $\mathbb{R}^n = \{(x_1, \cdots, x_n) | x_i \in \mathbb{R}, (i = 1, \cdots, n)\}$ and $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a metric function, which is defined by

$$d(p, q) = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}. \tag{2.1}$$

Then $\mathbb{E}^n = (\mathbb{R}^n, d)$ is said to be Euclidean space. We write $\mathbb{R}^n$ instead of $\mathbb{E}^n$ if we have no confusion.

Definition 2.1 A *differentiable curve* in $\mathbb{R}^n$ is a differentiable map $\gamma : I \to \mathbb{R}^n$ of an open interval $I = (a, b) \subset \mathbb{R}$ into $\mathbb{R}^n$.

Sometimes the curve γ means the image of the curve.

Definition 2.2 Let I and J be open intervals. Suppose $\gamma : I \to \mathbb{R}^n$ is a differentiable curve and $\theta : J \to I$ is a diffeomorphism from J to I with $d\theta/ds \neq 0$. Then the map $\tilde{\gamma} = \gamma \circ \theta : J \to \mathbb{R}^n$ is said to be a *reparametrization* of γ.

Definition 2.3 Any differentiable curve $\gamma : I \to \mathbb{R}^n$ is said to be a *regular curve* if γ' never vanishes.

Let $<, >$ be the natural inner product on $\mathbb{R}^n$, and $||\mathbf{v}|| = \sqrt{< \mathbf{v}, \mathbf{v} >}$ the length of $\mathbf{v} \in \mathbb{R}^n$.
A regular curve segment is a function $\alpha : [a, b] \to \mathbb{R}^n$ such that there exists a regular curve $\gamma : (c, d) \to \mathbb{R}^n$ with $\gamma(t) = \alpha(t)$ for all $t \in [a, b] \subset (c, d)$.
Sometimes we call the curve instead of curve segment.

Definition 2.4 The *arclength function* $s(t)$ of $\gamma : [a, b] \to \mathbb{R}^n$ is defined by

$$s(t) = \int_a^t ||\gamma'(u)||du. \tag{2.2}$$

Trivially, $s(b)$ is the length of γ, which is denoted by $L(\gamma)$.

Proposition 2.5 *The length of a regular curve is invariant under reparametrization.*

Proof. Let $\gamma = [a,b] \to \mathbb{R}^n$ be a regular curve segment, and $\tilde{\gamma} = [c,d] \to \mathbb{R}^n$ be a reparametrization of γ. Let $\theta : [c,d] \to [a,b]$ be a differentiable function with $\tilde{\gamma}(t) = (\gamma \circ \theta)(t)$. Then we must show

$$L(\gamma) = L(\tilde{\gamma}).$$

Case 1 : If $\theta'(t) > 0$, then $\theta(c) = a, \theta(d) = b$, and

$$\|\tilde{\gamma}'(t)\| = \|\theta'(t) \cdot \gamma'(\theta(t))\| = |\theta'(t)|\|\gamma'(\theta(t))\| = \theta'(t)\|\gamma'(\theta(t))\|.$$

By integration by substitution, we have

$$L_a^b(\gamma) = \int_a^b \|\gamma'(u)\|du = \int_c^d \|\gamma'(\theta(t))\|\theta'(t)dt = \int_c^d \|\tilde{\gamma}'(t)\|dt = L_c^d(\tilde{\gamma}).$$

Case 2 : If $\theta'(t) < 0$, then $\theta(c) = b, \theta(d) = a$, and

$$\|\tilde{\gamma}'(t)\| = -\theta'(t)\|\gamma'(\theta(t))\|.$$

We have

$$\begin{aligned} L_a^b(\gamma) &= \int_a^b \|\gamma'(u)\|du \\ &= \int_d^c \|\gamma'(\theta(t))\|\theta'(t)dt \\ &= -\int_c^d \|\gamma'(\theta(t))\|\theta'(t)dt \\ &= \int_c^d \|\tilde{\gamma}'(t)\|dt \\ &= L_c^d(\tilde{\gamma}). \quad \square \end{aligned}$$

Theorem 2.6 *Any regular curve can be reparametrized so as to have unit speed everywhere.*

Proof. Let $\gamma : (a,b) \to \mathbb{R}^n$ be a regular curve and $s(t)$ be the arc length function. Since $s'(t) = ||\gamma^{'}(t)||$ is positive, s is strictly increasing. So s has inverse function $s^{-1} = h$. Therefore, by the inverse function theorem, it is smooth and $\frac{dh}{ds} = \frac{1}{||\gamma'(t)||}$. So if we put $\tilde{\gamma}(s) = (\gamma \circ h)(s)$, then $\tilde{\gamma}'(s) = \gamma'(h(s))\frac{dh}{ds} = \frac{\gamma'(t)}{||\gamma'(t)||}$, which implies $\tilde{\gamma}'$ is a unit vector. $\square$

Proposition 2.7 *There is no curve joining two points in $\mathbb{R}^n$ shorter than the line segment between them.*

Proof. Let $p, q \in \mathbb{R}^n$ with $p \neq q$. Let $\alpha : [a, b] \to \mathbb{R}^n$ be any curve such that $\alpha(a) = p$ and $\alpha(b) = q$. If we put $u = \frac{q-p}{|q-p|}$, then

$$\int_a^b \alpha'(t) \cdot u dt = \int_a^b (\alpha(t) \cdot u)' dt = \alpha(b) \cdot u - \alpha(a) \cdot u = (q-p) \cdot u = ||q-p||.$$

On the other hand, by the Schwarz inequality, we have

$$d(p, q) = ||q-p|| = \int_a^b \alpha'(t) \cdot u dt \leq \int_a^b ||\alpha'(t)|| dt = L(\alpha),$$

which implies the result. □

Definition 2.8 Let $\gamma : I \to \mathbb{R}^n$ be a unit speed curve and $T = \gamma'(s)$ a unit tangent vector field. The *principal curvature* of γ is defined by

$$\kappa := ||\frac{dT}{ds}|| = ||\gamma''(s)||. \tag{2.3}$$

Proposition 2.9 *A curve γ in $\mathbb{R}^n$ is a straight line if and only if $\kappa = 0$.*

Proof. By (2.3), $\kappa = 0$ if and only if $\gamma'' = 0$, which implies $\gamma'(t) = \mathbf{p}$=constant. Then, by integrating,

$$\gamma(t) = t\mathbf{p} + \mathbf{q},\ \mathbf{q} \in \mathbb{R}^n$$

which means γ is a straight line. □

2.2 Euclidean isometry

Let $\{e_1, \cdots, e_n\}$ be the standard basis of $\mathbb{R}^n$.

Definition 2.10 An *isometry* of $\mathbb{R}^n$ is a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$d(F(p), F(q)) = d(p, q) \tag{2.4}$$

for all point $p, q \in \mathbb{R}^n$.

Definition 2.11 Given point $\mathbf{a} \in \mathbb{R}^n$, $T_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T_{\mathbf{a}}(\mathbf{p}) = \mathbf{p} + \mathbf{a}$ is called the *translation* by $\mathbf{a}$.

Proposition 2.12 *If F is an isometry of $\mathbb{R}^n$ such that $F(0) = 0$, then F is an orthogonal transformation.*

Proof. Since F is an isometry, we have that for any $\mathbf{p}$ and $\mathbf{q}$

$$||F(\mathbf{p}) - F(\mathbf{q})|| = ||\mathbf{p} - \mathbf{q}||,$$

which implies

$$||F(\mathbf{p})||^2 - 2 < F(\mathbf{p}), F(\mathbf{q}) > +||F(\mathbf{q})||^2 = ||\mathbf{p}||^2 - 2 < \mathbf{p}, \mathbf{q} > +||\mathbf{q}||^2.$$

Since F preserves norms, we have

$$< F(\mathbf{p}), F(\mathbf{q}) >=< \mathbf{p}, \mathbf{q} > .$$

It remains to show that F is linear. Let $\mathbf{p} \in \mathbb{R}^n$. Then $\mathbf{p}$ is expressed by

$$\mathbf{p} = \sum_{i=1}^{n} p_i \mathbf{e_i}.$$

Since F preserves inner products, $\{F(\mathbf{e_1}), \cdots, F(\mathbf{e_n})\}$ is an orthonormal basis. Thus

$$F(\mathbf{p}) = \sum_{i=1}^{n} < F(\mathbf{p}), F(\mathbf{e_i}) > F(\mathbf{e_i}).$$

Since

$$< F(\mathbf{p}), F(\mathbf{e_i}) >=< \mathbf{p}, \mathbf{e_i} >= p_i,$$

we have

$$F(\mathbf{p}) = \sum_{i=1}^{n} p_i F(\mathbf{e_i}).$$

Using this identity, it is easy to check the linearity condition

$$F(a\mathbf{p} + b\mathbf{q}) = aF(\mathbf{p}) + bF(\mathbf{q}),$$

where a and $b \in \mathbb{R}$. Hence the proof is completed. □

Theorem 2.13 *If F is an isometry of $\mathbb{R}^n$, then there exist a unique translation $T_{\mathbf{a}}$ and a unique orthogonal transformation A such that $F = T_{\mathbf{a}}A$, $\mathbf{a} \in \mathbb{R}^n$.*

Proof. Let $T_{\mathbf{a}}$ be the translation by $\mathbf{a} = F(0)$. Then $T_{\mathbf{a}}^{-1}$ is the translation by $-F(0)$. That is, $T_{\mathbf{a}}^{-1} = T_{-\mathbf{a}}$. Since $T_{\mathbf{a}}^{-1}$ is an isometry, $T_{\mathbf{a}}^{-1}F$ is an isometry with

$$(T_{\mathbf{a}}^{-1}F)(0) = 0.$$

Thus, by Proposition 2.12, $T_{\mathbf{a}}^{-1}F$ is an orthogonal transformation, say A, i.e., $T_{\mathbf{a}}^{-1}F = A$. So $F = T_{\mathbf{a}}A$. Suppose that $F = \bar{T}_{\mathbf{b}}\bar{A}$, where $\bar{T}_{\mathbf{b}}$ is a translation and $\bar{A}$ is an orthogonal transformation. Since $T_{\mathbf{a}}A = \bar{T}_{\mathbf{b}}\bar{A}$, $A = T_{\mathbf{a}}^{-1}\bar{T}_{\mathbf{b}}\bar{A}$. Since A and $\bar{A}$ are linear transformations, it follows that $(T_{\mathbf{a}}^{-1}\bar{T}_{\mathbf{b}})(0) = 0$, which means $T_{\mathbf{a}}^{-1}\bar{T}_{\mathbf{b}} = I$. Hence $\bar{T}_{\mathbf{b}} = T_{\mathbf{a}}$ and so $A = \bar{A}$. Hence the proof is completed. □

Definition 2.14 Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. Then the *derivative map* $(F_*)_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ at p of F is defined by

$$(F_*)_p V = \frac{d}{dt}(F \circ \alpha)(t)|_{t=0},$$

where $\alpha(0) = p$ and $\alpha'(0) = V$.

It is well-known that $(F_*)_p$ is a linear transformation.

Lemma 2.15 (See [5]) *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be tangent vectors* $\mathbb{R}^3$ *at* $\mathbf{p}$. *If* F *is an isometry of* $\mathbb{R}^3$, *then*

$$(F_*)_p(\mathbf{v} \times \mathbf{w}) = (sgnF)(F_*)_p(\mathbf{v}) \times (F_*)_p(\mathbf{w}),$$

where $sgnF = \pm 1$ *is the sign of* F.

3 Frenet formulas

3.1 Frenet formula in $\mathbb{R}^3$

Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve with $\kappa \neq 0$, and $T = \gamma'$. Define $N = \frac{1}{\kappa}T'$, which is said to be a *normal vector field* along γ. And $B = T \times N$ is said to be a *binormal vector field* along γ. Then $\{T, N, B\}$ is called the *Frenet frame* along γ on $\mathbb{R}^3$.

Theorem 3.1 (**Frenet Formulas**). *For a unit speed curve $\gamma(s)$ with $\kappa > 0$, the derivatives of the Frenet frame are given by*

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{3.1}$$

where $\tau = - < N, B' >$ is the torsion of the curve β.

Proof. It is trivial $T' = \kappa N$. Since $\{T, N, B\}$ is an orthonormal basis, any vector field $\mathbf{V}$ along γ can be expressed by

$$\mathbf{V} =< T, \mathbf{V} > T+ < N, \mathbf{V} > N+ < B, \mathbf{V} > B.$$

Hence

$$\begin{aligned} N' &=< T, N' > T+ < N, N' > N+ < B, N' > B \\ &=< T, N' > T+ < B, N' > B \\ &= - < T', N > T- < B', N > B \\ &= -\kappa T + \tau B. \end{aligned}$$

Similarly, we have

$$\begin{aligned} B' &=< T, B' > T+ < N, B' > N \\ &= - < T', B > T+ < N, B' > N \\ &= -\kappa < N, B > T+ < N, B' > N \\ &= -\tau N. \quad \square \end{aligned}$$

Theorem 3.2 *Let $\gamma(s)$ be a unit speed curve with $\kappa > 0$. Then γ is a plane curve if and only if $\tau = 0$.*

Proof. Suppose $\tau = 0$. By the Frenet Formulas, $B' = 0$, so B is a constant. It is sufficient to show that for all s,

$$< \gamma(s) - \gamma(0), B >= 0.$$

By differentiating, we have

$$< \gamma(s) - \gamma(0), B >' =< \gamma'(s), B >=< T, B >= 0.$$

Hence $< \gamma(s) - \gamma(0), B >= constant$, which means that $< \gamma(s) - \gamma(0), B >= 0$ because at $s = 0$, it is zero.

Conversely, suppose γ is a plane curve. Hence, for any constant unit vector u, we have

$$< \gamma(s) - \gamma(0), n >= 0.$$

By differentiating, $< \gamma'(s), u >= 0$ and $< \gamma''(s), u >= 0$. That is, $< T, u >= 0$ and $< \kappa N, u >= 0$. These mean that $u \perp T$ and $u \perp N$. So $u = \pm B$, which means $B' = 0$. Hence $\tau = 0$. $\square$

Corollary 3.3 *A curve γ is a part of a circle if and only if $\kappa > 0$ is constant and $\tau = 0$.*

Proof. Suppose γ is part of a circle. Then γ is a plane curve and for any s

$$||\gamma(s) - p||^2 = r^2, p = \gamma(0),$$

where r is constant. By Theorem 3.2, $\tau = 0$. By differentiating, we have

$$< T, \gamma(s) - p >= 0. \tag{3.2}$$

If we differentiate again, then we have

$$0 =< T', \gamma(s) - p > + < T, T >= \kappa < N, \gamma(s) - p > +1.$$

Hence we have

$$\kappa < N, \gamma(s) - p >= -1, \tag{3.3}$$

which means $\kappa > 0$ and $< N, \gamma(s) - p > \neq 0$. By differentiating (3.3), we have

$$
\begin{aligned}
0 &= \kappa' < N, \gamma(s) - p > + \kappa < -\kappa T + \tau B, \gamma(s) - p > \\
&= \kappa^{'} < N, \gamma(s) - p >,
\end{aligned}
$$

because $\tau = 0$ and (3.2). This means that $\kappa' = 0$, by (3.3) and so κ is constant.
Conversely, suppose $\kappa > 0$ is constant and $\tau = 0$. If we put

$$\beta(s) = \gamma(s) + \frac{1}{\kappa} N,$$

then $\beta'(s) = 0$ because of $\tau = 0$. This implies β is constant. If we put $\beta(s) = p$, then

$$||\gamma(s) - p|| = || - \frac{1}{\kappa} N|| = \frac{1}{\kappa},$$

so p is the center of a circle γ with radius $\frac{1}{\kappa}$. $\square$

Definition 3.4 A regular curve γ is a *helix*, if $< T, \mathbf{u} >$ is constant for some fixed unit vector $\mathbf{u}$.

Theorem 3.5 *A unit speed curve $\gamma(s)$ with $\kappa \neq 0$ is a helix if and only if $\frac{\tau}{\kappa}$ is constant.*

Proof. Assume γ is a helix. Since $< T, \mathbf{u} >$ is constant, we may write $< T, \mathbf{u} > = \cos\theta$, where θ is some fixed angle.
Case 1 : If $\theta = \lambda\pi$ $(\lambda = 0, 1)$, then $T(s) = \pm\mathbf{u}$. Since $\mathbf{u}$ is constant, $T' = \kappa N = 0$, and then $\kappa = 0$, which is contradiction.
Case 2 : If $\theta \neq \lambda\pi$ $(\lambda = 0, 1)$, then $0 = < T, \mathbf{u} >' = < \kappa\mathbf{N}, \mathbf{u} >$. So $N \perp \mathbf{u}$. If we put

$$\mathbf{u} = < T, \mathbf{u} > T + < B, \mathbf{u} > B,$$

then

$$
\begin{aligned}
0 = \mathbf{u}^{'} &= \cos\theta T' + \sin\theta B' \\
&= \cos\theta\kappa N + \sin\theta(-\tau N) \\
&= (\kappa\cos\theta - \tau\sin\theta) N.
\end{aligned}
$$

Hence $\frac{\tau}{\kappa} = \frac{\cos\theta}{\sin\theta}$ is constant.

Conversely, assume that $\frac{\tau}{\kappa} = \mathbf{c}$, for constant $\mathbf{c}$. Let $\mathbf{c} = \cot\theta$ with $0 < \theta < \pi$, and let $\mathbf{u} = \cos\theta T + \sin\theta B$. Then

$$\begin{aligned}
\mathbf{u}' &= \cos\theta T' + \sin\theta B' \\
&= \kappa\cos\theta N - \tau\sin\theta N \\
&= (\cos\theta - \frac{\cos\theta}{\sin\theta}\sin\theta)\kappa N \\
&= 0.
\end{aligned}$$

Hence $\mathbf{u}$ is constant vector and $< T, \mathbf{u} >= \cos\theta$, which is constant. Hence γ is a helix. $\square$

Remark 3.6 *In $\mathbb{R}^2$, we give the geometric meaning of k of the unit speed curve γ. Since γ is a unit speed curve in $\mathbb{R}^2$, we can consider*

$$\gamma'(s) = (\cos\theta(s), \sin\theta(s)),$$

where θ is an angle between γ' and $e_1 = (1, 0)$. Also, we take

$$N = (-\sin\theta(s), \cos\theta(s))$$

and then

$$\kappa N = T' = \theta'(s)(-\sin\theta(s), \cos\theta(s)) = \theta'(s)N.$$

So we have

$$\kappa = \theta'(s).$$

3.2 The sphere curves

A curve γ is called the *sphere curve* if γ satisfies $||\gamma(s) - m||^2 = r^2$ for some constants m and r.

Theorem 3.7 *Let γ be a unit speed curve with $\kappa \neq 0$ and $\tau \neq 0$. Then γ is a sphere curve if and only if $(\frac{1}{\kappa})^2 + (\frac{\kappa'}{\kappa^2\tau})^2 = r^2$, $r > 0$ is constant.*

Proof. Let γ be a sphere curve on the sphere of radius r and center m. That is, $< \gamma(s) - m, \gamma(s) - m >= r^2$. By differentiating,

$$< \gamma(s) - m, T >= 0.$$

By differentiating, we also have

$$\begin{aligned}0 =< \gamma(s) - m, T >^{'} &=< T, T > + < \gamma(s) - m, T' > \\ &= 1 + < \gamma(s) - m, \kappa N >,\end{aligned}$$

which implies $-1 = \kappa < \gamma(s) - m, N >$ and $\kappa \neq 0$. Now, for $a, b, c \in \mathbb{R}$, if we put

$$\gamma(s) - m = aT + bN + cB,$$

then

$$a =< \gamma(s) - m, T >= 0$$

and

$$b =< \gamma(s) - m, N >= -\frac{1}{\kappa}.$$

By differentiating, we have for $\rho = \frac{1}{\kappa}$,

$$\begin{aligned}-\rho^{'} &=< \gamma(s) - m, N >^{'} \\ &=< T, N > + < \gamma(s) - m, N' > \\ &=< \gamma(s) - m, -\kappa T + \tau B > \\ &= -\kappa < \gamma(s) - m, T > + \tau < \gamma(s) - m, B > \\ &= \tau < \gamma(s) - m, B > .\end{aligned}$$

So $c =< \gamma(s) - m, B >= -\rho^{'}\sigma$, where $\sigma = \frac{1}{\tau}$. Hence

$$\gamma(s) - m = -\rho N - \rho^{'}\sigma B,$$

which yields

$$\begin{aligned}r^2 = ||\gamma(s) - m||^2 &= || - \rho N - \rho^{'}\sigma B||^2 \\ &= \rho^2 + (\rho^{'}\sigma)^2.\end{aligned}$$

Conversely, suppose $\rho^2 + (\rho^{'}\sigma)^2 = r^2$ and $\rho^{'} \neq 0$. Then

$$2\rho^{'}\rho + 2(\rho^{'}\sigma^{'} + \rho^{''}\sigma)(\rho^{'}\sigma) = 0.$$

Since $\rho^{'} \neq 0$, we have

$$\rho^{'}\sigma^{'} + \rho^{''}\sigma = -\frac{\rho}{\sigma}.$$

Let $\gamma(s) + \rho N + \rho' \sigma B = m(s)$. Then, by differentiating, we have

$$\begin{aligned} m' &= \gamma'(s) + \rho' N + \rho N' + (\rho'' \sigma + \rho' \sigma')B + \rho' \sigma B' \\ &= T + \rho' N + \rho(-\kappa T + \tau B) - \rho \tau B + \rho' \sigma(-\tau N) \\ &= 0. \end{aligned}$$

Hence m is constant and so

$$||\gamma(s) - m||^2 = \rho^2 + (\rho' \sigma)^2 = r^2.$$

Thus $\gamma(s)$ is a sphere curve. □

Let γ be a unit speed curve. Since $||T(s)|| = 1$, $T(s)$ is a sphere curve. Similarly $N(s)$, $B(s)$ are sphere curves. Hence $T(s)$ is called the *tangent spherical image* of γ. And $N(s)$ and $B(s)$ are called the *normal spherical* and the *binormal spherical image* of γ, respectively.

Theorem 3.8 *Let γ be a unit speed curve in $\mathbb{R}^3$. Then*

(1) γ is a straight line if and only if the tangent spherical image of γ is a constant.

(2) γ is a plane curve if and only if the binormal spherical image of γ is a constant.

(3) γ is a helix if and only if the tangent spherical image is an arc of a circle.

Proof. (1) Suppose that $T(s)$ is a constant. Then $\kappa(s) = ||T'(s)|| = 0$. Thus $\gamma(s)$ is a straight line. Conversely, suppose that $\gamma(s)$ is a straight line. Then $\kappa(s) = 0$. Hence $\kappa(s) = ||T'(s)|| = 0$, which means T is constant.

(2) Suppose that $\gamma(s)$ is a plane curve. Then $\tau(s) = 0$, and so $B'(s) = 0$. Hence $B(s)$ is constant. Conversely, suppose that $B(s)$ is constant. Then

$$\tau(s) = - < B'(s), N(s) >= - < 0, N(s) >= 0.$$

Thus $\gamma(s)$ is a plane curve.

(3) Suppose that $\gamma(s)$ is a helix. Then $< T, \mathbf{u} >= C$ is a constant for some constant unit vector $\mathbf{u}$. Since

$$\begin{aligned} < T - T(t_0), \mathbf{u} > &=< T, \mathbf{u} > - < T(t_0), \mathbf{u} > \\ &= C - C = 0, \end{aligned}$$

$T(t)$ is a plane curve. Hence $T(t)$ lies on the sphere and plane, which means $T(t)$ is a part of a circle. Conversely, suppose that $T(t)$ is a part of a circle. Then $T(t)$ is a plan curve. So $< T - T(t_0), \mathbf{u} >= 0$ for some constant unit vector $\mathbf{u}$. Hence $< T, \mathbf{u} >=< T(t_0), \mathbf{u} >$ is constant. Thus $\gamma(s)$ is a helix. □

3.3 Bertrand curves

Definition 3.9 Two curves α and β are called *Bertrand curves* if for each s_0, the normal line to α at $s = s_0$ is the same as the normal line to β at $s = s_0$. That is, $N_\alpha = \pm N_\beta$. In this case, we say that β is a *Bertrand mate* of α.

Proposition 3.10 *The distance between corresponding points of a pair of Bertrand curves is constant.*

Proof. Let α be a unit speed curve and β be a Bertrand mate of α. Then there is a function $\lambda(s)$ such that $\beta(s) = \alpha(s) + \lambda(s)N_\alpha(s)$. Generally, β is not unit speed. We must show $||\beta(s) - \alpha(s)|| = \lambda = constant$. In fact,we have

$$\begin{aligned}\beta^{'}(s) &= T_\alpha(s) + \lambda^{'}(s)N_\alpha(s) + \lambda(s)N_\alpha'(s)\\ &= (1 - \lambda\kappa_\alpha)T_\alpha(s) + \lambda^{'}(s)N_\alpha(s) + \lambda\tau_\alpha B_\alpha(s).\end{aligned}$$

And so

$$\lambda^{'}(s) =< \beta^{'}(s), N_\alpha(s) >=< \beta^{'}(s), \pm N_\beta(s) >= 0.$$

Hence $\lambda(s)$ is constant. □

Proposition 3.11 *The angle between the tangents to two Bertrand curves at corresponding points is constant.*

Proof. Since $N_\alpha = \pm N_\beta$, we have

$$T_\alpha^{'} = \kappa_\alpha N_\alpha = \pm\kappa_\alpha N_\beta, \quad T_\beta^{'} = \kappa_\beta N_\beta = \pm\kappa_\beta N_\alpha.$$

Then

$$\begin{aligned}< T_\alpha(s), T_\beta(s) >^{'} &=< T_\alpha'(s), T_\beta(s) > + < T_\alpha(s), T_\beta'(s) >\\ &=< \pm\kappa_\alpha N_\beta(s), T_\beta(s) > + < T_\alpha(s), \pm\kappa_\beta N_\alpha(s) >= 0.\end{aligned}$$

Hence $< T_\alpha(s), T_\beta(s) >$ is constant. So the proof is completed. □

Theorem 3.12 *Let α be a unit speed curve with $\kappa\tau \neq 0$. Then there is a Bertrand mate $\beta(s)$ of α if and only if there are constants $\lambda \neq 0$ and μ with $\frac{1}{\lambda} = \kappa + \mu\tau$.*

Proof. Let β be a Bertrand mate of α. Then for some constant $\lambda \neq 0$, $\beta(s) = \alpha(s) + \lambda N_\alpha(s)$. Since $\beta'(s) = (1 - \lambda\kappa_\alpha)T_\alpha + \lambda\tau_\alpha B_\alpha$, we have

$$\begin{aligned}\cos\theta = < T_\alpha, T_\beta > &= < T_\alpha, \frac{\beta'(s)}{||\beta'(s)||} > \\ &= \frac{1}{||\beta'(s)||}(1 - \lambda\kappa_\alpha),\end{aligned}$$

$$\begin{aligned}\sin\theta = ||T_\alpha \times T_\beta|| &= ||T_\alpha \times \frac{\beta'(s)}{||\beta'(s)||}|| \\ &= \frac{1}{||\beta'(s)||}||T_\alpha \times \beta'(s)|| \\ &= \frac{1}{||\beta'(s)||}\lambda\tau_\alpha \neq 0,\end{aligned}$$

where θ is constant by Proposition 3.11. So $\frac{\cos\theta}{\sin\theta} = \frac{1-\lambda\kappa}{\lambda\tau} = \mu$ is constant. This implies that $\frac{1}{\lambda} = \kappa + \mu\tau$. Conversely, let $\beta(s) = \alpha(s) + \lambda N_\alpha(s)$. Then

$$\beta'(s) = \lambda\tau(\mu T_\alpha + B_\alpha)$$

and

$$\beta'' = \lambda\tau'\mu T_\alpha + (\lambda\tau\mu\kappa - \lambda\tau^2)N_\alpha + \lambda\tau' B_\alpha.$$

So we have

$$\beta'(s) \times \beta''(s) = \lambda\tau\mu(\lambda\tau\mu\kappa - \lambda\tau^2)B_\alpha - (\lambda\tau\mu\kappa - \lambda\tau^2)\lambda\tau T_\alpha.$$

By the direct calculation, we have

$$(\beta'(s) \times \beta''(s)) \times \beta'(s) = (\lambda\tau\mu\kappa - \lambda\tau^2)(\lambda\tau)^2(1 + \mu^2)N_\alpha.$$

Thus

$$N_\beta = \frac{(\beta'(s) \times \beta''(s)) \times \beta'(s)}{||(\beta'(s) \times \beta''(s)) \times \beta'(s)||} = \pm N_\alpha.$$

That is, β is a Bertrand mate of α. $\Box$

3.4 Involutes and Evolutes

Let α be a unit speed curve on an interval (a, b). Then β is an *involute* of α if $\beta(s) = \alpha(s) + (c - s)T_\alpha(s)$, where c is a constant and $T_\alpha = \alpha'$. And β is an *evolute* of α if α is an involute of β.

Proposition 3.13 *(1) An involute of a plane curve lies in that same plane.*

(2) An involute of a helix is a plane curve.

Proof. (1) Let α be a plane curve and $\beta(s)$ be an involute of α. Then, by definition,

$$\beta(s) = \alpha(s) + (c - s)T_\alpha(s). \tag{3.4}$$

From (3.4), we have

$$< \beta(s) - \alpha(s), B_\alpha(s) >=< (c - s)T_\alpha(s), B_\alpha(s) >= 0.$$

Since B_α is constant, β is also plane curve, which lies in the same plane.

(2) Let $\alpha(s)$ be a helix and $\beta(s)$ be a involute curve. Then, by (3.4), we have

$$\beta'(s) = (c - s)T'_\alpha(s)$$

and

$$||\beta'(s)|| = \kappa_\alpha(c - s).$$

Hence we have

$$T_\beta = \frac{\beta'(s)}{||\beta'(s)||} = \frac{(c - s)T'_\alpha(s)}{\kappa_\alpha(c - s)} = \frac{1}{\kappa_\alpha}T'_\alpha(s),$$

which implies $T'_\alpha(s) = \kappa_\alpha T_\beta$. Since $\alpha(s)$ is helix, $< T_\alpha, \mathbf{u} >$=constant for some constant unit vector $\mathbf{u}$. So we have

$$0 =< T_\alpha, \mathbf{u} >' = \kappa_\alpha < T_\beta, \mathbf{u} > .$$

If $\kappa_\alpha = 0$, then $\alpha(s)$ is straight line. Hence by (1), it is a plane curve. If $\kappa_\alpha \neq 0$, then $< T_\beta, \mathbf{u} >= 0$. By differentiating,

$$0 =< T_\beta, \mathbf{u} >' = \kappa_\beta < N_\beta, \mathbf{u} > .$$

If $\kappa_\beta = 0$, then $\beta(s)$ is straight line and so β is a plane curve. If $\kappa_\beta \neq 0$, then $< N_\beta, \mathbf{u} >= 0$. Hence $\mathbf{u} = \pm B_\beta$ and

$$0 =< N'_\beta, \mathbf{u} >= -\kappa_\beta < T_\beta, \mathbf{u} > +\tau_\beta < B_\beta, \mathbf{u} >= \pm\tau_\beta.$$

Hence $\beta(s)$ is plane curve. Thus, an involute of a helix is a plane curve. □

Theorem 3.14 *Let α be a unit speed curve and β be an evolute of α. Then $\beta(s) = \alpha(s) + \frac{1}{\kappa}N + \mu B$, where $\mu = \frac{1}{\kappa}\cot(\int_a^s \tau ds + constant)$.*

Proof. Let q be a point on the evolute curve β corresponding to the point p on a reparametrized curve α. Since $\beta(q) - \alpha(p)$ is orthogonal to T_α, in the normal plane of α at q. Hence $\beta - \alpha = \lambda N + \mu B$. But

$$\begin{aligned}\beta^{'}(s) &= T_\alpha + \lambda^{'}N + \lambda N' + \mu^{'}B + \mu B' \\ &= (1 - \lambda\kappa)T_\alpha + (\lambda^{'} - \mu\tau)N + (\mu^{'} + \lambda\tau)B\end{aligned}$$

is the tangent to β and so parallel to $\beta - \alpha$. Thus

$$1 - \lambda\kappa = 0, \quad \frac{\lambda^{'} - \mu\tau}{\lambda} = \frac{\mu^{'} + \lambda\tau}{\mu}.$$

That is

$$\lambda = \frac{1}{\kappa}, \quad \tau = \frac{\lambda^{'}\mu - \lambda\mu^{'}}{\lambda^2 + \mu^2} = \frac{d}{ds}\cot^{-1}\frac{\mu}{\lambda}.$$

Thus

$$\int_a^s \tau ds + C = \cot^{-1}\frac{\mu}{\lambda}, \quad C : constant$$

$$\mu = \lambda\cot(\int_a^s \tau ds + C).$$

Therefore

$$\beta(s) = \alpha(s) + \frac{1}{\kappa}N + \mu B. \quad \square$$

3.5 Frenet formula in Minkowski space $\mathbb{R}^3_1$

Let us define the metric $<, >_\circ$ on $\mathbb{R}^3$ by

$$< X, Y >_\circ = -x_1y_1 + x_2y_2 + x_3y_3, \tag{3.5}$$

where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$. Then $\mathbb{R}^3_1 = (\mathbb{R}^3, <, >_\circ)$ is called the *Minkowski space (or Lorentz space)*.

Definition 3.15 A vector X on $\mathbb{R}^3_1$ is said to be *:*

$$\begin{array}{lll} space-like, & \text{if} & < X, X >_\circ > 0, \\ time-like, & \text{if} & < X, X >_\circ < 0, \\ light-like, & \text{if} & < X, X >_\circ = 0, X \neq 0. \end{array}$$

Definition 3.16 A regular curve $\gamma : I \to \mathbb{R}^3_1$ is called *space-like, time-like* and *light-like* if for $\forall t \in I$, $\gamma'(t)$ is space-like, time-like and light-like, respectively.

Example 3.17 The curve $\alpha(t) = (\cosh t, \sinh t, 0)$ is space-like and $\beta(t) = (\sinh t, \cosh t, 0)$ is time-like and $\gamma(t) = (t, t, 0)$ is light-like.

Definition 3.18 A *vector product* $V \times W$ on $\mathbb{R}^3_1$ is defined by

$$< V \times W, U >_\circ = det(V, W, U) \tag{3.6}$$

for all U. That is, for any $V = (v_1, v_2, v_3)$ and $W = (w_1, w_2, w_3)$,

$$V \times W = (-v_2 w_3 + v_3 w_2, -v_1 w_3 + v_3 w_1, v_1 w_2 - v_2 w_1).$$

Now, we can define three frames as follows. For two vectors e_1 and e_2 such that $< e_i, e_i >_\circ = \pm 1$, $< e_1, e_2 >_\circ = 0$, a third is defined by $e_3 = e_1 \times e_2$. Then $< e_3, e_3 >_\circ = 1$. So $\{e_1, e_2, e_3\}$ form an orthonormal frame field. Generally, if we put $\epsilon, \eta \in \{1, -1\}$ by $< e_1, e_1 >_\circ = \epsilon$, and $< e_2, e_2 >_\circ = \eta$, then $< e_3, e_3 >_\circ = -\epsilon\eta$. The frame $\{e_1, e_2, e_3\}$ is said to be an *orthonormal basis* of $\mathbb{R}^3_1$.

Lemma 3.19 *Any vector X on $\mathbb{R}^3_1$ can be uniquely decomposed as*

$$X = \epsilon < X, e_1 >_\circ e_1 + \eta < X, e_2 >_\circ e_2 - \epsilon\eta < X, e_3 >_\circ e_3, \tag{3.7}$$

where $\{e_1, e_2, e_3\}$ is an orthonormal basis.

Proof. Let $X = a_1 e_1 + a_2 e_2 + a_3 e_3$ for $a_i \in \mathbb{R}(i = 1, 2, 3)$. Then

$$< X, e_1 >_\circ = a_1 < e_1, e_1 >_\circ = a_1 \epsilon.$$

Hence $a_1 = \epsilon < X, e_1 >_\circ$. Similarly, $a_2 = \eta < X, e_2 >_\circ$ and $a_3 = -\epsilon\eta < X, e_3 >_\circ$.□

Theorem 3.20 (**Frenet formula in** $\mathbb{R}^3_1$). *Let γ be a space-like or time-like curve by unit speed and $< \gamma'', \gamma'' >_\circ \neq 0$. Then $\{T = \gamma', N = \frac{\gamma''}{\sqrt{|<\gamma'', \gamma''>_\circ|}}, B = T \times N\}$ form an orthonormal frame field and satisfies*

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa\eta & 0 \\ -\kappa\epsilon & 0 & -\tau\epsilon\eta \\ 0 & -\tau\eta & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{3.8}$$

where $\kappa = < T', N >_\circ$ and $\tau = < N', B >_\circ$ are called the curvature and torsion of the curve γ.

Proof. Let γ be a space-like (or time-like) with unit speed. We have

$$T' = \epsilon < T', T >_\circ T + \eta < T', N >_\circ N - \epsilon\eta < T', B >_\circ B,$$

because $< T, T >_\circ = \epsilon$ and $< N, N >_\circ = \eta$. By the definition of N, $T' = \gamma'' = \sqrt{| < \gamma'', \gamma'' >_\circ |} N$, and so $\sqrt{| < \gamma'', \gamma'' >_\circ |} = \eta < T', N >_\circ = \eta\kappa$. Hence $T' = \kappa\eta N$. Next, we have

$$N' = \epsilon < N', T >_\circ T + \eta < N', N >_\circ N - \epsilon\eta < N', B >_\circ B.$$

Since

$$< N', T >_\circ = - < T', N >_\circ = -\kappa,$$

$$< B', N >_\circ = - < N', B >_\circ = -\tau,$$

we have $N' = -\kappa\epsilon T - \tau\epsilon\eta B$. Similarly, we have $B' = -\tau\eta B$. □

3.6 Frenet formula in $\mathbb{R}^n$

Definition 3.21 Let $\gamma : I \to \mathbb{R}^n$ be a unit speed curve. The generalized *Frenet frame field* $\{v_1 = \gamma'(t), v_2, \cdots, v_n\}$ in $\mathbb{R}^n$ is defined as follows: if we define w_{j+1} by

$$w_{j+1} = v_j' - \sum_{i=1}^{j} < v_j', v_i > v_i, \tag{3.9}$$

then

$$v_{j+1} = \frac{1}{\kappa_{j+1}} w_{j+1}, \quad \kappa_{j+1} := |w_{j+1}|. \tag{3.10}$$

Theorem 3.22 (***Frenet formulas***) *Let $\gamma : I \to \mathbb{R}^n$ be a unit speed curve with $\kappa_j \neq 0$. Then we have*

$$\begin{pmatrix} v_1' \\ \vdots \\ \vdots \\ \vdots \\ v_n' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_2 & 0 & \cdots & 0 \\ -\kappa_2 & 0 & \kappa_3 & \cdots & 0 \\ 0 & -\kappa_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \kappa_n \\ 0 & 0 & 0 & -\kappa_n & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ \vdots \\ v_n \end{pmatrix}. \tag{3.11}$$

Proof. Let $V(t) = (v_1 \cdots v_n)^T$ be the orthogonal matrix since rows are orthonormal basis for $\mathbb{R}^n$. So

$$V(t)V(t)^T = I \Longrightarrow V'(t)V(t)^T + V(t)V'(t)^T = 0.$$

Since $v_j' = \kappa_{j+1}v_{j+1} + \sum_{i=1}^{j} < v_j', v_i > v_i$, we can write

$$V'(t) = K(t)V(t),$$

where K_{ij} are of $< v_j', v_i >$ or κ_{j+1}. $\square$

Theorem 3.23 *If $\kappa_n = 0$, then the curve lies in a hyperplane.*

Proof. Since $\kappa_i \neq 0 (i = 1, \cdots, n-1)$, we define an orthonormal vectors $\{v_1, \cdots, v_{n-1}\}$. Let $W = (v_1, \cdots, v_{n-1})^T$. Now we define unit v_n so that

$$\det(v_1, \cdots, v_n)^T = 1.$$

If we put $A = (v_1, \cdots, v_n)^T$, then

$$1 = \det(AA^T) =< v_n, v_n > - < v_1, v_n >^2 - \cdots - < v_{n-1}, v_n >^2 .$$

Since $< v_n, v_n >= 1$, we have $< v_1, v_n >= \cdots =< v_{n-1}, v_n >= 0$. Hence $< W, v_n >= 0$. Differentiating $< W, v_n >= 0$, we have

$$< W', v_n > + < W, v_n' >= 0.$$

Since $\kappa_n = 0$, W' is linear combination of $v_1, \cdots, v_{n-1}$. Hence $< W', v_n >= 0$. So

$$< W, v_n' >= 0.$$

i.e.,$< v_j, v_n' >= 0$ for $j = 1, \cdots, n-1$. Also, since $< v_n, v_n' >= 0$, $v_n' = 0$, i.e., v_n is constant. Now,

$$\begin{aligned} < \gamma, v_n >' &= < \gamma', v_n > + < \gamma, v_n' > \\ &= < v_1, v_n > + < \gamma, 0 >= 0. \end{aligned}$$

Hence we get $< \gamma, v_n >=$ constant. i.e., γ lies in hyperplane. $\square$

Theorem 3.24 (***Fundamental theorem of Frenet theory***). *Given any smooth $\kappa_2, \cdots, \kappa_n$ on $(a,b) \subset \mathbb{R}$ such that $\kappa_i > 0,\ i < n$. Then there exists a unit speed curve $\gamma : (a,b) \to \mathbb{R}^n$ with these curvatures and γ is unique up to oriented isometry of $\mathbb{R}^n$.*

Proof. Consider the matrix ordinary differential equation with an initial value. i.e.,

$$V' = KV, \quad V(a) = I.$$

From the fundamental theorem of ordinary differential equations (linear case), there exists a solution to this equation. □

Theorem 3.25 *Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be an isometry. Let β be a unit speed curve in $\mathbb{R}^3$ with $\kappa_\beta > 0$. If $\alpha = F \circ \beta$, then $\kappa_\beta = \kappa_\alpha$, $\tau_\beta = (sgnF)\tau_\alpha$.*

Proof. Since β is a unit speed curve, $\alpha = F \circ \beta$ is also a unit speed curve. Hence we have

$$T_\alpha = F_*(T_\beta).$$

Since F_* preserves both acceleration and norm, we have

$$\kappa_\alpha = ||\alpha''|| = ||F_*(\beta'')|| = ||\beta''|| = \kappa_\beta.$$

Moreover, by definition, $N = \frac{1}{\kappa}T'$. Hence

$$N_\alpha = \frac{\alpha''}{\kappa_\alpha} = \frac{F_*(\beta'')}{\kappa_\beta} = F_*(\frac{\beta''}{\kappa_\beta}) = F_*(N_\beta).$$

This implies that, by Lemma 2.15,

$$B_\alpha = T_\alpha \times N_\alpha = F_*(T_\beta) \times F_*(N_\beta) = (sgnF)F_*(T_\beta \times N_\beta) = (sgnF)F_*(B_\beta).$$

Hence, by the definition of the torsion,

$$\tau_\alpha =< N_\alpha', B_\alpha >=< F_*(N_\beta'), (sgnF)F_*(B_\beta) >= (sgnF) < N_\beta', B_\beta >= (sgnF)\tau_\beta. \quad \square$$

4 Bishop formulas

4.1 Bishop formula in $\mathbb{R}^3$

Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve.

Definition 4.1 A vector field N along a curve γ is called *normal* if $< N, \gamma' >= 0$ for all t.

Note that the Frenet vector fields $\{N, B\}$ are normals when they exist.

Definition 4.2 A normal vector field N on γ is called *parallel* if N' has no component perpendicular to the curve. i.e., $N' = \lambda\gamma'$, where λ is a real valued function.

Proposition 4.3 *There exist parallel orthonormal normal vector fields along any C^2 curve γ in $\mathbb{R}^3$.*

Proof. Let γ be a smooth curve. Let ω_1 and ω_2 be normal vector fields along γ such that

$$< \omega_i, \omega_j >= \delta_{ij}.$$

Now, we construct parallel orthonormal normal normal vector fields $\{N_1, N_2\}$. This is equivalent to exist an orthogonal 2×2 matrix $A = \{a_{ij}\}$ such that

$$N_i = \sum_j a_{ij}\omega_j$$

is parallel. Assume $N_i^{'} = \lambda_i\gamma^{'}$. Then $< N_i^{'}, \omega_j >= 0$ for all i, j. Equivalently,

$$< a_{ij}^{'}\omega_j + a_{ij}\omega_j^{'}, \omega_k >= 0.$$

That is

$$\delta_{jk}a_{ij}^{'} + < \omega_j^{'}, \omega_k > a_{ij} = 0.$$

By the existence of the solutions of linear ordinary differential equation, the solutions (a_{ij}) exist. This means that $\{N_i = \sum_j a_{ij}\omega_j\}$ are parallel orthogonal normal vector fields. □

Definition 4.4 Let $\{N_1, N_2\}$ be the parallel orthonormal normal vector fields along γ with $N_1^{'} = \bar{\kappa}_1\gamma^{'}, N_2^{'} = \bar{\kappa}_2\gamma^{'}$. These $\bar{\kappa}_1$ and $\bar{\kappa}_2$ are called the *Bishop curvatures* of γ and $\{T, N_1, N_2\}$ are called the *Bishop frames*.

Theorem 4.5 (**Bishop Formula**). *For a unit speed curve $\gamma \in \mathbb{R}^3$, we have*

$$\begin{pmatrix} T' \\ N_1' \\ N_2' \end{pmatrix} = \begin{pmatrix} 0 & -\bar{\kappa}_1 & -\bar{\kappa}_2 \\ \bar{\kappa}_1 & 0 & 0 \\ \bar{\kappa}_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ N_1 \\ N_2 \end{pmatrix}. \tag{4.1}$$

Proof. Let γ be a unit speed curve and $T = \gamma'$. Then

$$\begin{aligned} T' &= < T', T > T + < T', N_1 > N_1 + < T', N_2 > N_2 \\ &= -\bar{\kappa}_1 N_1 - \bar{\kappa}_2 N_2. \end{aligned}$$

Others are trivial from the definition. □

Theorem 4.6 *Let γ be a unit speed curve in $\mathbb{R}^3$. Then $\kappa^2 = \bar{\kappa}_1^2 + \bar{\kappa}_2^2$.*

Proof. Let γ be a unit speed curve. Then, by the Bishop's formula, we have

$$\begin{aligned} \kappa^2 &= ||\gamma''||^2 = ||T'||^2 \\ &= \bar{\kappa}_1^2 < N_1, N_1 > + 2\bar{\kappa}_1\bar{\kappa}_2 < N_1, N_2 > + \bar{\kappa}_2^2 < N_2, N_2 > \\ &= \bar{\kappa}_1^2 + \bar{\kappa}_2^2. \quad \Box \end{aligned}$$

Theorem 4.7 *Let $\gamma : I \to \mathbb{R}^3$ be a smooth curve. Then $\bar{\kappa}_1 = -\kappa \cos\theta$, $\bar{\kappa}_2 = \kappa \sin\theta$, where $\theta(t) = \int \tau(u)du$+constant.*

Proof. Let $\{T, N, B\}$ be a Frenet frame and $\{T, N_1, N_2\}$ be Bishop frame. Then

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

and

$$\begin{pmatrix} T' \\ N_1' \\ N_2' \end{pmatrix} = \begin{pmatrix} 0 & -\bar{\kappa}_1 & -\bar{\kappa}_2 \\ \bar{\kappa}_1 & 0 & 0 \\ \bar{\kappa}_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ N_1 \\ N_2 \end{pmatrix}.$$

Since $\{N, B\} \perp T$ and $\{N_1, N_2\} \perp T$, we can put

$$\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} N \\ B \end{pmatrix}.$$

So

$$\begin{aligned} N_1' &= -\theta' \sin\theta N + \cos\theta N' + \theta' \cos\theta B + \sin\theta B' \\ &= -\theta' \sin\theta N + \cos\theta(-\kappa T + \tau B) + \theta' \cos\theta B + \sin\theta(-\tau N). \end{aligned}$$

Hence

$$\bar{\kappa}_1 T = -\kappa\cos\theta T + (-\theta' \sin\theta - \tau\sin\theta)N + (\theta' \cos\theta + \tau\cos\theta)B.$$

Similarly,

$$\bar{\kappa}_2 T = \kappa\sin\theta T + (-\theta' \cos\theta - \tau\cos\theta)N + (-\theta' \sin\theta - \tau\sin\theta)B.$$

Thus

$$\bar{\kappa}_1 = -\kappa\cos\theta,\ \ \bar{\kappa}_2 = \kappa\sin\theta, \tag{4.2}$$

$$\theta' \sin\theta + \tau\sin\theta = 0,\ \ \tau\cos\theta + \theta' \cos\theta = 0.$$

Then $\theta' = -\tau$, and

$$\theta(t) = -\int_a^t \tau(u)du + C. \quad \Box \tag{4.3}$$

Theorem 4.8 *Let γ be a unit speed curve in $\mathbb{R}^3$ with $\kappa > 0$. Then* $(\bar{\kappa}_1')^2 + (\bar{\kappa}_2')^2 = (\kappa')^2 + \kappa^2\tau^2$.

Proof. By Theorem 4.7, we have

$$\bar{\kappa}_1 = -\kappa\cos\theta = -\kappa\cos(-\int_a^t \tau(u)du + C),$$

$$\bar{\kappa}_2 = \kappa\sin\theta = -\kappa\sin(-\int_a^t \tau(u)du + C).$$

By differentiating (4.2),

$$\bar{\kappa}_1' = -\kappa' \cos\theta - \kappa\tau\sin\theta \quad , \quad \bar{\kappa}_2' = \kappa' \sin\theta - \kappa\tau\cos\theta.$$

Hence we have $(\bar{\kappa}_1')^2 + (\bar{\kappa}_2')^2 = (\kappa')^2 + \kappa^2\tau^2$. $\Box$

Notation : $\bar{\kappa}(t) = (\bar{\kappa}_1(t), \bar{\kappa}_2(t))$ is considered as a curve in $\mathbb{R}^2$.

Corollary 4.9 *Let γ be a unit speed curve. Then γ is a straight line if and only if $\bar{\kappa} = 0$. And γ is a circle if and only if $\bar{\kappa}$ = constant $\neq 0$.*

Proof. This follows from Proposition 2.9 and Corollary 3.3. □

Corollary 4.10 *Let $\frac{\bar{\kappa}_2}{\bar{\kappa}_1}$ = constant. Then γ is plane curve.*

Proof. From (4.2), we have

$$\frac{\bar{\kappa}_2}{\bar{\kappa}_1} = -\tan\theta = constant.$$

So $\theta' = 0$, which implies $\tau = 0$. Therefore, γ is a plane curve. □

4.2 Bishop formula in $\mathbb{R}^n$

Let $\gamma : I \to \mathbb{R}^n$ be a unit speed curve and $\{N_1, \cdots, N_{n-1}\}$ be a fixed parallel normal vector fields. Let $T = \gamma'$ be the unit tangent vector field. Define for any $i = 1, \cdots, n-1$, by

$$\bar{\kappa}_i = < N_i', T > = < N_i', \gamma' > . \tag{4.4}$$

Since N_i is parallel, i.e., $N_i' = \bar{\kappa}_i \gamma'$, we have the Bishop formula.

Theorem 4.11 *For a unit speed curve $\gamma \in \mathbb{R}^n$, we have*

$$\begin{pmatrix} T' \\ \bar{\kappa}_1 T \\ \cdots \\ \bar{\kappa}_{n-1} T \end{pmatrix} = \begin{pmatrix} 0 & -\bar{\kappa}_1 & \cdots & -\bar{\kappa}_{n-1} \\ \bar{\kappa}_1 & 0 & 0 & 0 \\ \cdots & 0 & \cdots & \cdots \\ \bar{\kappa}_{n-1} & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} T \\ N_1 \\ \cdots \\ N_{n-1} \end{pmatrix}. \tag{4.5}$$

Proof. It is trivial from the following fact.

$$\begin{aligned} T' &= < T', T > T + < T', N_1 > N_1 + \cdots + < T', N_{n-1} > N_{n-1} \\ &= -\bar{\kappa}_1 N_1 - \bar{\kappa}_2 N_2 - \cdots - \bar{\kappa}_{n-1} N_{n-1}. \quad \square \end{aligned}$$

Proposition 4.12 *Let $\gamma : I \to \mathbb{R}^n$ be a unit speed curve. Then*

$$\kappa^2 = \bar{\kappa}_1^2 + \bar{\kappa}_2^2 + \cdots + \bar{\kappa}_{n-1}^2.$$

Proof. It is trivial from the definition. $\square$

Definition 4.13 Regard $\bar{\kappa}(t) = (\bar{\kappa}_1(t), \cdots, \bar{\kappa}_{n-1}(t)) \in \mathbb{R}^{n-1}$ as a curve. We call $\bar{\kappa}$ as the *normal development* of γ.

Theorem 4.14 *A curve γ lies in an affine subspace of codimension m in $\mathbb{R}^n$ if and only if its normal development $\bar{\kappa}$ lies in a linear subspace of codimension m in $\mathbb{R}^{n-1}$.*

Proof. First, we prove in case $m = 1$. Then subspaces of higher codimension can be obtained by intersecting codimension 1 subspaces(hyperplanes). In fact, for $m = 1$, γ lies in hyperplane if and only if $< \gamma, a >$ = constant, where $a \neq 0$ is a constant vector. By differentiating, we have

$$< \gamma', a >= 0.$$

Hence $a = \sum_{i=1}^{n-1} c_i N_i$, where $\{N_i\}$ are Bishop vector fields. By differentiating,

$$0 = \sum c_i' N_i + \sum c_i N_i' = \sum (c_i' N_i + c_i \bar{\kappa}_i \gamma').$$

Then $\sum_i c_i \bar{\kappa}_i = 0$ and $c_i' = 0$. Hence we have

$$< \bar{\kappa}, c >= 0, \;\; c = (c_1, \cdots, c_{n-1}) = constant,$$

which means $(\bar{\kappa}_1, \cdots, \bar{\kappa}_{n-1}) \in$ linear subspace (hyperplane) in $\mathbb{R}^{n-1}$.
Generally, for higher codimension m, let a_i and b_i $(i = 1, \cdots, m)$ are constant such that $< \gamma, a_i >= b_i$. Then $a_i = \sum_{j=1}^{n-1} c_{ij} N_j$ and $< \gamma', a_i >= 0$ for all i. Hence $< \bar{\kappa}, c_i >= 0$, where $c_i = (c_{i1}, \cdots, c_{in-1})$. This implies that the proof is completed. $\square$

Theorem 4.15 *A curve γ lies on a sphere of radius $r > 0$ in $\mathbb{R}^n$ if and only if its normal development $\bar{\kappa}$ lies on a hyperplane distance $\frac{1}{r}$ from $0 \in \mathbb{R}^{n-1}$.*

Proof. Let $\gamma : I \to \mathbb{R}^n$ be a unit speed curve. Then γ lies on a sphere of radius r. That is $||\gamma - a||^2 = r^2$. By differentiating, we have

$$0 =< \gamma', (\gamma - a) > .$$

Hence $\gamma - a = \sum c_i N_i$ where $\{N_i\}$ are Bishop vector fields. By differentiating,

$$\gamma' = \sum c_i' N_i + c_i N_i' = \sum c_i' N_i + c_i \bar{\kappa}_i \gamma'.$$

Then $c_i' = 0$ and $\sum_i c_i \bar{\kappa}_i = 1$. Hence $\bar{\kappa}$ lies on hyperplane orthogonal to c.
Moreover, since $r^2 = ||\gamma - a||^2 = \sum c_i^2 = ||c||^2$, the distance from 0 to hyperplane is $< \bar{\kappa}, (\frac{c}{r}) >= \frac{1}{r}$. $\square$

Corollary 4.16 *A curve γ lies on a sphere of radius $r > 0$ in $\mathbb{R}^3$ if and only if $\bar{\kappa}$ lies on a straight line with a distance $\frac{1}{r}$ from 0.*

Remark 4.17 (Conclusions)
(1) In Frenet theory, any curve must be at least C^{n-1} in $\mathbb{R}^n$. Moreover, if any κ_i vanishes at single point, we can not define higher κ_i's and v_i's.
(2) In Frenet theory, if $\kappa_n = 0$, then curve lies in an affine subspace (Theorem 3.23). *But we do not know when the curve lies on the sphere.*
(3) In Bishop theory, we know when the curve lies on the sphere (Theorem 4.15).

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〈국문초록〉

유클리드 공간상의 곡선에 대한 Bishop 곡률

본 논문에서는 유클리드 공간상의 곡선의 성질에 대해 연구하였다. 첫째로, 유클리드 공간과 로렌츠 공간 각각에서의 Frenet 공식을 알아보았다. 그리고 Frenet 이론과 함께 여러가지 곡선에 대해 알아보았다. 또한, Bishop 이론을 소개하고 Frenet 이론과 Bishop 이론 사이의 관계를 연구하였다.

감사의 글

지난 2년간의 대학원 생활을 돌이켜 한 편의 논문으로 대신하기엔 아쉬움이 많이 남지만, 이렇게 잘 마무리 할 수 있게 되어서 무척 기쁩니다. 특히, 부족한 저를 많은 관심과 격려로 이끌어 주시고, 끝까지 포기하지 않게 조언을 아끼지 않으셨던 정승달 교수님께 깊은 감사를 드립니다. 교수님이 안계셨다면 대학원 생활을 어떻게 했을지 생각만 해도 아찔합니다.

정승달 교수님 정말 감사합니다!!!

또한, 미흡한 저의 논문을 심사해 주시고 대학에 들어온 후, 그리고 대학원 과정까지 많은 가르침을 주신 수학과 양영오 교수님, 송석준 교수님, 방은숙 교수님, 윤용식 교수님, 유상욱 교수님, 진현성 교수님께도 감사의 말씀을 드립니다.

그리고 공부할 때마다 눈앞이 캄캄했던 저에게 빛이 되어주신 민주 언니와 금란 언니에게도 감사의 마음을 전합니다. 대학원 생활에 있어서 서로에게 든든한 버팀목이 되었던 희란이, 대학원 선배이자 믿음직스러운 내 친구 은아, 늘 자신감이 부족했던 저에게 용기와 힘을 주었던 연정 언니에게도 고마운 마음을 전합니다.

Compact Set 화이팅!!!

무엇보다도 믿음으로 저에게 큰 힘이 되어주시고 물심양면으로 고생하신 부모님과 언니, 그리고 항상 옆에서 한결같은 마음으로 지켜준 광필 오빠에게도 깊은 감사의 말을 전하고 싶습니다.

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