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ON THE CLASS OF GENERALIZED PARANORMAL OPERATORS

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ON THE CLASS OF GENERALIZED PARANORMAL OPERATORS

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일반화된 PARANORMAL 작용소들의 집합에 관한 연구

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< abstract >

On the class of generalized paranormal operators

In this paper, we shall study the various characteristics of the M-paranormal operators which generalizes a paranormal operators, and those of kth roots of a paranormal on a Hilbert space H. The main results of the characteristics are as follows:

(1) If S is unitarily equivalent to any $M-$paranormal operator T, then S is M-paranormal.

(2) In general, the product TS of commuting M-paranormal operators T, S is not M-paranormal. But the following holds; Let T and S be commuting M-paranormal operators. Then the product TS is M-paranormal if one of the following holds;

(a) $\|TSx\|\|x\| \geq \sqrt{M}\|Tx\|\|Sx\|$ for any $x \in H$.

(b) $\|T^2Sx\|\|x\| \geq M\|T^2x\|\|Sx\|$ for any $x \in H$.

(3) Let T and S be double commuting M-paranormal operators and let $M > \frac{1}{2}$. Then the product TS is M-paranormal if

$$(2M-1)\|T^2S^2x\|\|x\| \geq \|T^2x\|\|S^2x\|$$

for any $x \in H$.

(4) If an M-paranormal operator T commutes with an isometrc operator S, then TS is M-paranormal.

(5) Let λ and μ be distinct eigenvalues of a M-paranormal operator T and $0 < M \leq 1$. Then $\ker(T-\lambda) \perp \ker(T-\mu)$.

(6) We have the following implications among paranormal operators, kth roots of paranormal operators and algebraically paranormal operators:

$$\begin{aligned} \text{paranormal} &\subseteq \text{the } k\text{th roots of paranormal operators} \\ &\subseteq \text{algebraically paranormal} \end{aligned}$$

(7) The kth roots of a paranormal operator T is closed in the norm topology and a proper subclass of $B(H)$.

(8) Let T be a kth root of a $M-$ paranormal operator. If T commutes with an isometric operator S, then TS is also a kth root of $M-$ paranormal.

(9) Let T be a weighted shift with nonzero weights $\{\alpha_n\}$ $(n = 1, 2, \cdots)$. Then T is a kth root of $M-$paranormal operator if and only if

$$|\alpha_n||\alpha_{n+1}|\cdots|\alpha_{n+k-1}| \leq M|\alpha_{n+k}||\alpha_{n+k+1}|\cdots|\alpha_{n+2k-1}|$$

for $n = 1, 2, 3, \cdots$.

(10) If T is a kth root of a paranormal operator with $0 \in \pi_{00}(T^n)$, then T is a Weyl operator.

0. Introduction

In the theory of non-normal operators on Hilbert spaces, it is important to seek ways to reduce the problem to the normal operator case. Many mathematicians have tried to extend the significant properties of normal operators to the case of non-normal operators in various way since early 1960. Some classes of non-normal operators are closely related to normal operators, and analogy and difference between such non-normal operators and normal operators have been discussed.

Let H be a Hilbert space and let $B(H)$ be the set of all bounded linear operators on H. By T. Saito([43]), T. Furuta([22]), etc., the following non-normal operators have been defined as follows;

An operator $T \in B(H)$ is called *normal* if $T^*T = TT^*$; *quasinormal* if T commutes T^*T, i.e., $T(T^*T) = (T^*T)T$; *subnormal* if T has a normal extension (i.e., there exists a Hilbert space K containing H as a subspace and a normal operator B on K such that $Tx = Bx$ for all $x \in H$); *hyponormal* if $T^*T - TT^* = D \geq 0$, or equivalently $\|Tx\| \geq \|T^*x\|$ for $x \in H$; *seminormal* if $T^*T - TT^* = D$, $D \geq 0$ or $D \leq 0$ (or equivalently T or T^* is hyponormal), and *normaloid* if $\|T\| = r(T)$, where $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ denotes the spectral radius of T . An operator T is called *paranormal* or equivalently of class N if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for every $x \in H$.

We have the following implications ([43]), but the converse of the implications are not reversible([26]).

$$\text{Normal} \subset \text{Quasinormal} \subset \text{Subnormal} \subset \text{Hyponormal} \subset \text{Paranormal} \subset \text{Normaloid}.$$

B. L. Wadhwa ([57]) introduced the class of M-hyponormal operators and V. Istratescu ([30]) has studied some structure theorems for a subclass of M-hyponormal operator: An operator T is called *M-hyponormal* if there exists a real number $M > 0$ such that $M\|(T-\lambda)x\| \geq \|(T-\lambda)^*x\|$ for any unit vector x in H and for any complex number λ. Every hyponormal operator is $M-$hyponormal, but the converse is not true in general: for example, consider the weighted shift S on l_2 given by

$$S(x_1, x_2, \cdots) = (0, 2x_1, x_2, x_3, \cdots).$$

Then S is $M-$hyponormal, but not hyponormal. The M-hyponormality of operators has been studied by many mathematicians([2], [19], [35], [40], [51], [57]).

On the other hand, an operator T is called *M-paranormal* if $M\|T^2x\| \geq \|Tx\|^2$ for any unit vector x in H. In particular if $M = 1$, the class of M-paranoramal operators becomes the class of paranormal operators as studied by T. Ando ([1]) and T. Furuta ([21]). T. Ando([1]) has characterized the paranormal operator as follows:

Theorem. *(Ando) An operator T is paranormal if and only if*

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0$$

for all $\lambda > 0$.

Every paranormal operator is normaloid. However this result is not valid for M-paranormal operator if $M > 1$. Also, similarity need not preserve M-hyponormality and M-hyponormal operators need not be normaloid.

The organization of this thesis is as follows:

In section 1, we introduce basic properties of various spectra(spectrum, point spectrum, approximate point spectrum, essential spectrum, Weyl specrum etc.) of a bounded linear operator and the spectral mapping theorem.

In section 2, we give well known results of hyponormal operators and M-hyponormal operators on a Hilbert space H. Also we shall give some properties of algebraically $M-$hyponormal operators.

In section 3, we shall study certain properties of M-paranormal operators. In particular, we shall give an essentially characterization of M-paranormal operators in the following way;

Theorem 3.8. *An operator T is M-paranormal if and only if*

$$M^2T^{*2}T^2 + 2\lambda T^*T + \lambda^2 I \geq 0 \quad \textit{for all real } \lambda.$$

Also we shall give an example due to T. Ando([1]) that there is a paranormal operator such that some translation is not paranormal. And we discuss the conditions under which, the sum, the product and the inverse (if it exists)of M-paranormal operators become M-paranormal. The question of inverse can be readily answered. We show that $\ker(T - \lambda) \perp \ker(T - \mu)$ for distinct eigenvalues λ, μ of a M-paranormal operator T where $0 < M \leq 1$.

In section 4, we shall study a new class of operators called a kth roots of G-operator: An operator $T \in L(H)$ is a *kth root of a G-operator* if T^n is a G-operator. In particular, if a G-operator is paranormal, then T is called the kth root of a paranormal operator. We shall show the following results:

(1) The kth roots of a paranormal operator T is a proper subclass of $B(H)$.

(2) If $T \in B(H)$ is a kth root of a paranormal operator and T is invertible, then T^{-1} is a kth root of a paranormal operator.

(3) Unitary equivalence preserves kth root of paranormality i.e., If $S \in B(H)$ is a kth root of a paranormal operator and S is unitarily equivalent to T, then T is a kth root of a paranormal operator.

(4) The set of all the kth roots of hyponormal operators is closed in the norm topology.

(5) A weighted shift T with nonzero weights $\{\alpha_n\}$ $(n = 1, 2, \cdots)$ is a kth root of $M-$paranormal operator if and only if

$$|\alpha_n||\alpha_{n+1}|\cdots|\alpha_{n+k-1}| \leq M|\alpha_{n+k}||\alpha_{n+k+1}|\cdots|\alpha_{n+2k-1}|$$

for $n = 1, 2, 3, \cdots$.

Also we show that if T is a kth root of a of a paranormal operator with $0 \in \pi_{00}(T^k)$, then T is a Weyl operator. If S and T are commuting kth roots of paranormal operators respectively, we prove that ST is Weyl if and only if S and T are both Weyl.

1. Preliminaries and Basic Results

Let H be a Hilbert space and let $B(H)$ the set of all bounded linear operators on H. We denote the kernel of T and the range of T by $\ker T (= N(T))$ and $R(T)$ respectively. We note that $R(T)^{\perp} = N(T^*)$ for any operator $T \in B(H)$. Write $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$ for the spectrum of T, $\rho(T) = \sigma(T)^c$ for the resolvent of T, $\sigma_p(T) = \pi_o(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda) \neq \{0\}\}$ for the set of eigenvalues of T, $\pi_{0f}(T)$ for the points of $\sigma(T)$ that are eigenvalues of finite multiplicity, and $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity. If K is a subset of $\mathbb{C}$, we write iso K for the set of isolated points of K.

A complex number $\lambda \in \mathbb{C}$ is said to be an *approximate eigenvalue* of T if there exists a sequence $\{x_n\}$ with $\|x_n\| = 1$ such that $Tx_n - \lambda x_n \to 0$, i.e., $(T - \lambda)x_n \to 0$. Let

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an approximate eigenvalue of } T\}.$$

Then $\sigma_{ap}(T)$ is called the *approximate point spectrum* of T. Let $\sigma_{com}(T) = \{\lambda \in \mathbb{C} : R(T - \lambda) \text{ is not dense in } H\}$ be the *compression spectrum* of T.

An operator $T \in B(H)$ is said to be *Fredholm* if its range $R(T)$ is closed and both the null space $\ker T$ and $\ker T^*$ are finite dimensional. The *index* of a Fredholm operator T, denoted by $\operatorname{ind} T$ or $i(T)$ is defined by

$$\operatorname{ind}(T) = \dim\ker T - \dim\ker T^*.$$

The *essential spectrum* of T, denoted by $\sigma_e(T)$, is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}.$$

A Fredholm operator of index zero is called a *Weyl operator*. The *Weyl spectrum* of T, denoted by $\omega(T)$, is defined by

$$\omega(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}.$$

It was shown ([5]) that for any operator T,

$$\sigma_e(T) \subset \omega(T) \subset \sigma(T)$$

and $\omega(T)$ is a nonempty compact subset of $\mathbb{C}$. The spectral radius $r(T)$ of T is

$$\lim_{n\to\infty} \|T^n\|^{1/n} = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

An operator $T \in B(H)$ is said to be *Browder* if it is Fredholm "of finite ascent and descent", or equivalently [28] if T is Fredholm and $T-\lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. It is well known that $\sigma_e(T)$ and $\omega(T)$ are both compact. If $T \in B(H)$ for a finite dimensional Hilbert space H then T is always Fredholm, so that $\sigma_e(T) = \emptyset$. However, if dim $H = \infty$ then we can easily show that $\sigma_e(T)$ is non-empty by using Calkin algebra theory. Atkinson Theorem says that T is Fredholm if and only if T is invertible modulo compact operators.

Theorem 1.1. *Let H be a Hilbert space and let $T \in B(H)$. Then* $\ker T^* = (ran\ T)^{\perp}$.

Proof. If $x \in \ker T^*$, then $0 = < T^*x, y > = < x, Ty >$ for all $y \in H$ and hence x is orthogonal to ran T. i.e. $x \in (\text{ran } T)^{\perp}$. Conversely if $x \in (\text{ran}\, T)^{\perp}$ i.e., x is orthogonal to $\text{ran}\, T$, then $< T^*x, y > = < x, Ty > = 0$ for y in H, which implies $T^*x = 0$. Therefore $x \in \ker T^*$. □

Definition 1.2. *A vector $x \in H$ is said to be a proper vector for the operator T or equivalently eigenvector of T if $x \neq 0$, and $Tx = \mu x$ for a suitable scalar μ. A scalar μ is said to be a proper value for the operator T or equivalently eigenvalue of T if there exists a vector $x \neq 0$ such that $Tx = \mu x$, and the null space of the operator $T - \mu I$ is called the μ-th proper subspace of T, denoted by $N_T(\mu)$.*

That is,

$$N_T(\mu) = \{x \in H : Tx = \mu x\}.$$

Briefly speaking, $N_T(\mu)$ is called the μ-space of T. Thus $N_T(\mu)$ is different from $\{0\}$ if and only if μ is a proper value for T. A nonzero vector x is a proper vector for T if and only if x belongs to some μ-space of T.

Definition 1.3. *A closed linear subspace M of H is said to be invariant under the operator T if $T(M) \subseteq M$. A closed linear subspace M is said to reduce the operator T if both M and $M^{\perp}$ are invariant under T.*

Clearly, $\{0\}$ and H are invariant under every operator T.

Theorem 1.4. *([7]) If $T \in B(H)$ and M is a closed linear subspace of H, the following conditions are equivalent ;*

(1) *M reduces T.*
(2) *$M^{\perp}$ reduces T.*
(3) *M reduces T^*.*
(4) *M is invariant under both T and T^*.*

Theorem 1.5. *If S and T are operators such that $ST = TS$, then the μ-space of T is invariant under S, that is, $S(N_T(\mu) \subset N_T(\mu)$ for all μ.*

Proof. Let $x \in N_T(\mu)$. Then

$$T(Sx) = (TS)x = (ST)x = S(Tx) = S(\mu x) = \mu(Sx)$$

shows that $Sx \in N_T(\mu)$. □

From the above theorem, the μ-space of T is invariant under T.

Let M be a closed invariant subspace of T and $T|_M$ the restriction of T on M. If M is a reducing subspace of T, then T can be decomposed into the direct sum : $T = T|_M \oplus T|_{M^{\perp}}$, where $M^{\perp}$ is the orthogonal complement of M.

The projection with range $R(M)$ on a closed subspace M is the linear transformation P defined by $Pz = x$, for every vector z of the form $x + y$ with $x \in M$ and $y \in M^{\perp}$.

Corallary 1.6. *([8]) If T is an operator on H, M is a closed subspace of H and P is the projection onto M, then M is an invariant subspace for T if and only if $PTP = TP$ if and only if $M^{\perp}$ is an invariant subspace for T^*. Further, M is a reducing subspace for T if and only if $PT = TP$ if and only if M is an invariant subspace for both T and T^*.*

The well-known results on the spectrum are as follows;

Theorem 1.7. *([8],[13]) For any operator $T \in B(H)$,*

(1) *$\sigma(T)$ is a nonempty compact subset of $\mathbb{C}$.*
(2) *$\sigma_p(T) \subset \sigma_{ap}(T) \subset \sigma(T)$.*
(3) *$\sigma_{ap}(A)$ is a closed subset of $\sigma(T)$.*
(4) *$\partial\sigma(T) \subset \sigma_{ap}(T)$.*

Definition 1.8. *An operator T is said to be self-adjoint (or Hermition) if $T^* = T$; positive if $(Tx, x) \geq 0$ for all $x \in H$, denoted by $T \geq 0$; isometry if $\|Tx\| = \|x\|$ for all $x \in H$; unitary if $T^*T = TT^* = I$. An operator T is said to be unitarily equivalent to an operator S if $S = U^*TU$ for a unitary operator U. An operator S is said to be similar to the operator T if there exists an invertible operator A such that $T = A^{-1}SA$, denoted by $S \sim T$*

Clearly every positive operator is self-adjoint, and if $S \sim T$, then $S^* \sim T^*$.

Lemma 1.9. *([7])*

(1) *If T is a positive operator, then for all $x, y \in H$,*

$$|(Tx, y)|^2 \leq (Tx, x) \cdot (Ty, y).$$

(2) *If $S \leq T$ and R is any operator, then $R^*SR \leq R^*TR$. If $T \geq 0$ and S is any operator, then $S^*TS \geq 0$.*

Theorem 1.10. *([7])*

(1) *If S and T are self-adjoint, so is $S + T$.*
(2) *If T is self-adjoint and $\alpha \in \mathbb{R}$, then αT is self-adjoint.*
(3) *If T is any operator, then T^*T and $T + T^*$ are self-adjoint.*
(4) *If S and T are self-adjoint, then ST is self-adjoint if and only if $ST = TS$.*

Theorem 1.11. *The range $R(T)$ of an isometric operator T is a closed linear subspace of H.*

Proof. Clearly $R(T)$ is a linear subspace([7]). Suppose y is a limit point of $R(T)$. It suffices to show that $y \in R(T)$. Choose any sequence $y_n \in R(T)$ such that $y_n \to y$. Say $y_n = Tx_n$. Since $\|x_m - x_n\| = \|Tx_m - Tx_n\| = \|y_m - y_n\| \to 0$, $\{x_n\}$ is a Cauchy sequence. Since H is complete, $x_n \to x$ for some vector x. By the continuity of T, $y_n = Tx_n \to Tx$, that is, $Tx = \lim Tx_n = \lim y_n = y$. □

Theorem 1.12. *([7]) The following conditions on T are equivalent ;*

(1) *T is unitary.*
(2) *T^* is unitary.*
(3) *T and T^* are isometric.*
(4) *T is isometric and T^* is injective.*
(5) *T is isometric and surjective.*
(6) *T is bijective and $T^{-1} = T^*$.*

Theorem 1.13. *(The Spectral Mapping Theorem) If $T \in B(H)$ and f is analytic in a neighborhood of $\sigma(T)$, then $\sigma(f(T)) = f(\sigma(T))$.*

2. Hyponormal operators and M-hyponormal operators

Lemma 2.1. *Let T be a hyponormal operator on a Hilbert space H. Then*

(1) *$T-\lambda I$ and T^{-1} are hyponormal for each $\lambda \in \mathbb{C}$.*
(2) *$Tx=\lambda x$ implies $T^*x=\bar{\lambda}x$, for all $x \in H$, $\lambda \in \mathbb{C}$.*
(3) *$Tx=\lambda x$, $Ty=\mu y$ and $\lambda \neq \mu$ for all $x, y \in H$, $\lambda, \mu \in \mathbb{C}$ imply that x and y are orthogonal.*

Proof. (1) Since T is hyponormal, we have

$$\begin{aligned}(T-\lambda I)(T^*-\bar{\lambda}I) &= TT^*-\lambda T^*-\bar{\lambda}T+|\lambda|^2 I \\ &\leq T^*T-\lambda T^*-\bar{\lambda}T+|\lambda|^2 I \\ &= (T^*-\bar{\lambda}I)(T-\lambda I)\end{aligned}$$

for each $\lambda \in \mathbb{C}$, and so $T-\lambda I$ is hyponormal.

If T is invertible and $TT^* \leq T^*T$, then $TT^* = TIT^* \leq T^*T$ and so $I \leq T^{-1}T^*TT^{*-1}$. Since $A \geq I$ implies $A^{-1} \leq I$ for any operator for A, we get that $T^*T^{-1}T^{*-1}T \leq I$ i.e., $I - T^*T^{-1}T^{*-1}T \geq 0$. Thus

$$T^{*-1}T^{-1} - T^{-1}T^{*-1} = T^{*-1}\{I - T^*T^{-1}T^{*-1}T\}T^{-1} \geq 0$$

and so T^{-1} is hyponormal.

(2) Since $T-\lambda I$ is hyponormal by (1), we have

$$0 \leq \|(T-\lambda I)^*x\| \leq \|(T-\lambda I)x\| = 0.$$

Thus $\|(T-\lambda I)^*x\| = 0$ and so $T^*x = \bar{\lambda}x$.

(3) Since

$$\lambda(x,y) = (\lambda x, y) = (Tx, y) = (x, T^*y) = (x, \bar{\mu}y) = \mu(x,y),$$

$(\lambda-\mu)(x,y)=0$ implies $(x,y)=0$ $(\lambda \neq \mu)$, that is, x, y are orthogonal. □

Definition 2.2. *An operator $T \in B(H)$ is said to be nilpotent if $T^n = 0$ for some $n \in N$; quasinilpotent if $\|T^n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$.*

Evidently, if T is nilpotent then T is also quasinilpotent, and since the spectral radius $r(T)$ can be expressed as

$$r(T) = \lim \|T^n\|^{1/n}$$

it follows that $\sigma(T) = \{0\}$ if T is quasinilpotent.

Lemma 2.3. *Let T be a hyponormal operator on H.*

(1) *$\alpha T + \beta I$ is hyponormal for any complex numbers α and β.*
(2) *If $x \in H$ is any vector of H, then $\|Tx\| = \|T^*x\|$ if and only if $T^*Tx = TT^*x$.*
(3) *$M = \{x \in H : \|Tx\| = \|T^*x\|\}$ is a closed subspace of H.*
(4) *If $M \subset H$ is invariant under T, then $T|_M$ is hyponormal.*
(5) *For every positive integer n,*

$$\|T^n\| = \|T\|^n, \tag{2.1}$$

and so $r(T) = \|T\|$ i.e., T is normaloid.

Proof. (1) Since T is hyponormal,

$$(\alpha T + \beta I)^*(\alpha T + \beta I) - (\alpha T + \beta I)(\alpha T + \beta I)^* = |\alpha|^2(T^*T - TT^*) \geq 0$$

for any complex numbers α and β. Hence $\alpha T + \beta I$ is hyponormal.

(2) The proof of the sufficiency is obvious. If $\|Tx\| = \|T^*x\|$ for each vector $x \in H$, then $((T^*T - TT^*)x, x) = 0$ and hence for each vector $y \in H$,

$$|((T^*T - TT^*)x, y)|^2 \leq |((T^*T - TT^*)x, x)| \cdot |((T^*T - TT^*)y, y)| = 0$$

by the generalized Schwarz inequality for positive operators. Since y is arbitrary, we have $T^*Tx = TT^*x$ for each vector $x \in H$.

(3) By (2),

$$\begin{aligned} M &= \{x \in H : \|Tx\| = \|T^*x\|\} \\ &= \{x \in H : (T^*T - TT^*)x = 0\} = \ker(T^*T - TT^*) \end{aligned}$$

is clearly closed.

(4) Since M is invariant under T, $PTP = TP$ where P is the projection on M. Since T is hyponormal,

$$\|PT^*Px\| \leq \|T^*Px\| \leq \|TPx\| = \|PTPx\|$$

for each vector $x \in H$ and so PTP is hyponormal. Hence $T|_M$ is hyponormal.

(5) Let $n \in \mathbb{N}$ be any positive integer and $\xi \in H$ be fixed. Then

$$\|T^*T^n\xi\| \leq \|TT^n\xi\| \leq \|T^{n+1}\|\|\xi\|,$$

and hence $\|T^*T^n\xi\| \leq \|T^{n+1}\|\|\xi\|$.

Assume relation (2.1) holds for every positive integer $p \leq n$, and let us prove it for $n+1$. Then

$$\begin{aligned}\|T^n\|^2 = \|T^{*n}T^n\| &= \|T^{*n-1}T^*T^n\| \\ &\leq \|T^{*n-1}\|\|T^*T^n\| \\ &\leq \|T^{*n-1}\|\|T^{n+1}\| \\ &= \|T^{n-1}\|\|T^{n+1}\|.\end{aligned}$$

Since $\|T^{n-1}\| = \|T\|^{n-1}$ and $\|T^n\| = \|T\|^n$, we get $\|T^{n+1}\| \geq \|T\|^{n+1}$. The converse inequality being obvious, the proof is complete.

(Another Method) For each $x \in H$ with $\|x\| = 1$, we have

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) \leq \|T^*Tx\| \leq \|T^2x\|.$$

But then $\|T\|^2 \leq \|T^2\| \leq \|T\|^2$ which implies $\|T\|^2 = \|T^2\|$. Now

$$\begin{aligned}\|T^nx\|^2 = (T^nx, T^nx) &= (T^*T^nx, T^{n-1}x) \\ &\leq \|T^*T^nx\| \cdot \|T^{n-1}x\| \\ &\leq \|T^{n+1}x\| \cdot \|T^{n-1}x\|.\end{aligned}$$

Thus $\|T^n\|^2 \leq \|T^{n+1}\| \cdot \|T^{n-1}\|$, and combining this with the equality above, a simple induction argument yields $\|T\|^n = \|T^n\|$ for $n = 1, 2, \cdots$. Thus

$$r(T) = \lim \|T^n\|^{1/n} = \lim \|T\| = \|T\|.$$

□

Corollary 2.4. *The only quasinilpotent hyponormal operator is the zero operator.*

Proof. By hypothesis $\sigma(T) = \{0\}$ and so $\|T\| = r(T) = 0$. Hence T is zero. □

Theorem 2.5. *Let T be the weighted shift operator defined by $Te_n = \alpha_n e_{n+1}$ with weights $\{\alpha_n\}_{n=0}^{\infty}$ for each positive integer n. Then T is hyponormal if and only if the weight sequence α_n is monotonically increasing.*

P. Fan([18]) and S. L. Campbel([11]) showed the following examples.

Example 2.6.

(1) ([18]) Let T be a bilateral shift defined by

$$Te_n = \begin{cases} e_{n-1} & \text{for } n \leq 2 \\ 2e_{n-1} & \text{for } n \geq 3. \end{cases}$$

Then T is a hyponormal operator.

(2) ([11]) Let T be the unilateral weighted shift with weight sequence $\{1, \frac{1}{2}, 1, 1, \cdots\}$. Then T is not a hyponormal operator.

Theorem 2.7. *Let λ be a point in the resolvent set of a hyponormal operator T. Then*

$$\|(T-\lambda)^{-1}\| = \frac{1}{dist(\lambda, \sigma(T))}.$$

Proof. Let $\alpha \in \sigma((T-\lambda)^{-1})$. By the spectral mapping theorem, we infer $\mu = \alpha^{-1} + \lambda \in \sigma(T)$, and conversely every $\beta \in \sigma(T)$ is of the form $\alpha^{-1} + \lambda$, with $\alpha \in \sigma((T-\lambda)^{-1})$. In fact,

$$\begin{aligned}\alpha \notin \sigma\left((T-\lambda)^{-1}\right) &\Leftrightarrow \alpha I - (T-\lambda)^{-1} \equiv S \ : \text{ invertible} \\ &\Leftrightarrow \alpha(T-\lambda) - I = (T-\lambda)S \ : \text{ invertible} \\ &\Leftrightarrow (T-\lambda) - \alpha^{-1}I = \alpha^{-1}(T-\lambda)S \ : \text{ invertible} \\ &\Leftrightarrow \alpha^{-1} \notin \sigma(T-\lambda)\end{aligned}$$

and so $\sigma\left((T-\lambda)^{-1}\right) = [\sigma(T-\lambda)]^{-1}$. Also

$$\alpha \in \sigma\left((T-\lambda)^{-1}\right) \Leftrightarrow \alpha^{-1} \in \sigma(T-\lambda) \Leftrightarrow \underbrace{\alpha^{-1}+\lambda}_{\beta} \in \sigma(T).$$

Therefore

$$\begin{aligned} r(T-\lambda)^{-1} &= \max\{|\alpha| \ : \ \alpha \in \sigma\left((T-\lambda)^{-1}\right)\} \\ &= \left(\min\{|\lambda-\beta| \ : \ \beta \in \sigma(T)\}\right)^{-1} \\ &= \operatorname{dist}(\lambda, \sigma(T))^{-1}.\end{aligned}$$

But $(T-\lambda)^{-1}$ is still a hyponormal operator (for instance from the factorization), so that

$$\|(T-\lambda)^{-1}\| = r((T-\lambda)^{-1}) = \operatorname{dist}(\lambda, \sigma(T))^{-1},$$

as desired. □

It is well-known ([25]) that T^2 may not be hyponormal when T is hyponormal. For example, if U is the unilateral shift on l^2 and $T = U^* + 2U$, then

$$T^*T - TT^* = 3I - 3UU^* = 3(I - UU^*) \geq 0.$$

Therefore T is hyponormal. Howerver if we take $x = (1, 0, -2, 0, \cdots)$, then $T^2x = (0, 0, -4, 0, -8, 0, \cdots)$, $(T^2)^*x = (-6, 0, -7, -2, 0, \cdots)$ and so

$$\|T^2x\|^2 = 80 < 89 = \|(T^*)^2x\|^2,$$

and so T^2 is not hyponormal.

The following another example is due to ([34]).

Example 2.8. Let H denote any Hilbert space and let $\mathcal{A}$ denote the set of all function $x = x(n)$ defined on integers with values in H and satisfying $\sum_{-\infty}^{\infty} \|x(n)\|^2 < \infty$. Then $\mathcal{A}$ becomes a Hilbert space with inner product

$$(x, y) = \sum (x(n), y(n)).$$

Next, let $\{P_n\}$ be a bounded sequence of nonnegative operators on H, so that $0 \le P_n \le (\text{constant}) \cdot I$, and define the operators U and P on $\mathcal{A}$ by

$$Ux(n) = x(n+1) \quad \text{and} \quad Px(n) = P_n x(n).$$

It is clear that U is unitary and that P is a non-negative bounded operator. Furthermore, if $T = UP$, then

$$Tx(n) = P_{n+1}x(n+1) \quad \text{and} \quad T^*x(n) = P_n x(n-1),$$

and hence $T^*Tx(n) = P_n^2 x(n)$ and

$$TT^*x(n) = P_{n+1}^2 x(n).$$

Consequently, $T^*T - TT^* \ge 0$ if and only if

$$(2.2) \qquad P_n^2 \ge P_{n+1}^2 \quad \text{for } n = 0, \pm 1, \pm 2, \cdots.$$

An easy calculation shows that

$$T^2 x(n) = P_{n+1}P_{n+2}x(n+2)$$

and $T^{*2}x(n) = P_n P_{n-1} x(n-2)$, and hence

$$T^{*2}T^2 x(n) = P_n P_{n-1}^2 P_n x(n)$$

and

$$T^2 T^{*2} x(n) = P_{n+1} P_{n+2}^2 P_{n+1} x(n).$$

Thus T^2 is a hyponormal operator if and only if

$$P_n P_{n-1}^2 P_n \geq P_{n+1} P_{n+2}^2 P_{n+1} \quad \text{for all } n. \tag{2.3}$$

It will be shown that (2.2) does not imply (2.3).

If H is two-dimensional, so that operators on H can be regarded as 2×2 matrices. Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $A \geq 0, B \geq 0$ and

$$A - B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geq 0.$$

But $A^2 - B^2 = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$ is not semi-definite. Let P_n be the non-negative square root of A for $n \leq 0$ and the non-negative square root of B for $n > 0$. Then $P_n^2 \geq P_{n+1}^2$, so that (2.2) holds and T is a hyponormal operator. But

$$P_0 P_{-1}^2 P_0 = A^2 \quad \text{and} \quad P_1 P_2^2 P_1 = B^2$$

so that (2.3) fails to hold for $n = 0$. Hence T^2 is not a hyponormal operator.

We recall that an operator T is *unitarily equivalent to an operator* S if $S = U^* T U$ for a unitary operator U.

In ([20]), T. Furuta and R. Nakamoto have proved the second part (2) of the following theorem;

Theorem 2.9.

(1) *An operator unitarily equivalent to a hyponormal operator is a hyponormal operator.*

(2) *([20]) A hyponormal operator unitarily equivalent to its adjoint is normal.*

Proof. (1) Suppose $S = U^*TU$, T hyponormal and U unitary. Now for every $x \in H$,

$$\|S^*x\| = \|U^*T^*Ux\| = \|T^*Ux\| \leq \|TUx\| = \|U^*TUx\| = \|Sx\|$$

and so S is hyponormal. □

Definition 2.10. *Two bounded linear operator S and T are doubly commutative(resp. weakly doubly commutative) if $TS = ST$ and $TS^* = S^*T$ (resp. $TS \neq ST$ but $TS^* = S^*T$).*

In the following lemma, we show that if two operators are weakly doubly commutative, then the sum and product of two hyponormal operators are hyponormal.

Lemma 2.11. *Let T and S be hyponormal operators such that $T^*S = ST^*$. Then*
(1) *$T + S$ is hyponormal.*
(2) *ST is hyponormal.*

Proof. (1) By hypothesis,

$$\begin{aligned}(T+S)^*(T+S) &= T^*T + T^*S + S^*T + S^*S \\ &\geq TT^* + T^*S + S^*T + S^*S \\ &\geq TT^* + TS^* + ST^* + SS^* \\ &= (T+S)(T^* + S^*).\end{aligned}$$

Thus $T + S$ is hyponormal.

(2) For every $x \in H$,

$$\begin{aligned}\|(ST)^*x\|^2 &= (T^*S^*x, T^*S^*x) = \|T^*S^*x\|^2 \\ &\leq \|TS^*x\|^2 = \|S^*Tx\|^2 \leq \|STx\|^2.\end{aligned}$$

Thus TS is hyponormal.

(Another Method) By the hyponormality and the hypothesis, we have

$$\begin{aligned}(ST)^*(ST) &= T^*(S^*S)T \geq T^*(SS^*)T \\ &= S(T^*T)S^* \geq S(TT^*)S^* = (ST)(ST)^*.\end{aligned}$$

Thus ST is a hyponormal operator. □

The sum and product of two double commuting hyponormal operators are easily shown to be a hyponormal operator. But the sum and product of two commuting hyponormal operators are not necessarily hyponormal. We attempt to find conditions under which the product of two hyponormal operators are also hyponormal.

If we replace one of the hyponormal operators by an isometric operator in Lemma 2.11(2), then the condition of commutativity is sufficient to ensure the hyponormality of their product.

Theorem 2.12. *If a hyponormal operator S commutes with an isometric operator T, then ST is a hyponormal operator.*

Proof. For any $x \in H$,

$$\begin{aligned}\|(ST)^*x\| &= \|T^*S^*x\| \leq \|S^*x\| \\ &\leq \|Sx\| = \|TSx\| = \|STx\|.\end{aligned}$$

Thus ST is a hyponormal operator. □

H. Weyl([58]) asserted that if T is a self-adjoint operator acting on a Hilbert space H, then $\omega(T)$ consists precisely of all points of $\sigma(T)$ except the isolated eigenvalues of finite multiplicity, that is,

$$\omega(T) = \sigma(T) - \pi_{00}(T).$$

Following L. A. Coburn([12]), we say that *Weyl's theorem holds for* T if $\omega(T) = \sigma(T) - \pi_{00}(T)$, or equivalently, if $\sigma(T) - \omega(T) = \pi_{00}(T)$.

There are several classes of operators for which Weyl's theorem holds:

(1) H. Weyl([58]) showed that Weyl's theorem holds for any self-adjoint operator.
(2) L. A. Coburn([12]) showed that Weyl's theorem holds for any hyponormal operator and any Toeplitz operator.
(3) S. K. Berberian([5],[6]) showed that Weyl's theorem holds for any seminormal operator.
(4) K. K. Oberai([37]) showed that if N is a nilpotent operator commuting with T and if Weyl's theorem holds for T then it also holds for $T + N$.
(5) A. Uchiyama([54]) recently showed that Weyl's theorem holds for any paranormal operators.

Theorem 2.13. *([12]) Weyl's theorem holds for hyponormal operators.*

Proof. If T is hyponormal then $T - \lambda I$ is hyponormal. Thus it suffices to show that $0 \in \sigma(T) - \omega(T)$ if and only if $0 \in \pi_{00}(T)$.

$(\Longrightarrow)$: Let $0 \in \sigma(T) - \omega(T)$. Then T is weyl but not invertible. Then $T(H)$ is closed, dim $T^{-1}(0) = \dim T(H)^{\perp} < \infty$ and $T^{-1}(0) \neq \{0\}$, so that $T(H)^{\perp} \neq \{0\}$. Since T is hyponormal, $\|Tx\| \geq \|T^{*}x\|$. In particular, $\ker T \subset \ker T^{*} = T(H)^{\perp}$. Thus $T = 0 \oplus B$, where B is invertible. Hence $\sigma(T) = \{0\} \cup \sigma(B)$. Since $0 \notin \sigma(B), 0 \in \text{iso}\,\sigma(T)$. Thus $0 \in \pi_{00}(T)$.

$(\Longleftarrow)$: Let $0 \in \pi_{00}(T)$. Then $0 \in \text{iso}\,\sigma(T)$ and $0 < \dim \ker T < \infty$. By hyponormality, $\ker T \subset T(H)^{\perp}$. So $T = 0 \oplus B$, where B is 1-1 and hyponormal. Also, B is invertible. Since

$$H = \ker T \oplus (\text{ket } T)^{\perp} = \ker T \oplus T(H),$$

$\ker T = T(H)$ and $(\ker T)^{\perp} = T(H)$. Thus $\dim \ker T = \dim T(H)^{\perp} < \infty$ and index $(T) = 0$. Since $0 \in \text{iso}\,\sigma(T)$, T is not invertible. Hence $0 \in \sigma(T) - \omega(T)$. □

Definition 2.14. *An operator $T \in B(H)$ is said to be isoloid if isolated points of $\sigma(T)$ are eigenvalues of T.*

Theorem 2.15. *If T is hyponormal, then T is isoloid.*

Proof. It suffices to show that if $0 \in \operatorname{iso}\sigma(T)$, then $0 \in \sigma_p(T)$. Choose $R > 0$ sufficiently small enough that 0 is the only point of $\sigma(T)$ contained in or on the circle $|\lambda| = R$. Define

$$P = \int_{|\lambda|=R} (\lambda I - T)^{-1} d\lambda.$$

Then P is a nonzero projection which commutes with T. Thus $T|_{PH}$ is hyponormal. Also $\sigma(T|_{PH}) = \{0\}$. That is, $T|_{PH}$ is a quasinilpotent hyponormal operator. Since the only quasinilpotent hyponormal operator is 0 by Corollary 2.4, it follows that $T|_{PH} = 0$, so that T is not one-to-one. Therefore $0 \in \sigma_p(T)$. □

Recall that an operator $T \in B(H)$ is $M-$ hyponormal if there exists $M > 0$ such that

$$\|(T-z)^* x\| \leq M \|(T-z)x\|$$

for all x in H and for all $z \in \mathbb{C}$. The notion of an M-paranormal operator is due to J. Stampfli and B. L. Wadhwa ([51]). Every hyponormal operator is $M-$hyponormal, but the converse is not true in general: for example, consider the weighted shift S on l_2 given by

$$S(x_1, x_2, \cdots) = (0, 2x_1, x_2, x_3, \cdots).$$

The examples of M-hyponormal non-hyponormal operators seem to be scare from the literature. B. Wadhwa([57]) gave an example of an M-hyponormal non-hyponormal weighted shift T on l_2 ;

$$T = \begin{pmatrix} 0 & & & & & & \\ 1 & 0 & & & & & \\ & 2 & 0 & & & & \\ & & 1 & 0 & & & \\ & & & 1 & 0 & & \\ & & & & 1 & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix}$$

The following facts are well-known : If $T \in B(H)$ is M-hyponormal, then

(1) $M^2(T-\lambda)^*(T-\lambda) \geq (T-\lambda)(T-\lambda)^*$ for any $\lambda \in \mathbb{C}$ and the converse is also true.
(2) $Tx = \lambda x$ implies $T^*x = \bar{\lambda}x$.
(3) $\|(T-\lambda)^{n+1}x\| \geq M^{-n(n+1)/2}\|(T-\lambda)x\|^{n+1}$.
(4) T is isoloid.
(5) Weyl's theorem holds for T.
(6) $M\|(T-\lambda)^{-1}x\| \geq \|(T^*-\bar{\lambda})^{-1}x\|$ for all $\lambda \in \rho(T)$ and for all $x \in H$.
(7) The only M-hyponormal quasinilpotent is zero.
(8) If $N \subseteq H$ is invariant for T, that is, $TN \subseteq N$, then $T|_N$ is also M-hyponormal.

Theorem 2.16. *Suppose T is the weighted shift with weight sequence $\{\alpha_n\}_{n=0}^{\infty}$. If $\{\alpha_n\}_{n=0}^{\infty}$ is eventually monotonically increasing, that is, $\{\alpha_n\}_{n=k}^{\infty}$ is monotonically increasing for some $k \in N$, then T is M-hyponormal.*

The following examples showed that similarity need not preserve M-hyponormality and M-hyponormal operators need not be normaloid.

Example 2.17. If on l_2

$$T = \begin{pmatrix} 0 & & & & & & \\ 2 & 0 & & & & & \\ & 2 & 0 & & & & \\ & & 2 & 0 & & & \\ & & & 2 & 0 & & \\ & & & & 2 & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & & & & & & \\ 1 & 0 & & & & & \\ & 4 & 0 & & & & \\ & & 1 & 0 & & & \\ & & & 4 & 0 & & \\ & & & & 1 & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix},$$

then T and S are similar while S is not M-hyponormal.

If on l_2

$$T = \begin{pmatrix} 0 & & & & & & \\ 1 & 0 & & & & & \\ & \frac{1}{2} & 0 & & & & \\ & & 1 & 0 & & & \\ & & & 1 & 0 & & \\ & & & & 1 & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix},$$

then T is M-hyponormal and transloid(i.e., $T - \lambda$ is normaloid for all $\lambda \in \mathbb{C}$), but not hyponormal.

If T is both Fredholm and $M-$hyponormal, then $i(T) \leq 0$. It was known that the mapping $T \to \omega(T)$ is upper semi-continuous, but not continuous at T([37]). However if $T_n \to T$ with $T_nT = TT_n$ for all $n \in \mathbb{N}$ then

$$\lim \omega(T_n) = \omega(T).$$

It was known that $\omega(T)$ satisfies the one-way spectral mapping theorem for analytic funcions: if f is analytic on a neighborhood of $\sigma(T)$ then

$$\omega(f(T)) \subset f(\omega(T)).$$

If T is normal then $\sigma_e(T)$ and $\omega(T)$ coincide. Thus if T is normal and f is analytic on a neighborhood of $\sigma(T)$, it follows that $\omega(f(T)) = f(\omega(T))$ since $f(T)$ is also normal.

We recall that for any operator $T \in B(H)$,

$$\sigma(T) - w(T) \subset \pi_{0f}(T) \quad \text{or equivaently} \quad \sigma(T) - \pi_{0f}(T) \subset w(T).$$

Theorem 2.18. *Let S and T be operators in $B(H)$. Suppose the indices of S and T are either both nonnegative or both nonpositive. Then*

$$S, T \ \ \textit{Weyl} \iff ST \ \ \textit{Weyl}.$$

Proof. If S, T are Weyl, then S, T are Fredholm and $i(S) = i(T) = 0$. Therefore ST is Fredholm and by the index product theorem,

$$i(ST) = i(S) + i(T) = 0.$$

Hence ST is Weyl.

Conversely, suppose that ST is Weyl and each index is nonpositive. Then ST is Fredholm and $i(ST) = 0$. Since we note that $\ker S^* \subseteq \ker(ST)^*$ and $i(S) \leq 0$, $\dim\ker S \leq \dim\ker S^* \leq \dim\ker(T^*S^*) = \dim\ker(ST)^* < \infty$, and so $\ker S$ and $\ker S^*$ are finite dimensional. Also the range $R(S)$ is are closed. Thus S is Fredholm. Therefore S and T are Fredholm. Since each index is nonpositive and

$$0 = i(ST) = i(S) + i(T), \quad i(S) = i(T) = 0.$$

Hence S and T are Weyl.

Suppose that ST is Weyl and each index is nonnegative. By the similar argument, S and T are Weyl. □

Corollary 2.19. *If S and T are $M-$hyponormal operators, then*

$$ST \quad Weyl \Longrightarrow S, T \quad Weyl.$$

Proof. Since S and T are $M-$hyponormal, we have $i(S) \leq 0$ and $i(T) \leq 0$. Thus by the above theorem 2.18, both S and T are Weyl. □

If the "$M-$hyponormal" condition is dropped in the above Corollary 2.19, then the backward implication may fail even though S and T commute: For example, if U is the unilateral shift on l_2, consider the following operators on $l_2 \oplus l_2$: $T_1 = U \oplus I$ and $T_2 = I \oplus U^*$. Since

$$T_1T_2 = (U \oplus I)(I \oplus U^*) = U \oplus U^*,$$

we have $i(T_1T_2) = i(U) + i(U^*) = 0$. Therefore T_1T_2 is Weyl. But since $i(T_1) = 1, i(T_2) = -1$, so T_1 and T_2 are not Weyl.

It is possible for the product of non-Weyl operators to be Weyl. For example, consider the unilateral shift on l_2. Then since $\dim\ker U = 0$ and $\dim\ker U^* = 1$, U and U^* are Fredholm operators of index -1 and 1 respectively and so U and U^* are not Weyl operators since $i(U) = -1$ and $i(U^*) = 1$. But UU^* is Fredholm and

$$i(UU^*) = i(U) + i(U^*) = -1 + 1 = 0$$

by the index product theorem. Thus UU^* is Weyl.

Theorem 2.20. *If T is $M-$hyponormal and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.*

Proof. Suppose that $p(t)$ is any polynomial. Let

$$p(T) - \lambda I = a_0(T - \mu_1 I)\cdots(T - \mu_n I).$$

Since T is $M-$hyponormal, $T - \mu_i I$ are commuting $M-$hyponormal operators for each $i = 1, 2, \cdots, n$. Thus

$$\begin{aligned}\lambda \notin \omega(p(T)) &\Longleftrightarrow p(T) - \lambda I = \text{Weyl}\\ &\Longleftrightarrow a_0(T - \mu_1 I)\cdots(T - \mu_n I) = \text{Weyl}\\ &\Longleftrightarrow T - \mu_i I = \text{Weyl for each } i = 1, 2, \cdots, n\\ &\Longleftrightarrow \mu_i \notin \omega(T) \text{ for each } i = 1, 2, \cdots, n\\ &\Longleftrightarrow \lambda \notin p(\omega(T))\end{aligned}$$

which says that $\omega(p(T)) = p(\omega(T))$.

If f is analytic on a neighborhood of $\sigma(T)$, then by Runge's theorem([14]), there is a sequence $(p_n(t))$ of polynomials converging uniformly in a neighborhood of $\sigma(T)$ to $f(t)$ so that $p_n(T) \to f(T)$. Since each $p_n(T)$ commutes with $f(T)$, by ([37])

$$f(\omega(T)) = \lim p_n(\omega(T)) = \lim \omega(p_n(T)) = \omega(f(T)).$$

□

Corollary 2.21. *If T is hyponormal and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.*

An operator $T \in B(H)$ is said to be *algebraically M-hyponormal* if there exists a nonconstant complex polynomial p such that $p(T)$ is M-hyponormal ; *pth root of a M-hyponormal operator* if T^p is M-hyponormal ; *polynomially M-hyponormal* if $p(T)$ is M-hyponormal for every complex polynomial p. Evidently, we have the implications:

$$\begin{aligned}\text{polynomially } M\text{-hyponormal} &\subseteq M\text{-hyponormal}\\ &\subseteq \text{the } p\text{th roots of } M\text{-hyponormal}\\ &\subseteq \text{algebraically } M\text{-hyponormal.}\end{aligned}$$

But the converse is not true in general : for example, if $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on two dimensional Hilbert space, then T^p is not M-hyponormal for any $p \in \mathbb{N}$, whereas $p(T) = 0$ with $p(z) = (z-1)^2$.

The following facts follow from the above definition and the well-known facts of M-hyponormal operators.

(1) If $T \in \mathcal{B}(H)$ is algebraically M-hyponormal, then so is $T - \lambda I$ for each $\lambda \in \mathbb{C}$.

(2) If $T \in \mathcal{B}(H)$ is algebraically M-hyponormal and $M \subseteq H$ is invariant under T, then $T|_M$ is algebraically M-hyponormal.

(3) Unitary equivalence preserves algebraic M-hyponormality.

The following Lemma gives the essential facts for algebraically M-hyponormal operators that we will need to prove the main theorem.

Lemma 2.22. *Suppose* $T \in \mathcal{B}(\mathcal{H})$.

(1) *If* T *is algebraically* $M-$*hyponormal and quasinilpotent, then* T *is nilpotent.*

(2) *If* T *is algebraically* $M-$*hyponormal, then* T *is isoloid.*

(3) *If* T *is algebraically* $M-$*hyponormal, then* T *has finite ascent.*

Proof. (1) Suppose $p(T)$ is $M-$hyponormal for some nonconstant polynomial p. Since $M-$hyponormality is translation-invariant, we may assume $p(0) = 0$. Thus we can write

$$p(\lambda) \equiv a_0 \lambda^m (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$$

where $m \neq 0$, $\lambda_i \neq 0$ for every $1 \leq i \leq n$. If T is quasinilpotent, then

$$\sigma(p(T)) = p(\sigma(T)) = p(\{0\}) = \{0\},$$

so that $p(T)$ is also quasinilpotent. Since the only $M-$hyponormal and quasinilpotent operator is zero, it follows that

$$a_0 T^m (T - \lambda_1 I) \cdots (T - \lambda_n I) = 0.$$

Since $T - \lambda_i I$ is invertible for every $1 \leq i \leq n$, we have that $T^m = 0$.

(2) Suppose $p(T)$ is $M-$hyponormal for some nonconstant polynomial p. Let $\lambda \in \operatorname{iso}\sigma(T)$. Then using the spectral decomposition, we can represent T as the direct sum $T = T_1 \oplus T_2$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) - \{\lambda\}$. By the preceding remark, $T_1 - \lambda I$ is also algebraically $M-$hyponormal. Since $T_1 - \lambda I$ is quasinilpotent, it follows from the statement (1) that $T_1 - \lambda I$ is nilpotent. Therefore $\lambda \in \sigma_p(T_1)$ and hence $\lambda \in \sigma_p(T)$. This shows that T is isoloid.

(3) If T is $M-$hyponormal, then it follows from the property in Introduction that $N(T-\lambda)^{n+1} \subseteqq N(T-\lambda)$. But since, in general, $N(T-\lambda) \subseteqq N(T-\lambda)^n$, it follows that $N(T-\lambda) = N(T-\lambda)^2$. Thus $M-$hyponormal operator is one of ascent 1. Suppose $p(T)$ is $M-$hyponormal for some nonconstant polynomial p. We may assume $p(0) = 0$. If $p(\lambda) \equiv a_0\lambda^m$ then $N(T^m) = N(T^{2m})$. Thus we write

$$p(\lambda) \equiv a_0\lambda^m(\lambda - \lambda_1)\cdots(\lambda - \lambda_n)$$

where $m \neq 0, \lambda_i \neq 0$ for $1 \leq i \leq n$. We then claim that

$$N(T^m) = N(T^{m+1}). \tag{2.4}$$

To show (2.4), let $x(\neq 0) \in N(T^{m+1})$. Then we can write

$$p(T)x = (-1)^n a_0\lambda_1\cdots\lambda_n T^m x.$$

Thus we have

$$\begin{aligned}
&|a_0\lambda_1\cdots\lambda_n|^2\|T^m x\|^2 = \|p(T)x\|^2 \\
&= (p(T)x, p(T)x) \leq \|p(T)^*p(T)x\|\|x\| \\
&\leq M\|p(T)^2x\|\|x\| \quad \text{(because } p(T) \text{ is } M-\text{hyponormal)} \\
&= M\|a_0^2(T-\lambda_1 I)^2\cdots(T-\lambda_n I)^2T^{2m}x\|\|x\| \\
&= 0,
\end{aligned}$$

which implies $x \in N(T^m)$. Therefore $N(T^{m+1}) \subseteqq N(T^m)$ and the reverse inclusion is evident. This completes the proof. □

Theorem 2.23. *Weyl's theorem holds for every algebraically $M-$ hyponormal operator.*

Proof. Suppose $p(T)$ is $M-$ hyponormal for some nonconstant polynomial p. We first prove that $\pi_{00}(T) \subseteqq \sigma(T) - \omega(T)$. Since algebraic $M-$hyponormality is translation-invariant, it suffices to show that

$$0 \in \pi_{00}(T) \Longrightarrow T \quad \text{is Weyl but not invertible.}$$

Suppose $0 \in \pi_{00}(T)$. By the spectral decomposition, we can represent T as the following 2×2 operator matrix with respect to the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^{\perp}$:

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} : \mathcal{K} \oplus \mathcal{K}^{\perp} \longrightarrow \mathcal{K} \oplus \mathcal{K}^{\perp}$$

where $\sigma(T_1) = \{0\}$ and $\sigma(T_2) = \sigma(T) - \{0\}$. But then T_1 is also algebraically $M-$ hyponormal and quasinilpotent. Thus by Lemma 2.22(1), T_1 is nilpotent. Thus we should have that $\dim \mathcal{K} < \infty$: if it were not so then $N(T_1)$ would be infinite dimensional, so that $0 \notin \pi_{00}(T)$, giving a contradiction. Therefore T_1 is a finite dimensional operator. Since finite dimensional operators are always Weyl it follows that T_1 is Weyl. But since T_2 is invertible we can conclude that T is Weyl. Therefore

$$\pi_{00}(T) \subseteqq \sigma(T) - \omega(T).$$

For the reverse inclusion, suppose $\lambda \in \sigma(T) - \omega(T)$. Thus $T - \lambda I$ is Weyl. Then by the "Index Product Theorem",

$$\begin{aligned} \dim N((T-\lambda I)^n) - \dim R((T-\lambda I)^n)^{\perp} &= \text{ind } ((T-\lambda I)^n) \\ &= n \,\text{ind } (T-\lambda I) = 0. \end{aligned}$$

Thus if $\dim N((T-\lambda I)^n)$ is a constant then so is $\dim R((T-\lambda I)^n)^{\perp}$. Consequently finite ascent forces finite descent. Therefore by Lemma 2.22(3), $T - \lambda I$ is Weyl of finite ascent and descent, and thus it is Browder. Therefore $\lambda \in \pi_{00}(T)$. This completes the proof. □

3. M-paranormal operators

We recall that an operator T is *hyponormal* if $T^*T - TT^* = D \geq 0$ or equivalently $\|Tx\| \geq \|T^*x\|$ for every vector $x \in H$; *paranormal* or equivalently *of class* N if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for every $x \in H$ and **-paranormal* if $\|T^*x\|^2 \leq \|T^2x\|\|x\|$ for every $x \in H$.

By ([21],[29],[32]), we have the proper inclusion relation among the classes of non-normal operators as follows :

$$\begin{aligned} \text{Normal} &\Longrightarrow \text{Quasinormal} \Longrightarrow \text{Subnormal} \\ &\Longrightarrow \text{Hyponormal} \\ &\Longrightarrow \text{Paranormal(or } ^*\text{-paranormal)} \Longrightarrow \text{Normaloid} \end{aligned}$$

The following example due to P. R. Halmos([26]) shows that the implications subnormal $\Rightarrow$ hyponormal $\Rightarrow$ paranormal are not reversible.

Example 3.1. ([26],[43]) Let H be a Hilbert space and let $K = \cdots \oplus H \oplus H \oplus \cdots$. K be a Hilbert space with inner product defined by

$$(x, y) = \sum_{n=-\infty}^{\infty} (x_n, y_n)$$

for $x = \{x_n\}_{-\infty}^{\infty}$, $y = \{y_n\}_{-\infty}^{\infty}$ with

$$\sum_{-\infty}^{\infty} \|x_n\|^2 < \infty, \quad \sum_{-\infty}^{\infty} \|y_n\|^2 < \infty.$$

Let $\{P_n\}_{-\infty}^{\infty}$ be a sequence of positive operators on H such that $\{\|P_n\|\}_{-\infty}^{\infty}$ is bounded, and define the operators U and P on K by

$$(3.1) \qquad (Ux)_n = x_{n-1}; \quad (Px)_n = P_n x_n$$

for $x = \{x_n\}_{-\infty}^{\infty} \in K$. Then U is a bilateral shift operator on K and

$$(3.2) \qquad (U^*x)_n = x_{n+1}; \quad (P^*x)_n = P_n^* x_n = P_n x_n..$$

Let $T = UP$. Then from (3.1) and (3.2), we have

$$(3.3) \qquad (T^*Tx)_n = P_n^2 x_n; \quad (TT^*x)_n = P_{n-1}^2 x_n$$

for $x = \{x_n\}_{-\infty}^{\infty} \in K$. Consequently, $T^*T - TT^* \geq 0$ if and only if $\{P_n^2\}$ is increasing. Again by (3.1) and (3.2), we have

$$(3.4) \quad (T^{*2}T^2x)_n = P_n P_{n+1}{}^2 P_n x_n; \quad (T^2T^{*2}x)_n = P_{n-1}P_{n-2}^2 P_{n-1} x_n$$

for $x = \{x_n\}_{-\infty}^{\infty} \in K$. Thus T^2 is hyponormal if and only if

$$(3.5) \qquad P_{n-1}P_{n-2}{}^2 P_{n-1} \leq P_n P_{n+1}^2 P_n$$

for all n.

Now let H be a two-dimensional Hilbert space and let

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

(acting on H). Then

$$D - C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geq 0$$

but

$$D^2 - C^2 = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$$

and $D^2 - C^2$ has negative determinant. Thus $D^2 - C^2$ is not semi-definite. Let

$$P_n = \begin{cases} \sqrt{C} & (n \leq 0) \\ \sqrt{D} & (n > 0). \end{cases}$$

Then $P_n^2 \leq P_{n+1}^2$ for $n = 0, \pm 1, \pm 2, \cdots$ and

$$P_0P_{-1}^2P_0 = C^2, \quad P_1{P_2}^2P_1 = D^2.$$

Thus $\{P_n\}$ is increasing, but does not satisfy (3.5). Hence T is hyponormal and T^2 is not hyponormal. Since every hyponormal is paranormal, T^2 is paranormal, so that T^2 is a non-hyponormal paranormal operator.

From this fact, we can see that T is not subnormal. In fact, if T is subnormal, then T has a normal extension B. Thus B^2 is a normal extension of T^2 and T^2 is subnormal, hence hyponormal. This is a contradiction.

B. L. Wadhwa in ([57]) introduced the class of M-hyponormal operator and V. Istratescu in ([30]) has studied some structure theorems for a subclass of M-hyponormal operators. The following definition of M-paranormal operators also appears in ([30]).

Definition 3.2. *An operator T is called M-paranormal if $M\|T^2x\| \geq \|Tx\|^2$ for any unit vector x in H; M^*-paranormal if $M\|T^2x\| \geq \|T^*x\|^2$ for any unit vector x in H.*

Recall that an operator T is M*-quasihyponormal* if

$$M\|T^2x\| \geq \|T^*Tx\|$$

for any unit vector x in H.

If $M = 1$, the class of M-paranoramal operators becomes the class of paranormal operators as studied by T. Ando ([1]) and T. Furuta ([21]). The purpose of the present paper is to study certain properties of M-paranormal operators.

Theorem 3.3.

(1) *Every M-hyponormal operator T is M-quasihyponormal.*
(2) *Every M-quasihyponormal operator T is M-paranormal.*

Proof. (1) Since T is M-hyponormal,

$$\begin{aligned} M\|T^2x\| &= M\|Tx\|\ \|T(\frac{Tx}{\|Tx\|})\| \\ &\geq \|Tx\|\ \|T^*(\frac{Tx}{\|Tx\|})\| \\ &= \|Tx\|\frac{\|T^*Tx\|}{\|Tx\|} = \|T^*Tx\|. \end{aligned}$$

(2) Since T is M-quasihyponormal, $M\|T^2x\| \geq \|T^*Tx\|$ for any unit vector x in H. We know that for any bounded linear operator T on H,

$$\|Tx\|^2 \leq \|T^*Tx\|$$

for any unit vector x in H. Therefore $M\|T^2x\| \geq \|T^*Tx\| \geq \|Tx\|^2$ for any unit vector x in H. □

we have the proper inclusion relation among the classes of non-normal operators as follows :

$$\begin{aligned} \text{Hyponormal} &\Longrightarrow M-\text{hyponormal} \\ &\Longrightarrow M-\text{quasinormal} \Longrightarrow M-\text{paranormal}. \end{aligned}$$

Lemma 3.4. *If T is paranormal, then $\|T^n\| = \|T\|^n$ for every positive integer n, and so T is normaloid(i.e., $\|T\| = r(T)$).*

Proof. Let T be a paranormal operator on H. It is sufficient to show that $\|T^n\| = \|T\|^n$ for all $n = 1, 2, \cdots$. Suppose that

$$\|T^kx\| \geq \|Tx\|^k$$

for all unit vectors $x \in H$ and $k = 1, 2, \cdots, n$. Then by induction,

$$\begin{aligned} \|T^{n+1}x\| &= \|Tx\| \left\| T^n \frac{Tx}{\|Tx\|} \right\| \\ &\geq \|Tx\| \left\| T\left(\frac{Tx}{\|Tx\|}\right) \right\|^n \\ &= \|Tx\|^{1-n}\|T^2x\|^n \\ &\geq \|Tx\|^{1-n}\|Tx\|^{2n} = \|Tx\|^{n+1} \end{aligned}$$

for any unit vector $x \in H$. Thus by the mathematical induction, $\|T^n\| = \|T\|^n$ for $n = 1, 2, \cdots$.

From the well-known fact that the spectral radius of an operator T is equal to $\lim_{n\to\infty} \sqrt[n]{\|T^n\|}$, we have $\|T\| = r(T)$ and so T is normaloid. $\square$

An operator $T \in B(H)$ is said to be ***nilpotent*** if there is a positive integer n such that $T^n = 0$; ***quasinilpotent*** if $\sigma(T) = \{0\}$. Since the spectral radius is defined by

$$r(T) = \sup\{\, |\lambda| : \lambda \in \sigma(T)\} = \lim_{n\to\infty} \|T^n\|^{1/n},$$

it follows that $r(T) = 0$ if T is quasinilpotent.

Corollary 3.5. *The only quasinilpotent paranormal operator is the zero operator.*

Proof. $\|T\| = r(T) = 0$ by Lemma 3.4. $\square$

Every paranormal operator is normaloid. However the following operator T shows that this result is not valid for M-paranormal operator if $M > 1$.

Example 3.6. Let T be an operator on a three-dimensional Hilbert space defined by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

with respect to the orthonormal basis $\{e_1, e_2, e_3\}$. Then $\sigma(T) = \{0, 1\}$ and

$$\|T\| = r(T) = 1$$

so that T is normaloid. But since $1 = \|Te_2\|^2$ and $\|T^2 e_2\| = 0$, $M\|T^2 e_2\| < \|Te_2\|^2 = 1$. Hence T is not M-paranormal.

The classes of M^*-paranormal operators and M-paranormal operators are independent by the following example;

Example 3.7.([3]) Let $(e_n)_{n=1}^{\infty}$ be an orthonormal of a Hilbert space H. Define a bilateral weighted shift T on H with weight $\{\alpha_n\}$ given by

$$\alpha_n = \begin{cases} \frac{3}{7} & \text{if } n \le -1 \\ \frac{2}{7} & \text{if } n = 0 \\ \frac{n}{n+1}\frac{6}{7} & \text{if } n \ge 1. \end{cases}$$

Then T is $\frac{7}{6}^*$-paranormal, but T is not $\frac{7}{6}$-paranormal.

We begin with a characterization of M-paranormal operators in the following way;

Theorem 3.8. *([2]) A bounded linear operator T is M-paranormal if and only if*

$$M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \ge 0$$

for all $\lambda > 0$.

Proof. We know that for positive numbers b and c, $c - 2b\lambda + \lambda^2 \ge 0$ for all $\lambda > 0$ if and only if $b^2 \le c$. Let $b = \|Tx\|^2$ and $c = M^2\|T^2x\|^2$, $\|x\| = 1$. Then by definition of M-paranormal, T is M-paranormal if and only if $b^2 \le c$. This means that T is M-paranormal if and only if

$$M^2\|T^2x\|^2 - 2\lambda\|Tx\|^2 + \lambda^2 \ge 0$$

for each $\lambda > 0$ and for each vector x with $\|x\| = 1$. This proves the proof. □

Equivalently, putting $A = (TT^*)^{\frac{1}{2}}$ and $B = (T^*T)^{\frac{1}{2}}$ we see that T is M-paranormal if and only if $M^2AB^2A - 2\lambda A^2 + \lambda^2 \ge 0$ for each number $\lambda > 0$.

Corollary 3.9. *([1]) A bounded linear operator T is paranormal if and only if*

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0$$

for all $\lambda > 0$.

Corollary 3.10. *Let T be a weighted shift with weights $\{\alpha_n\}$. Then T is M-paranormal if and only if*

$$|\alpha_n| \leq M|\alpha_{n+1}|$$

for each positive integer n.

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of a Hilbert space H.

($\Longrightarrow$) Suppose T is $M-$ paranormal. Then

$$\|Te_n\| = \|\alpha_n e_{n+1}\| = |\alpha_n|$$

and

$$\|T^2 e_n\| = \|T(\alpha_n e_{n+1})\| = |\alpha_n|\|\alpha_{n+1}e_{n+2}\| = |\alpha_n||\alpha_{n+1}|$$

for each positive integer n. Since T is $M-$ paranormal,

$$\|Te_n\|^2 \leq M\|T^2 e_n\|,$$

and so $|\alpha_n| \leq M|\alpha_{n+1}|$ for each positive integer n.

($\Longleftarrow$) Suppose $|\alpha_n| \leq M|\alpha_{n+1}|$ for each positive integer n. Then for each positive integer n, we have

$$\begin{aligned} M\|T^2 e_n\| - \|Te_n\|^2 &= M|\alpha_n||\alpha_{n+1}| - |\alpha_n|^2 \\ &= |\alpha_n|(M|\alpha_{n+1}| - |\alpha_n|) \geq 0. \end{aligned}$$

Therefore $M\|T^2 e_n\| \geq \|Te_n\|^2$ for each positive integer n, and so T is $M-$ paranormal. □

It can be easily shown that every M-hyponormal operator T is M-paranormal because

$$\|Tx\|^2 = (T^*Tx, x) \leq \|T^*(Tx)\|\|x\| \leq M\|T^2x\|\|x\|$$

for any vector $x \in H$. However the converse need not be true. Indeed if $\{e_n\}$ is an orthonormal basis for a separable Hilbert space and if T is a weighted bilateral shift defined as

$$Te_n = \frac{1}{2^{|n|}}e_{n+1}$$

for each n, then T is not M-hyponormal for any $M > 0$ ([40, Corollary 5] but T is M-paranormal for any $M \geq 2$. We also notice that T is not a paranormal operator.

We shall give an essentially characterization of M-paranormal operators in the following way;

Theorem 3.11. *An operator T is M-paranormal if and only if*

$$M^2T^{*2}T^2 + 2\lambda T^*T + \lambda^2 I \geq 0 \quad \textit{for all real } \lambda.$$

Proof. Let x be any unit vector in H. Then

$$\begin{aligned}
&M^2T^{*2}T^2 + 2\lambda T^*T + \lambda^2 I \geq 0 \quad \text{for all real } \lambda \\
&\Leftrightarrow ((M^2T^{*2}T^2 + 2\lambda T^*T + \lambda^2 I)x, x) \geq 0 \quad \text{for all real } \lambda \\
&\Leftrightarrow M^2\|T^2x\|^2 + 2\lambda\|Tx\|^2 + \lambda^2\|x\|^2 \geq 0 \quad \text{for all real } \lambda \\
&\Leftrightarrow \|Tx\|^4 \leq M^2\|T^2x\|^2 \\
&\Leftrightarrow T \ \text{ is } M\text{-paranormal.}
\end{aligned}$$

□

We recall that the smallest positive integer n, for which $N(T^n) = N(T^{n+1})$ is *the ascent of* T, where $N(T)$ denotes the null space of T. It is well known that the ascent of a normal operator is 0 or 1. I. Sheth ([44]) has proved that if T is hyponormal then the ascent of T is 0 or 1. We generalize this result to $M-$paranormal operator.

Theorem 3.12. *Let T be any $M-$paranormal operator. Then*

(1) *The restriction $T|_N$ to its invariant subspace N is M-paranormal.*
(2) *λT is $M-$paranormal for every complex number λ.*
(3) *If T is a invertible, then T^{-1} is also M-paranormal.*
(4) *If S is unitarily equivalent to T, then S is M-paranormal.*
(5) *The ascent of T is 0 or 1.*

Proof. (1) Let x be any vector of N. Then we have

$$\|T|_N x\|^2 = \|Tx\|^2 \le M\|T^2x\|\|x\| = M\|(T|_N)^2x\|\|x\|.$$

This implies that $T\,|_M$ is M-paranormal.

(2) It is sufficient to show that

$$\|(\lambda T)x\|^2 \le M\|(\lambda T)^2 x\|$$

for all unit vector $x \in H$ and $\lambda \in \mathbb{C}$. Thus

$$\|(\lambda T)x\|^2 = |\lambda|^2\|Tx\|^2 \le |\lambda|^2 M\|T^2x\| = M\|(\lambda T)^2x\|.$$

(3) Since T is M-paranormal, we have $M\|T^2x\|\|x\| \ge \|Tx\|^2$ for each vector x. This can be replaced by

$$\frac{M\|x\|}{\|Tx\|} \ge \frac{\|Tx\|}{\|T^2x\|}$$

for each vector x in H. Now replacing x by $T^{-2}x$, we have

$$M\|x\|\|T^{-2}x\| \ge \|T^{-1}x\|^2$$

for each vector x in H. This shows that T^{-1} is M-paranormal.

(4) Since S is unitarily equivalent to T, there exists a unitary operator U such that $S = U^*TU$. It is sufficient to show that

$$M^2S^{*2}S^2 + 2\lambda S^*S + \lambda^2 I \ge 0$$

for all real λ. Then since T is M-paranormal,

$$\begin{aligned} &M^2S^{*2}S^2 + 2\lambda S^*S + \lambda^2 I \\ &= M^2(U^*T^*U)^2(U^*TU)^2 + 2\lambda(U^*T^*U)(U^*TU) + \lambda^2(U^*U) \\ &= U^*(M^2T^{*2}T^2 + 2\lambda T^*T + \lambda^2 I)U \geq 0. \end{aligned}$$

Therefore S is M-paranormal.

(5) Let x be any vector in $N(T^2)$. Then $T^2x = 0$. Since T is $M-$paranormal,

$$||Tx||^2 \leq M||T^2x||||x|| = 0$$

and so $Tx = 0$, i.e., $x \in N(T)$. Hence $N(T^2) \subseteq N(T) \subseteq N(T^2)$. This completes the result. □

Corollary 3.13. *Let T be any paranormal operator. Then*

(1) *The restriction $T|_N$ to its invariant subspace N is paranormal.*
(2) *λT is paranormal for every complex number λ.*
(3) *If T is a invertible, then T^{-1} is also paranormal.*
(4) *If S is unitarily equivalent to T, then S is paranormal.*

The following example due to T. Ando([1]) shows that there is a paranormal operator such that some translation is not paranormal.

Example 3.14. ([1]) Let C and D be operators on a two-dimensional Hilbert space defined by

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

Then $D \geq C \geq 0$, but

$$26D^2 - 25C^2 = \begin{pmatrix} 105 & 130 \\ 130 & 160 \end{pmatrix}$$

is not positive. Let n be a positive integer such that $4 \times 25^n < 26^n$. Then $4^{1/n}D^2 - C^2$ is not positive. Let $A = (C \otimes C \otimes \cdots \otimes C)^{\frac{1}{2}}$ and $B = (D \otimes D \otimes \cdots \otimes D)^{\frac{1}{2}}$ be operators on a 2^n-dimensional Hilbert space H. Then $D \geq C$ is equivalent to $B^2 \geq A^2$. Let $K = \Sigma_1^\infty \oplus H$, and let T be an operator on K defined by

$$T\{x_n\} = \{y_n\}$$

$$y_n = \begin{cases} 0 & (n = 1) \\ Ax_{n-1} & (n = 2, 3, 4) \\ Bx_{n-1} & (n \geq 5) \end{cases}$$

for $\{x_n\}$ and $\{y_n\} \in K$. By this definition,

$$(T^*T - TT^*)\{x_1, x_2, x_3, \cdots\} = \{A^2x_1, 0, 0, 0, (B^2 - A^2)x_5, 0, 0, \cdots\}.$$

Hence T is hyponormal, so that T^2 is paranormal.

Suppose that $T^2 - \lambda I$ is paranormal for all λ. Let x and y be vectors of H. Then we have

$$\|T^2\bar{x}_0\|^2 = \|A^2x\|^2 + \|B^2y\|^2. \tag{3.6}$$

$$(T^4\bar{x}_0, \bar{x}_0) = (A^4x, y)$$

for $\bar{x_0} = \{x, 0, 0, 0, y, 0, 0, \cdots\} \in K$. Applying Corollary 3.9, we have

$$(T^{2*} - re^{-i\theta_1})^2(T^2 - re^{i\theta_1})^2 - 2r^2(T^{2*} - re^{-i\theta_1})(T^2 - re^{i\theta_1}) + r^4I \geq 0$$

for all $r > 0$ and $0 \leq \theta < 2\pi$. Expanding the left hand side, dividing by r^2 and letting $r \to +\infty$, we have

$$e^{-2i\theta}(T^2)^2 + e^{2i\theta}(T^{2*})^2 + 2T^{2*}T^2 \geq 0.$$

Hence we have $\|T^2\bar{x}\|^2 \geq -Re\{e^{-2i\theta}(T^4\bar{x}, \bar{x})\}$ for $\bar{x} \in K$. Since $0 \leq \theta < 2\pi$ is arbitrary, we get

$$|(T^4\bar{x}, \bar{x})| \leq \|T^2\bar{x}\|^2 \tag{3.7}$$

for all $\bar{x} \in K$. From (3.6) and (3.7), we have

$$\|A^2x\|^2 + \|B^2y\|^2 \geq |(A^4x, y)|$$

for all $x, y \in H$. Since $x, y \in H$ are arbitrary, this implies that

$$2\|A^2x\|\|B^2y\| \geq |(A^4x, y)|$$

for all $x, y \in H$. Setting $x = y$, we have

$$2\|B^2x\| \geq \|A^2x\|,$$

and hence $4B^4 \geq A^4$, a contradiction, since if $4B^4 \geq A^4$, then $4^{\frac{1}{n}}D^2 \geq C^2$. Therefore T^2 is paranormal and $T^2 - \lambda I$ is not paranormal for some λ.

Now we discuss the conditions under which, the sum and the product of M-paranormal operators become M-paranormal. The question of inverse can be readily answered. The sum of two M-paranormal even commuting or double commuting (A and B are said to be double commuting if A commutes with B and B^*) operators may not be M-paranormal as can be seen by the following example;

Example 3.15. Let

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

be operators on 2-dimensional space. Then T and S are both $\sqrt{2}$-paranormal while

$$T + S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is not so because $\|(T+S)x\|^2 = 1 > 0 = \|(T+S)^2x\|$ for some $x = (0, 1)$.

Theorem 3.16. *If T is any $M-$paranormal operator, then $T \otimes I$ and $I \otimes T$ are both $M-$paranormal.*

Proof.

$$\begin{aligned} &M^2[(T \otimes I)^*]^2(T \otimes I)^2 - 2\lambda(T \otimes I)^*(T \otimes I) + \lambda^2(I \otimes I) \\ &= [M^2 T^{*2} T^2 - 2\lambda T^* T + \lambda^2] \otimes I \end{aligned}$$

since T is $M-$paranormal. □

The properties of normaloid and hyponormal are invariant under the tensor product operators. But paranormality is not in this case. We find an example of a M-paranormal operator T such that $T \otimes T$ is not M-paranormal.

Example 3.17. Let H be a 2-dimensional Hilbert space and let K be the direct sum of a denumerable copies of H. Let A and B be any two positive operators on H. Define an operator $T = T_{A'B'n}$ on K as

$$T < x_1, x_2, \cdots > = < 0, Ax_1, Ax_2, \cdots, Ax_n, Bx_{n+1}, Bx_{n+2}, \cdots >,$$

we can compute to find that T is M-paranormal iff $M^2AB^2A - 2\lambda A^2 + \lambda^2 \geq 0$ for each $\lambda > 0$. Set

$$C = \begin{pmatrix} M & M \\ M & 2M \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}.$$

Then both C and D are positive and for $\lambda > 0$,

$$M^2D - 2\lambda C + \lambda^2 = \begin{pmatrix} (M-\lambda)^2 & 2M(M-\lambda) \\ 2M(M-\lambda) & (2M-\lambda)^2 + 4M^2 \end{pmatrix}.$$

This operator is also seen to be positive. Now let $A = C^{\frac{1}{2}}$ and $B = (C^{-\frac{1}{2}}DC^{-\frac{1}{2}})^{\frac{1}{2}}$. Taking $T = T_{A'B'n}$ as mentioned above, we find that T is $M-$paranormal. We claim that $T \otimes T$ is not M-paranormal. Let if possible

$$M^2[(T \otimes T)^*]^2 - 2\lambda[T \otimes T)^*(T \otimes T) + \lambda^2(I \otimes I) \geq 0$$

for each $\lambda > 0$. Putting $\lambda = 1$, we get that

$$M^2[T^{*2}T^2 \otimes T^{*2}T^2] - 2[T^*T \otimes T^*T] + I \otimes I \geq 0.$$

Thus the compression of this operator to the canonical image of $H \otimes H$ in $K \otimes K$ is also positive. But the compression coincides with

$$M^2(D\otimes D) - 2(C\otimes C) + I\otimes I = \begin{pmatrix} 1-M^2 & 0 & 0 & 2M^2 \\ 0 & 4M^2+1 & 2M^2 & 12M^2 \\ 0 & 2M^2 & 4M^2+1 & 12M^2 \\ 2M^2 & 12M^2 & 12M^2 & 56M^2+1 \end{pmatrix}$$

which is not positive.

Lemma 3.18. *Let T be an M-paranormal operator. Then we have the followings;*

(1) $M^2\|T^3x\| \geq \|T^2x\|\|Tx\|$ *for every unit vector x in H.*

(2) *For every positive integer k and every unit vector x in H,*

$$(P_k) \qquad M^{2k-1}\|T^{k+1}x\|^2 \geq \|T^kx\|^2\|T^2x\|.$$

Proof. (1) For a unit vector x in H, we may assume $Tx \neq 0$. Then

$$\begin{aligned} M^2\|T^3x\| &= M^2\|Tx\| \left\| T^2\frac{Tx}{\|Tx\|} \right\| \geq M\|Tx\| \left\| T\frac{Tx}{\|Tx\|} \right\|^2 \\ &= M\frac{\|T^2x\|^2}{\|Tx\|} \geq \frac{\|T^2x\|\|Tx\|^2}{\|Tx\|} = \|T^2x\|\|Tx\|. \end{aligned}$$

(2) For the case $k = 1$,

$$M\|T^2x\|^2 = M\|T^2x\|\|T^2x\| \geq \|Tx\|^2\|T^2x\|$$

and (P_1) is clear. Now suppose that (P_k) is valid for k and we assume that $\|Tx\| \neq 0$. Then

$$
\begin{aligned}
M^{2k+1}\|T^{k+2}x\|^2 &= M^{2k+1}\|Tx\|^2\left\|T^{k+1}(\frac{Tx}{\|Tx\|})\right\|^2\\
&\geq M^2\|Tx\|^2\left\|T^k(\frac{Tx}{\|Tx\|})\right\|^2\|T^2(\frac{Tx}{\|Tx\|})\|\\
&= M^2\|T^{k+1}x\|^2\frac{\|T^3x\|}{\|Tx\|}\\
&\geq \|T^{k+1}x\|^2\|T^2x\|
\end{aligned}
$$

by (1) and (P_k). So (P_{k+1}) is valid and the proof is completed by the mathematical induction.

(Another method) Since T is $M-$paranormal, $\|Tx\|^2 \leq M\|T^2x\|$ for any unit vector x. Putting $\frac{T^kx}{\|T^kx\|}$ instead of x, we have

$$
\begin{aligned}
&\left\|T\left(\frac{T^kx}{\|T^kx\|}\right)\right\|^2 \leq M\left\|T^2\left(\frac{T^kx}{\|T^kx\|}\right)\right\|\\
\Leftrightarrow &\frac{\|T^{k+1}x\|^2}{\|T^kx\|^2} \leq M\frac{\|T^{k+2}x\|}{\|T^kx\|}.
\end{aligned}
$$

Therefore $\frac{\|T^{k+1}x\|^2}{\|T^kx\|} \leq M\|T^{k+2}x\|$. Squaring both sides, we have

$$
\frac{1}{M^2}\cdot\frac{\|T^{k+1}x\|^4}{\|T^kx\|^2} \leq \|T^2(T^kx)\|^2 \quad (k = 1, 2, \cdots).
$$

Hence

$$
\begin{aligned}
\frac{\|T^{k+1}x\|^2}{\|T^k x\|^2} = \frac{\|T^2(T^{k-1}x)\|^2}{\|T^k x\|^2} &\geq \frac{1}{\|T^k x\|^2}\cdot\frac{1}{M^2}\cdot\frac{\|T^k x\|^4}{\|T^{k-1}x\|^2} \\
&= \frac{1}{M^2}\cdot\frac{\|T^k x\|^2}{\|T^{k-1}x\|^2} \\
&= \frac{1}{M^2}\cdot\frac{1}{\|T^{k-1}x\|^2}\|T^2(T^{k-2}x)\|^2 \\
&\geq \frac{1}{M^2}\cdot\frac{1}{\|T^{k-1}x\|^2}\cdot\frac{1}{M^2}\cdot\frac{\|T^{k-1}x\|^4}{\|T^{k-2}x\|^2} \\
&= \frac{1}{M^4}\cdot\frac{1}{\|T^{k-2}x\|^2}\|T^{k-1}x\|^2 \\
&\geq \frac{1}{M^6}\cdot\frac{\|T^{k-2}x\|^2}{\|T^{k-3}x\|^2} \geq \cdots \\
&\geq \frac{1}{M^{2(k-1)}}\cdot\frac{\|T^2x\|^2}{\|Tx\|^2} \\
&\geq \frac{1}{M^{2(k-1)}}\cdot\frac{\|T^2x\|^2}{M\|T^2x\|} = \frac{1}{M^{2k-1}}\|T^2x\|.
\end{aligned}
$$

Therefore

$$\|T^{k+1}x\|^2 \geq \frac{1}{M^{2k-1}}\|T^k x\|^2\|T^2x\|$$

for any unit vector x. □

Corollary 3.19. *Let T be a paranormal operator. Then*

(1) $\|T^3x\| \geq \|T^2x\|\|Tx\|$ *for every unit vector x in H.*

(2) *For every positive integer k and every unit vector x in H,*

$$\|T^{k+1}x\|^2 \geq \|T^k x\|^2\|T^2x\|.$$

Theorem 3.20. *If T is M-paranormal, then T^2 is M^4-paranormal.*

Proof. Let λ be any real number. Then since T is M-paranormal, we have

$$
\begin{aligned}
&M^2T^{*3}T^3 + 2\lambda T^{*2}T^2 + \lambda^2T^*T \\
&= T^*(M^2T^{*2}T^2 + 2\lambda T^*T + \lambda^2 I)T \geq 0.
\end{aligned}
$$

Therefore $\left(\left(M^2T^{*3}T^3 + 2\lambda T^{*2}T^2 + \lambda^2T^*T\right)x, x\right) \geq 0$ for any unit vector $x \in H$, and so

$$M^2\|T^3x\|^2 + 2\lambda\|T^2x\|^2 + \lambda^2\|Tx\|^2 \geq 0.$$

Since

$$\begin{aligned}\|T^3x\|^2 = \|T(T^2x)\|^2 &= \left\|\frac{T(T^2x)}{\|T^2x\|}\|T^2x\|\right\|^2 \\ &= \|T^2x\|^2\left\|T\frac{T^2x}{\|T^2x\|}\right\|^2 \\ &\leq \|T^2x\|^2 M\left\|T^2\left(\frac{T^2x}{\|T^2x\|}\right)\right\| \\ &= M\|T^2x\|\|T^4x\|,\end{aligned}$$

we have

$$M^3\|T^2x\|\|T^4x\| + 2\lambda\|T^2x\|^2 + \lambda^2\|Tx\|^2 \geq 0$$

for any unit vector $x \in H$ and so

$$D/4 = \|T^2x\|^4 - M^3\|T^2x\|\|T^4x\|\|Tx\|^2 \leq 0.$$

Since T is M-paranormal,

$$\|T^2x\|^4 - M^3\|T^2x\|\|T^4x\|M\|T^2x\| \leq 0.$$

Dividing both by $\|T^2x\|^2$, we have $\|T^2x\|^2 \leq M^4\|T^4x\|$ for some $M > 0$ and so T^2 is M^4-paranormal. □

Theorem 3.21. *Let T and S be commuting M-paranormal operators. Then the product TS is M-paranormal if one of the following holds;*

(1) $\|TSx\|\|x\| \geq \sqrt{M}\|Tx\|\|Sx\|$ *for any* $x \in H$.
(2) $\|T^2Sx\|\|x\| \geq M\|T^2x\|\|Sx\|$ *for any* $x \in H$.

Proof. First, suppose that (1) holds. Since T is M-paranormal,

$$M\|T^2(S^2x)\|\|S^2x\| \geq \|T(S^2x)\|^2.$$

Thus we have

$$\begin{aligned} M\|T^2S^2x\|\|S^2x\|\|Sx\|^2\|x\| &\geq \|TS^2x\|^2\|Sx\|^2\|x\| \\ &\geq M\|TSx\|^2\|S^2x\|^2\|x\| \\ &\geq \|TSx\|^2\|S^2x\|\|Sx\|^2 \end{aligned}$$

for any vector $x \in H$. Therefore $M\|T^2S^2x\|\|x\| \geq \|TSx\|^2$, and so TS is M-paranormal.

Secondly, suppose that (2) holds. Then

$$\begin{aligned} M\|T^2S^2x\|\|T^2x\|\|x\| &\geq \|S(T^2x)\|^2\|x\| \\ &= \|T^2Sx\|\|T^2Sx\|\|x\| \\ &\geq M\|T^2Sx\|\|T^2x\|\|Sx\| \\ &\geq \|TSx\|^2\|T^2x\|. \end{aligned}$$

Therefore $M\|T^2S^2x\|\|x\| \geq \|TSx\|^2$ and so TS is M-paranormal. □

Theorem 3.22. *Let T and S be double commuting M-paranormal operators and let $M > \frac{1}{2}$. Then the product TS is M-paranormal if*

$$(2M-1)\|T^2S^2x\|\|x\| \geq \|T^2x\|\|S^2x\|$$

for any $x \in H$.

Proof. Suppose that $(2M-1)\|T^2S^2x\|\|x\| \geq \|T^2x\|\|S^2x\|$ for any $x \in H$. Then

$$\begin{aligned} &M^2(TS)^{*2}(TS)^2 + \lambda^2\rho^2 + \lambda^2 MS^{*2}S^2 + \rho^2 MT^{*2}T^2 \\ &= (MS^{*2}S^2 + \rho^2)(MT^{*2}T^2 + \lambda^2) \\ &\geq 4M\lambda\rho(TS)^*(TS) \end{aligned}$$

for any $\lambda, \rho > 0$. Thus

$$\begin{aligned} M^2\|T^2S^2x\|^2 + \lambda^2\rho^2\|x\|^2 + M\lambda^2\|S^2x\|^2 + M\rho^2\|T^2x\|^2 \\ \geq 4M\lambda\rho\|TSx\|^2. \end{aligned}$$

Put $\lambda\rho = \frac{M\|T^2S^2x\|}{\|x\|}$ and $\frac{\rho}{\lambda} = \frac{\|S^2x\|}{\|T^2x\|}$. Then $\lambda\|S^2x\| = \rho\|T^2x\|$,

$$\rho^2 = \frac{M\|T^2S^2x\|\|S^2x\|}{\|T^2x\|\|x\|},$$

and so

$$M^2\|T^2S^2x\|^2 + M^2\|T^2S^2x\|^2 + 2M\rho^2\|T^2x\|^2 \geq 4M\frac{M\|T^2S^2x\|}{\|x\|}\|TSx\|^2.$$

Therefore

$$M^2\|T^2S^2x\|^2 + M\rho^2\|T^2x\|^2 \geq 2M^2\frac{1}{\|x\|}\|T^2S^2x\|\|TSx\|^2,$$

i.e.,

$$M^2\|T^2S^2x\|^2 + M\cdot\frac{M\|T^2S^2x\|\|S^2x\|}{\|T^2x\|\|x\|}\|T^2x\|^2 \geq 2M^2\frac{1}{\|x\|}\|T^2S^2x\|\|TSx\|^2$$

and so we have

$$\|T^2S^2x\|\|x\| + \|S^2x\|\|T^2x\| \geq 2\|TSx\|^2.$$

By hypothesis,

$$\|T^2S^2x\|\|x\| + (2M-1)\|T^2S^2x\|\|x\| \geq 2\|TSx\|^2.$$

Hence $M\|T^2S^2x\|\|x\| \geq \|TSx\|^2$. This completes the proof. □

Recall that an operator T is *isometric* if $\|Tx\| = \|x\|$ for all $x \in H$. It is easy to verify that every isometric operator is hyponormal.

Theorem 3.23.

(1) *([2])If an M-paranormal operator T double commutes with a hyponormal operator S, then the product TS is M-paranormal.*

(2) *If a paranormal operator T double commutes with an M-hyponormal operator S, then TS is M-paranormal.*

(3) *If an M-paranormal operator T commutes with an isometrc operator S, then TS is M-paranormal.*

Proof. (1) Let $\{E(t)\}$ be the resolution of the identity for the self-adjoint operator S^*S. By hypothesis T^*T and $T^{*2}T^2$ both commute with every $E(t)$. Since S is hyponormal, $S^*S \geq SS^*$. Hence for each $\lambda > 0$,

$$\begin{aligned}
&M^2[(TS)^*]^2(TS)^2 - 2\lambda(TS)^*(TS) + \lambda^2 \\
&= M^2(T^{*2}T^2)(S^{*2}S^2) - 2\lambda(T^*T)(S^*S) + \lambda^2 \\
&\geq M^2(T^{*2}T^2)(S^*S)^2 - 2\lambda(T^*T)(S^*S) + \lambda^2 \\
&= \int_0^\infty (t^2M^2T^{*2}T^2 - 2\lambda tT^*T + \lambda^2)dE(t) \\
&\geq 0,
\end{aligned}$$

since T is M-paranormal. Hence TS is M-paranormal by Theorem 3.8.

(2) If S is a M-hyponormal operator, then $M^2S^*S \geq SS^*$ ([57]). Now if T is any operator double commuting with S, then

$$\begin{aligned}
&M^2(TS)^{*2}(TS)^2 - 2\lambda(TS)^*(TS) + \lambda^2 I \\
&\geq T^{*2}T^2(S^*S)^2 - 2\lambda(T^*T)(S^*S) + \lambda^2 I
\end{aligned}$$

for each λ. Using this and arguing as in (1), the proof is completed.

(3) If $A = TS$, then we have for any real λ there exists $M > 0$ such that

$$\begin{aligned}
&M^2A^{*2}A^2 + 2\lambda A^*A + \lambda^2 I \\
&= M^2S^*T^*S^*T^*TSTS + 2\lambda S^*T^*TS + \lambda^2 I.
\end{aligned}$$

Using $TS = ST$, $T^*S^* = S^*T^*$ and $S^*S = I$, we get

$$M^2A^{*2}A^2 + 2\lambda A^*A + \lambda^2 I = M^2T^{*2}T^2 + 2\lambda T^*T + \lambda^2 I \geq 0.$$

Hence $A = TS$ is M-paranormal. □

Corollary 3.24.

(1) *If a paranormal operator T double commutes with a hyponormal operator S, then the product TS is paranormal.*

(2) *([46]) If a paranormal operator T commutes with an isometrc operator S, then TS is paranormal.*

Theorem 3.25. *([2]) Let T and S be double commuting operators. Let one of T and S be paranormal and other be M-paranormal. Then the product TS is M-paranormal if there are a self-adjoint operator A and bounded positive Borel function $f(t)$ and $g(t)$ such that*

$$(f(t) - f(s))(g(t) - g(s)) \geq 0 \quad (-\infty < t, s < \infty),$$

and one of the following holds;

(1) $f(A) = T^*T$ *and* $g(A) = S^*S$.
(2) $f(A) = T^{*2}T^2$ *and* $g(A) = S^*S$.

Proof. First of all, we remark that the assumption implies

$$(f(A)g(A)x, x) \cdot (x, x) \geq (f(A)x, x) \cdot (g(A)x, x).$$

Because, let $\{E(t)\}$ be the resolution of identity for A. Then

$$\begin{aligned}
&(f(A)g(A)x, x) \cdot (x, x) - (f(A)x, x) \cdot (g(A)x, x) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{f(t)g(t) - f(t)g(s)\} d(E(t)x, x) d(E(s)x, x) \\
&= \int \int_{t>s} (f(t) - f(s))(g(t) - g(s)) d(E(t)x, x) d(E(s)x, x) \geq 0.
\end{aligned}$$

Double commutativity, when applied to (1), (2), yield respectively
(1^*) $\|TSx\|\|x\| \geq \|Tx\|\|Sx\|$.
(2^*) $\|T^2Sx\|\|x\| \geq \|T^2x\|\|Sx\|$.

Let (1^*) holds. Without loss of generality, we may assume that T is M-paranormal and S is paranormal. Then

$$\begin{aligned}
M\|T^2S^2x\|\|S^2x\|\|Sx\|^2\|x\| &\geq \|TS^2x\|^2\|Sx\|^2\|x\| \\
&\geq \|TSx\|^2\|S^2x\|^2\|x\| \\
&\geq \|TSx\|^2\|S^2x\|\|Sx\|^2.
\end{aligned}$$

Therefore

$$M\|T^2S^2x\|\|x\| \geq \|TSx\|^2.$$

This implies that TS is M-paranormal.

Let (2^*) holds. Assume that T is M-paranormal and S is paranormal. Then

$$\begin{aligned}\|T^2S^2x\|\|T^2x\|\|x\| &\geq \|S(T^2x)\|^2\|x\| \\ &= \|T^2Sx\|\|T^2Sx\|\|x\| \\ &\geq \|T^2Sx\|\|T^2x\|\|Sx\| \\ &\geq \frac{1}{M}\|TSx\|^2\|T^2x\|.\end{aligned}$$

Therefore $M\|T^2S^2x\|\|x\| \geq \|TSx\|^2$ for any vector x. This implies that TS is M-paranormal. □

Motivated by M-power class considered by V. Istratescu([30]), we consider the subclass S of M-paranormal operators satisfying

$$\|T^nx\|^2 \leq M\|T^{2n}x\|$$

for each $n \geq 1$ and for all unit vector $x \in H$. We can easily prove the following:

Theorem 3.26. *The followings are valid:*

(1) *If $T \in S$, then the spectral radius $r(T)$ of T satisfies*

$$\frac{1}{M}\|T\| \leq r(T).$$

(2) *If $T \in S$ and is invertible, then $T^{-1} \in S$.*

(3) *If $T \in S$ and $z \in \rho(T)$, the resolvent set of T, then*

$$\|(T-z)^{-1}\| \leq \frac{M}{d(z,\sigma(T))}.$$

(4) *If $T \in S$ and is quasinilpotent then $T = 0$.*

(5) *If $T \in S$, then the set*

$$M_T = \{x : \|T^nx\| \leq M\|x_n\|, n = 1, 2, \cdots\}$$

is a closed invariant subspace for T and also for operators commuting with T.

Proof. (1) Suppose $T \in S$. Then $\|T^n x\|^2 \leq M\|T^{2n}x\|$ for each $n \geq 1$ and all unit vector $x \in H$. If $n = 1$, then $\|T^2 x\| \geq \frac{1}{M}\|Tx\|^2$, and if $n = 2$, then

$$\|T^4 x\| \geq \frac{1}{M}\|T^2 x\|^2 \geq \frac{1}{M}(\frac{1}{M}\|Tx\|^2)^2 = \frac{1}{M^3}\|Tx\|^4.$$

By the same method, $\|T^8 x\| \geq \frac{1}{M^7}\|Tx\|^8$ if $n = 4$. In general, we have $\|T^{2n}x\| \geq \frac{1}{M^{2n-1}}\|Tx\|^{2n}$ for each $n \geq 1$. Therefore

$$\|T^{2n}\| = \sup_{\|x\|=1} \|T^{2n}x\| \geq \frac{1}{M^{2n-1}} \sup_{\|x\|=1} \|Tx\|^{2n} = \frac{1}{M^{2n-1}}\|T\|^{2n},$$

and so

$$\begin{aligned} r(T) &= \lim_{n\to\infty} \|T^{2n}\|^{1/2n} \geq \lim_{n\to\infty} (\frac{1}{M^{2n-1}}\|T\|^{2n})^{1/2n} \\ &= \lim_{n\to\infty} \frac{1}{M^{(2n-1)/2n}}\|T\| = \frac{1}{M}\|T\|. \end{aligned}$$

(2) Suppose $T \in S$ and is invertible. Then T^k is M-paranormal and so $\|T^n x\|^2 \leq M\|T^{2n}x\|\|x\|$ for all $x \in H$. We need to show that

$$\|(T^{-1})^n x\|^2 \leq M\|(T^{-1})^{2n}x\|\|x\|. \tag{3.8}$$

In (3.8),

$$\frac{M\|x\|}{\|T^n x\|} \geq \frac{\|T^n x\|}{\|T^{2n}x\|}.$$

Replacing x by $T^{-2n}x$,

$$\frac{M\|T^{-2n}x\|}{\|T^n(T^{-2n}x)\|} \geq \frac{\|T^n(T^{-2n}x)\|}{\|T^{2n}(T^{-2n}x)\|}.$$

Therefore $\|(T^{-1})^n x\|^2 \leq M\|(T^{-1})^{2n}x\|\|x\|$ for all $x \in H$.

(4) Since $T \in S$ and is quainilpotent, by (1),

$$\frac{1}{M}\|T\| \leq r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} = 0.$$

Therefore $T = 0$. □

Theorem 3.27. *Let T be M-paranormal. The followings are valid;*

(1)

$$\|T^n x\|^2 \leq M\|T^{n+1}x\|\|T^{n-1}x\|$$

for every positive integer n and for every unit vector x.

(2)

$$M^{\frac{n(n-1)}{2}}\|T^n x\| \geq \|Tx\|^n$$

for any unit vector $x \in H$ and for every positive integer $n \geq 2$.

Proof. (1) (Mathematical Induction) If $n = 1$, then it holds immediately by the definition of M-paranormal. Suppose our results holds for $n = k$. Then for every unit vector x, we have

$$\|T^k x\|^2 \leq M\|T^{k+1}x\|\|T^{k-1}x\|.$$

We shall show that $\|T^{k+1}x\|^2 \leq M\|T^{k+2}x\|\|T^k x\|$ for every unit vector x. We have

$$\begin{aligned} M\|T^{k+2}x\| &= M\left\|T^{k+1}\left(\frac{Tx}{\|Tx\|}\right)\right\|\|Tx\| \\ &\geq \frac{\left\|T^k\left(\frac{Tx}{\|Tx\|}\right)\right\|^2}{\left\|T^{k-1}\left(\frac{Tx}{\|Tx\|}\right)\right\|}\|Tx\| \\ &= \frac{\|T^{k+1}x\|^2}{\|T^k x\|} \end{aligned}$$

and hence $\|T^{k+1}x\|^2 \leq M\|T^{k+2}x\|\|T^k x\|$ $(k = 1, 2, 3, \cdots)$. This completes the proof.

(2) (Mathematical Induction) If $n = 2$, then it holds by definition of $M-$paranormal. Suppose that it holds for $n = k$. Then

$$M^{\frac{k(k-1)}{2}}\|T^k x\| \geq \|Tx\|^k$$

for $k = 1, 2, \cdots, n$ and for any unit vector $x \in H$. It is sufficient to show that

$$M^{\frac{k(k+1)}{2}}\|T^{k+1}x\| \geq \|Tx\|^{k+1}$$

for any unit vector $x \in H$. Thus we have

$$\begin{aligned} M^{\frac{k(k+1)}{2}}\|T^{k+1}x\| &= M^{\frac{k(k+1)}{2}}\left\|T^k\frac{Tx}{\|Tx\|}\right\|\|Tx\| \\ &\geq M^k\left\|T\left(\frac{Tx}{\|Tx\|}\right)\right\|^k\|Tx\| \\ &= M^k\|T^2x\|^k\frac{1}{\|Tx\|^{k-1}} \\ &\geq \|Tx\|^{2k}\frac{1}{\|Tx\|^{k-1}} = \|Tx\|^{k+1}. \end{aligned}$$

By mathematical induction, our results holds for every positive integer $n \geq 2$ and for any unit vector $x \in H$. That is, $M^{\frac{n(n-1)}{2}}\|T^nx\| \geq \|Tx\|^n$ for every positive integer $n \geq 2$ and for any unit vector $x \in H$. □

Corollary 3.28. *Let T be paranormal. Then we have the followings;*

(1)

$$\|T^nx\|^2 \leq \|T^{n+1}x\|\|T^{n-1}x\|$$

for every positive integer n and for every unit vector x.

(2)

$$\|T^nx\| \geq \|Tx\|^n$$

for any unit vector $x \in H$ and for every positive integer n.

J. Stampili([49]) proved that if T is hyponormal with its spectrum on the unit circle, then T is unitary. We generalize this result to paranormal operators.

Theorem 3.29. *([32]) If T is paranormal with its spectrum on the unit circle, then T is unitary.*

Proof. Since T is a paranormal operator, the result $\|T\| =$ spectral radius of T follows from the fact that spectral radius of $T = \lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}$. Since the spectrum of T lies on the unit circle, $0 \notin \sigma(T)$ and so T is invertible. Then by Corollary 3.12(3), T and T^{-1} are paranormal operators. Therefore by Lemma 3.4,

$$\|T\| = \|T^{-1}\| = \sup\{|\lambda| : \lambda \in \sigma(T)\} = 1.$$

Now $\|x\| = \|T^{-1}Tx\| \leq \|Tx\| \leq \|x\|$, that is, $\|Tx\| = \|x\|$ for all $x \in H$. Since T is invertible, T is unitary. This completes the proof.

(Another method) If $\sigma(T)$ lies on the unit circle, then $\|T\| = \|T^{-1}\| = 1$ since T is normaloid. We have

$$\begin{aligned} \|x\| \geq \|Tx\| &= \|T^{-1}x\| \left\| T^2 \frac{T^{-1}x}{\|T^{-1}x\|} \right\| \\ &\geq \|T^{-1}x\| \left\| T \frac{T^{-1}x}{\|T^{-1}x\|} \right\|^2 \\ &= \frac{\|x\|^2}{\|T^{-1}x\|} \geq \|x\|. \end{aligned}$$

Hence $\|Tx\| = \|x\|$ for $x \in H$ and T is a unitary operator. □

Theorem 3.30. *Let λ and μ be distinct eigenvalues of a M-paranormal operator T and $0 < M \leq 1$. Then* $\ker(T - \lambda) \perp \ker(T - \mu)$.

Proof. Without loss of generality, we may assume that $|\lambda| \leq 1$ and $\mu = 1$. Let $T = \begin{pmatrix} \lambda & A \\ 0 & B \end{pmatrix}$ on $H = \ker(T - \lambda) \oplus \overline{\operatorname{ran}(T - \lambda)^*}$. Let

$$x = x_1 \oplus x_2 \in \ker(T - \lambda) \oplus \overline{\operatorname{ran}(T - \lambda)^*}$$

be an arbitrary (non-zero) eigenvector of T with respect to 1. Then $Tx = x$ implies that $\lambda x_1 + Ax_2 = x_1$ and $Bx_2 = x_2$. Hence $Ax_2 = (1 - \lambda)x_1$. If $x_2 = 0$, then $x_1 = 0$ since $1 - \lambda \neq 0$, and this is a contradiction for $x \neq 0$. Thus we have $x_2 \neq 0$ and we may assume $\|x_2\| = 1$. We shall show that $x_1 = 0$. Put $x' = 0 \oplus x_2$. Then

$$\begin{aligned} Tx' = (1 - \lambda)x_1 \oplus x_2, \quad & T^2x' = (1 - \lambda^2)x_1 \oplus x_2, \cdots, \\ & T^n x' = (1 - \lambda^n)x_1 \oplus x_2. \end{aligned}$$

Since T is M-paranormal and $\|x'\| = 1$, by the above Lemma 3.27(2) we have $M^{\frac{n(n-1)}{2}}\|T^n x'\| \geq \|Tx'\|^n$ for every positive integer $n \geq 2$ and for any unit vector $x' \in H$. Hence

$$\left(\sqrt{|1-\lambda|^2\|x_1\|^2+1}\,\right)^n \leq M^{\frac{n(n-1)}{2}}\sqrt{|1-\lambda^n|^2\|x_1\|^2+1}$$

for every positive integer $n \geq 2$. And this is impossible unless $x_1 = 0$ because if $x_1 \neq 0$, the left-side of the above inequality tends to ∞ as $n \to \infty$ and the right-side is uniformly bounded by $\sqrt{4\|x_1\|^2+1}$. So we have $x_1 = 0$ and

$$x = 0 \oplus x_2 \in \overline{\text{ran}(T-\lambda)^*} = (\ker(T-\lambda))^{\perp}.$$

This completes the proof. □

Corollary 3.31. *Let λ and μ be distinct eigenvalues of a paranormal operator T. Then* $\ker(T-\lambda) \perp \ker(T-\mu)$.

Lemma 3.32. *([54]) Let T be a paranoral operator and let $\lambda \in \sigma(T)$ be an isolated point. Then the Riesz projection*

$$E = \frac{1}{2\pi i}\int_{\partial D}(z-T)^{-1}dz \tag{3.9}$$

satisfies EH = $\ker(T-\lambda)$, where D is a closed disk with its center λ and satisfies $D \cap \sigma(T) = \{\lambda\}$.

Proof. Since E commutes with T, the range $R(E)$ is T-invariant, so the restriction $T|_{EH}$ of T to EH is also paranormal. It is easy to see that the spectrum $\sigma(T|_{EH}) = \{\lambda\}$. $T|_{EH} = \lambda$ immediately from this, see ([55], [56]). Hence $EH \subset \ker(T-\lambda)$.

Conversely, let $x \in \ker(T-\lambda)$ be any vector. Then by Cauchy's integration theorem,

$$\begin{aligned} Ex &= \frac{1}{2\pi i}\int_{\partial D}(z-T)^{-1}x\,dz = \frac{1}{2\pi i}\int_{\partial D}(z-\lambda)^{-1}x\,dz \\ &= \left\{\frac{1}{2\pi i}\int_{\partial D}\frac{1}{z-\lambda}dz\right\}x = x. \end{aligned}$$

Hence we have also $\ker(T-\lambda)$. This completes the proof. □

Theorem 3.33. *If T is paranormal, then T is isoloid.*

Proof. Let $\lambda \in \sigma(T)$ be an isolated point. Then the Riesz projection

$$E = \frac{1}{2\pi i}\int_{\partial D}(z-T)^{-1}dz \tag{3.9}$$

is invariant closed subspace of T and $\sigma(T|_{EH}) = \{\lambda\}$, where D is a closed disk with its center λ and satisfies $D \cap \sigma(T) = \{\lambda\}$. If $\lambda = 0$, then $\sigma(T|_{EH}) = \{0\}$. Since $T|_{EH}$ is paranormal by the above Corollary 3.12(1), $T|_{EH} = 0$ by Lemma 3.4. Therefore 0 is an eigenvalue of T. If $\lambda \neq 0$, then $T|_{EH}$ is an invertible paranormal operator and hence $(T|_{EH})^{-1}$ is also paranormal by Corollary 3.13. By Lemma 3.4, we see $\|T|_{EH}\| = |\lambda|$ and $\|(T|_{EH})^{-1}\| = |\frac{1}{\lambda}|$. Let $x \in EH$ be an arbitrary vector. Then $\|x\| \leq \|(T|_{EH})^{-1}\|\|(T|_{EH}x\| = \frac{1}{|\lambda|}\|T|_{EH}x\| \leq \frac{1}{|\lambda|}|\lambda|\|x\| = \|x\|$. This implies that $\frac{1}{\lambda}T|_{EH}$ is unitary with its spctrum $\sigma(\frac{1}{\lambda}T|_{EH}) = \{1\}$. Hence $T|_{EH} = \lambda$ and λ is an eigenvalue of T. This completes the proof. □

Lemma 3.34. *([54]) Let $T = \begin{pmatrix} \lambda & S \\ 0 & T_1 \end{pmatrix}$ on $\mathcal{H} = \ker(T-\lambda) \oplus \overline{ran(T-\lambda)^*}$ be paranormal, where $\lambda \in \sigma_p(T)$ is an arbitrary point. Then $\ker(T_1-\lambda) = \{0\}$.*

Proof. Suppose $\ker(T_1-\lambda) \neq \{0\}$. Then for each non-zero $x \in \ker(T_1-\lambda)$, we have $Sx \neq 0$, otherwise $(T-\lambda)x = Sx \oplus (T_1-\lambda)x = 0$ and hence $x \in \overline{\mathrm{ran}(T-\lambda)^*} \cap \ker(T-\lambda) = \{0\}$.

First, we consider the case $\lambda = 0$. Then $Tx = Sx \oplus T_1x = Sx$ since $x \in \ker T_1$ and $T^2x = TSx = 0$ since $Sx \in \ker T$. Paranormality of T implies

$$\|Sx\|^2 = \|Tx\|^2 \leq \|T^2x\|\|x\| = 0,$$

hence $Sx = 0$ and this is a contradiction. Therefore $\ker T_1 = \{0\}$.

Next, we consider the case $\lambda \neq 0$. Without loss of generality, we may assume that $\|x\| = 1$. It is easy to see that

$$T^n x = n\lambda^{n-1} Sx \oplus \lambda^n x \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.10}$$

Since T is paranormal by Corollary 3.28, we have

$$\|T^n x\|^2 \leq \|T^{n+1}x\| \|T^{n-1}x\|$$

for every positive integer n. So we have

$$\begin{aligned} & n^2|\lambda|^{2n-2}\|Sx\|^2 + |\lambda|^{2n} \\ & \leq \sqrt{(n+1)^2|\lambda|^{2n}\|Sx\|^2 + |\lambda|^{2n+2}}\sqrt{(n-1)^2|\lambda|^{2n-4}\|Sx\|^2 + |\lambda|^{2n-2}} \end{aligned}$$

and
(3.11)

$$\begin{aligned} & \left(n^2|\lambda|^{2n-2}\|Sx\|^2 + |\lambda|^{2n}\right)^2 \\ & \leq \{(n+1)^2|\lambda|^{2n}\|Sx\|^2 + |\lambda|^{2n+2}\}\{(n-1)^2|\lambda|^{2n-4}\|Sx\|^2 + |\lambda|^{2n-2}\}. \end{aligned}$$

The left side of (3.11) is

$$n^4|\lambda|^{4n-4}\|Sx\|^4 + 2n^2|\lambda|^{4n-2}\|Sx\|^2 + |\lambda|^{4n},$$

and the right side of (3.11) is

$$\left(n^2 - 1\right)^2 |\lambda|^{4n-4}\|Sx\|^4 + \left\{(n+1)^2 + (n-1)^2\right\} |\lambda|^{4n-2}\|Sx\|^2 + |\lambda|^{4n}.$$

Hence, we have

$$\left\{n^4 - \left(n^2 - 1\right)^2\right\} |\lambda|^{4n-4}\|Sx\|^4 \leq \left\{(n+1)^2 + (n-1)^2 - 2n^2\right\} |\lambda|^{4n-2}\|Sx\|^2$$

and therefore

$$\|Sx\|^2 \leq \frac{2}{2n^2 - 1}|\lambda|^2 \to 0 \quad (\text{as} \quad n \to \infty).$$

This also contradicts the fact that $Sx \neq 0$. Hence $\ker(T_1 - \lambda) = \{0\}$. □

Theorem 3.35. *([54]) Weyl's theorem holds for paranormal operators.*

Proof. First we show that $\sigma(T) \setminus w(T) \subset \pi_{00}(T)$. Let $\lambda \in \sigma(T) \setminus w(T)$ be arbitrary. Then $T - \lambda$ is a Fredholm operator with index 0. Hence $\ker(T-\lambda)$ is a (non-zero) finite demensimal subspace. Let $T = \begin{pmatrix} \lambda & S \\ 0 & T_1 \end{pmatrix}$ on $H = \ker(T-\lambda) \oplus \operatorname{ran}(T-\lambda)^*$ be 2×2 matrix representation. Then $\ker(T_1 - \lambda) = \{0\}$ by Lemma 3.34 and $\operatorname{ind}(T_1 - \lambda) = \operatorname{ind}(T - \lambda) = 0$ since S is a finite rank operator. Hence $T_1 - \lambda$ is an invertible operator on $\operatorname{ran}(T-\lambda)^*$. So we have $\lambda \notin \sigma(T_1)$ and therefore λ is an isolated point in $\sigma(T)\,(\subset \sigma(T_1) \cup \{\lambda\})$. Hence $\sigma(T) \setminus w(T) \subset \pi_{00}(T)$.

Next we show that $\pi_{00}(T) \subset \sigma(T) \setminus w(T)$. Let $\lambda \in \pi_{00}(T)$ be an arbitrary point. Then the Riesz projection E defined by (3.13) satisfies $EH = \ker(T-\lambda)$ by Lemma 3.32. It is clear that $(1-E)H$ is T-invariant subspace such that $\sigma(T|_{(1-E)H}) \not\ni \lambda$. If we use the decomposition $H = (1-E)H + EH$, we have

$$(T-\lambda)H = (T-\lambda)(1-E)H + (T-\lambda)EH = (1-E)H..$$

Hence $\operatorname{ran}(T-\lambda)$ is closed and

$$\ker(T-\lambda)^* \cong H/\operatorname{ran}(T-\lambda) \cong EH = \ker(T-\lambda).$$

This implies that $T - \lambda$ is a Fredholm operator with index 0 which is not invertible, hence $\lambda \in \sigma(T) \setminus w(T)$. □

Theorem 3.36. *([54]) Let T be a paranormal operator with $w(T) = \{0\}$. Then T is compact and normal.*

Proof. Since Weyl's theorem holds for T, each element in $\sigma(T) \setminus w(T) = \sigma(T) \setminus \{0\}$ is an eigenvalue of T with finite multiplicity, and is isolated in $\sigma(T)$. This implies that $\sigma(T) \setminus \{0\}$ is a finite set or a countable infinite set with its accumulating point is only 0. Put $\sigma(T) \setminus \{0\} = \{\lambda_n\}$, where $\lambda_n \neq \lambda_m$ whenever $n \neq m$ and $\{|\lambda_n|\}$ is a non-increasing sequence. Since T is normaloid, we have $|\lambda_1| = \|T\|$. By the general theory, $(T-\lambda_1)x = 0$ implies $(T-\lambda_1)^* x = 0$. In fact,

$$\begin{aligned} \| \left(\|T\|^2 - T^*T\right)^{\frac{1}{2}} x\|^2 &= \|T\|^2\|x\|^2 - \|Tx\|^2 \\ &= \|T\|^2\|x\|^2 - \|\lambda_1 x\|^2 = 0. \end{aligned}$$

Thus $\lambda_1 T^* x = T^* T x = \|T\|^2 x = |\lambda_1|^2 x$ and $T^* x = \overline{\lambda_1} x$. Hence $\ker(T - \lambda_1)$ is a reducing subspace of T. Let E_1 be the orthogonal projection onto $\ker(T - \lambda_1)$. Then $T = \lambda_1 \oplus T_1$ on $H = E_1 H \oplus (1 - E_1)H$. Since T_1 is paranormal and $\sigma_p(T) = \sigma_p(T_1) \cup \{\lambda_1\}$, we have $\lambda_2 \in \sigma_p(T_1)$. By the same argument as above, $\ker(T_1 - \lambda_2)$ is a finite dimensional reducing subspace of T which is included in $(1 - E_1)H$. Put E_2 be the orthogonal projection onto $\ker(T - \lambda_2)$. Then

$$T = \lambda_1 E_1 \oplus \lambda_2 E_2 \oplus T_2$$

on

$$H = E_1 H \oplus E_2 H \oplus (1 - E_1 - E_2)H.$$

By repeating above argument, each $\ker(T - \lambda_n)$ is a reducing subspace of T and

$$\|T - \bigoplus_{k=1}^{n} \lambda_k E_k\| = \|T_n\| = |\lambda_{n+1}| \to 0$$

as $n \to \infty$. Here E_k is the orthogonal projection onto $Ker(T - \lambda_k)$ and $T = (\bigoplus_{k=1}^{n} \lambda_k E_k) \oplus T_n$ on

$$H = \left(\bigoplus_{k=1}^{n} E_k H \right) \oplus \left(1 - \sum_{k=1}^{n} E_k \right) H.$$

Hence $T = \bigoplus_{k=1}^{\infty} \lambda_k E_k$ is compact and normal because each E_k is a finite rank orthogonal projection which satisfies $E_k E_l = 0$ whenever $k \neq l$ and $\lambda_n \to 0$ as $n \to \infty$. □

4. The kth roots of paranormal operators

In this section we shall study a new class of operators called a kth roots of G-operator: An operator $T \in B(H)$ is a *kth root of a G-operator* if T^k is a G-operator. In particular, if a G-operator is paranormal, then T is called *the kth root of a paranormal operator* if T^k is paranormal. Also we show that if T is a kth root of a paranormal operator with $0 \in \pi_{00}(T^k)$, then T is a Weyl operator. If S and T are commuting kth roots of paranormal operators respectively, we prove that ST is Weyl if and only if S and T are both Weyl.

Lemma 4.1. *(1) ([5],[27]) Every hyponormal operator T on a finite dimensional Hilbert space is normal.*

(2) If T is paranormal, then T^k is paranormal for every positive integer k.

Proof. (2) (Method 1) It is sufficient to show that if T and T^k is paranormal, then T^{k+1} is also paranormal. We may assume $\|T^2x\| \neq 0$. Then

$$\begin{aligned}
\|T^{2(k+1)}x\| &= \left\|T^{2k}\left(\frac{T^2x}{\|T^2x\|}\right)\right\| \|T^2x\| \\
&\geq \left\|T^{k}\left(\frac{T^2x}{\|T^2x\|}\right)\right\|^2 \|T^2x\| \\
&= \frac{\|T^{k+2}x\|^2}{\|T^2x\|} \geq \frac{\|T^{k+1}x\|^2\|T^2x\|}{\|T^2x\|} \\
&= \|T^{k+1}x\|^2
\end{aligned}$$

by (P_{k+1}) of Lemma 3.19(2). Hence T^{k+1} is paranormal.

(Method 2) We give the proof by induction. First we prove that T^2 is also paranormal. Since T is paranormal, we have, for any real λ,

$$T^{*3}T^3 + 2\lambda T^{*2}T^2 + \lambda^2 T^*T = T^*(T^{*2}T^2 + 2\lambda T^*T + \lambda^2 I)T \geq 0$$

which is equivalent to

$$\|T^3x\|^2 + 2\lambda\|T^2x\|^2 + \lambda^2\|Tx\|^2 \geq 0$$

for every unit vector x. As $\|T^3x\|^2 = \|T(T^2x)\|^2 \le \|T^4x\| \cdot \|T^2x\|$, we have for every real λ,

$$\|T^4x\| \cdot \|T^2x\| + 2\lambda\|T^2x\|^2 + \lambda^2\|Tx\|^2 \ge 0.$$

Hence

$$\|T^2x\|^4 \le \|T^4x\| \cdot \|T^2x\| \cdot \|Tx\|^2 \le \|T^4x\| \cdot \|T^2x\|^2.$$

Therefore $\|T^2x\|^2 \le \|T^4x\|$, so that T^2 is paranormal.

Now assuming that T^k is paranormal, we show that T^{k+1} is also paranormal. Therefore we have for any real λ

$$\begin{aligned} &T^{*(2k+1)}T^{2k+1} + 2\lambda T^{*(k+1)}T^{k+1} + \lambda^2 T^*T \\ &= T^*(T^{*(2k)}T^{2k} + 2\lambda T^{*(k)}T^k + \lambda^2 I)T \ge 0. \end{aligned}$$

This implies that

$$\|T^{2k+1}x\|^2 + 2\lambda\|T^{k+1}x\|^2 + \lambda^2\|Tx\|^2 \ge 0$$

for every unit vector x. Hence $\|T^{k+1}x\|^4 \le \|T^{2k+1}x\|^2\|Tx\|^2$. Now T being paranormal, by Lemma 3.19(2) we have $\|T^{n+1}x\|^2 \ge \|T^nx\|^2\|T^2x\|$ for any positive integer n. Thus

$$\|T^{k+1}x\|^4 \le \|T^{2k+1}x\|^2\|Tx\|^2 \le \|T^{2k+1}x\|^2\|T^2x\| \le \|T^{2(k+1)}x\|^2$$

and T^{k+1} is paranormal.

(Method 3) Suppose T^k $(k \ge 1)$ is also paranormal. Then

$$\|T^{2k}x\|\|x\| \ge \|T^kx\|^2$$

for all $x \in H$. Since T is paranormal, we have

$$\begin{aligned} \|T^{2(k+1)}x\|\|x\| = \|T^{2k}(T^2x)\|\|x\| &\ge \frac{\|T^{k+2}x\|^2}{\|T^2x\|}\|x\| \\ &= \frac{1}{\|T^2x\|}\|T^2(T^kx)\|^2\|x\| \\ &\ge \frac{1}{\|T^2x\|}\left(\frac{\|T^{k+1}x\|^2}{\|T^kx\|}\right)^2\|x\| \\ &= \|T^{k+1}x\|^2 \cdot \frac{\|x\|}{\|T^2x\|} \cdot \frac{\|T^{k+1}x\|^2}{\|T^kx\|^2}, \end{aligned}$$

and

$$\begin{aligned}\frac{\|T^{k+1}x\|^2}{\|T^kx\|^2} &= \frac{\|T^2(T^{k-1}x)\|^2}{\|T^kx\|^2} \\ &\geq \frac{1}{\|T^kx\|^2}\left(\frac{\|T^kx\|^2}{\|T^{k-1}x\|}\right)^2 = \frac{\|T^kx\|^2}{\|T^{k-1}x\|^2} \\ &\geq \cdots \geq \frac{\|T^2x\|^2}{\|Tx\|^2}.\end{aligned}$$

Thus we have

$$\begin{aligned}\|T^{2(k+1)}x\|\|x\| &\geq \|T^{k+1}x\|^2 \cdot \frac{\|x\|}{\|T^2x\|} \cdot \frac{\|T^2x\|^2}{\|Tx\|^2} \\ &= \|T^{k+1}x\|^2 \cdot \frac{\|T^2x\|\|x\|}{\|Tx\|^2} \geq \|T^{k+1}x\|^2.\end{aligned}$$

This shows that T^{k+1} is paranormal. □

Example 4.2. Let H be k-dimensional Hilbert space. Define T on H as

$$T = (a_{ij})$$

where $a_{ij} = 0$ if $i \geq j$ and $a_{ij} = 1$ if $i < j$. Then T^k is hyponormal and so T is a kth root of a hyponormal operator . But $TT^* \neq T^*T$. Therefore T is not hyponormal since every hyponormal operator T on a finite dimensional Hilbert space is normal([5],[27]).

From Example 4.2, we can deduce that if T is any nilpotent operator of order k, i.e., $T^k = 0$, then T is a kth root of a paranormal operator, but it is not necessarily a paranormal operator. Also it is well-known ([34]) that T^2 may not be hyponormal when T is hyponormal.

An operator $T \in B(H)$ will be called *algebraically paranormal* if there exists a nonconstant complex polynomial p such that $p(T)$ is paranormal; *polynomially paranormal* if $p(T)$ is paranormal for every complex polynomial p.

Theorem 4.3.

(1) *Every paranormal operator T is a kth root of a paranormal operator.*

(2) *Every kth root of a paranormal operator T is algebraically paranormal.*

Proof. (1) Suppose T is a paranormal operator. Then by Lemma 4.1(1), T^k is a paranormal operator for every positive integer k. Therefore T is a kth root of a paranormal operator.

(2) Suppose T is a kth root of a paranormal operator. Then T^k is a paranormal operator. Putting $p(x) = x^k$. Then $p(T) = T^k$ is paranormal. Therefore T is algebraically paranormal. □

The converse of the above Theorem is not true by the following examples;

Example 4.4. (1) Let T be an operator on a two-dimensional Hilbert space defined by $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then T is the square root of a paranormal operator. But T is not paranormal since $\|Tx\|^2 = 1 > 0 = \|T^2x\|$ for some $x = (0, 1)$.

(2) If

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

on two dimensional Hilbert space and we put $p(z) = (z-1)^2$, then

$$p(T) = (T - I)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = 0.$$

Hence T is algebraically paranormal since $p(T)$ is paranormal. But T is not the square root of a paranormal operator: In fact, if we take $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then

$$T^2x = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and so $\|T^2x\|^2 = 5$.

On the other hand, since $T^4 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$, we have

$$T^4x = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

and so $\|T^4x\| = \sqrt{17}$. Therefore $\|T^2x\|^2 \nleq \|T^4x\|$. This means T^2 is not a paranormal operator.

From the above Theorem and examples, we have the following implications:

$$\begin{aligned} \text{polynomial paranormal} &\subseteq \text{ paranormal} \\ &\subseteq \text{the } k\text{th roots of paranormal operators} \\ &\subseteq \text{algebraically paranormal.} \end{aligned}$$

Theorem 4.5. *Let $T \in B(H)$ be a kth root of a paranormal operator. Then we have the followings;*

(1) *If T is quasinilpotent, then T is nilpotent.*
(2) *If S is unitarily equivalent to T, then S is a kth root of a paranormal operator.*
(3) *If T is invertible, then T^{-1} is a kth root of a paranormal operator.*
(4) *T^n is a kth root of a paranormal operator for every positive integer n.*
(5) *λT is also kth root of a paranormal operator for every complex number λ.*
(6) *The restriction $T|_M$ to its invariant subspace M is kth root of a paranormal operator.*

Proof. (1) Since T is quasinilpotent, $\sigma(T) = \{0\}$. By the spectral mapping theorem, we get that

$$\sigma(T^k) = \{\sigma(T)\}^k = \{0\}.$$

Hence T^k is quasinilpotent. Since T^k is paranormal and quasinilpotent, by Corollary 3.5, T^k is a zero operator. Therefore T is nilpotent.

(2) Since S is unitarily equivalent to T, there exists a unitary operator U such that $S = U^*TU$. Thus $S^k = (U^*TU)^k = U^*T^kU$ and so S^k is unitarily equivalent to T^k. Since T^k is paranormal by hypothesis, S^k is paranormal and hence S is a kth root of a paranormal operator.

(3) By hypothesis. T^k is an invertible paranormal operator and so $(T^k)^{-1}$ is paranormal. Hence $(T^{-1})^k$ is paranormal i.e., T^{-1} is a kth root of a paranormal operator.

(4) By hypothesis T^k is paranormal and so $(T^k)^n = (T^n)^k$ is paranormal for every positive integer n. Hence T^n is a kth root of a paranormal operator for every positive integer n.

(5) It is sufficient to show that

$$\|(\lambda T)^k x\|^2 \leq \|(\lambda T)^{2k} x\|$$

for all unit vector $x \in H$ and every complex number λ. Since T^k is paranormal,

$$\|(\lambda T)^k x\|^2 = |\lambda|^{2k} \|T^{2k}\|^2 \leq |\lambda|^{2k} \|T^{2k} x\| = \|(\lambda T)^{2k} x\|$$

for all unit vector $x \in H$ and every complex number λ. Therefore λT is also kth root of a paranormal operator for every complex number λ.

(6) Let $x \in M$. Then we have

$$\begin{aligned} \|(T|_M)^k x\|^2 = \|(T^k|_M) x\|^2 = \|T^k x\|^2 &\leq \|T^{2k} x\| \|x\| \\ &= \|(T^{2k}|_M) x\| \|x\| = \|(T|_M)^{2k} x\| \|x\|. \end{aligned}$$

This implies that $(T|_M)^k$ is paranormal, and so $T|_M$ is kth root of a paranormal operator. □

Corollary 4.6. *Let $T \in B(H)$ be a kth root of a hyponormal operator. Then we have the followings;*

(1) *If T is quasinilpotent, then T is nilpotent.*

(2) *If S is unitarily equivalent to T, then S is a kth root of a hyponormal operator.*
(3) *If T is invertible, then T^{-1} is a kth root of a hyponormal operator.*
(4) *T^n is a kth root of a hyponormal operator for every positive integer n.*

If T is hyponormal, then T is normaloid, i.e., $\|T^k\| = \|T\|^k$ for each natural number k. This is not true in the case of a kth root of a hyponormal operator. This can be seen as follows; Let T be the operator on a k-dimensional Hilbert space H in Example 4.2. Then T^k is hyponormal and so T is a kth root of a hyponormal operator. Also $\|T^k\| = 0$. However, it is easy to show that $\|T\|^k = 1$. Hence $\|T\|^k \neq \|T^k\|$. Thus T is not normaloid.

If T is paranormal, then T is normaloid, but the converse is not true ([43]). This is not true in the case of a kth root of a paranormal operator by Example 4.4(1).

Theorem 4.7. *The kth roots of a paranormal operator T is a proper subclass of $B(H)$.*

Proof. Since T^k is a paranormal operator, $\ker T^k = \ker T^{2k}$. Hence we have $\ker T^k = \ker T^{k+1}$ since

$$\ker T^k \subseteq \ker T^{k+1} \subseteq \cdots \subseteq \ker T^{2k}.$$

Let U^* be any unilateral backward shift on $l^2(\mathbb{N})$. Since $\ker(U^*)^k \neq \ker(U^*)^{k+1}$ for any $k \in \mathbb{N}$, $(U^*)^k$ is not paranormal. Therefore U^* is not a kth root of a paranormal operator. □

Theorem 4.8. *Let T be a weighted shift with nonzero weights $\{\alpha_n\}$ $(n = 1, 2, \cdots)$. Then T is a kth root of $M-$paranormal operator if and only if*

$$|\alpha_n||\alpha_{n+1}|\cdots|\alpha_{n+k-1}| \leq M|\alpha_{n+k}||\alpha_{n+k+1}|\cdots|\alpha_{n+2k-1}|$$

for $n = 1, 2, 3, \cdots$.

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of a Hilbert space H.

($\Longrightarrow$) Suppose T is a kth root of $M-$paranormal operator. Then T^k is $M-$paranormal operator. Therefore $\|T^k e_n\|^2 \leq M\|T^{2k}e_n\|$ $(n = 1, 2, \cdots)$. Here

$$T^k e_n = \alpha_n \alpha_{n+1} \cdots \alpha_{n+(k-1)} e_{n+k}$$

and

$$T^{2k} e_n = \alpha_n \alpha_{n+1} \cdots \alpha_{n+(k-1)} \alpha_{n+k} \cdots \alpha_{n+(2k-1)} e_{n+2k}$$

for $k = 1, 2, \cdots$. Then $\|T^k e_n\|^2 \leq M\|T^{2k}e_n\|$ $(n = 1, 2, \cdots)$, and so

$$|\alpha_n|^2 |\alpha_{n+1}|^2 \cdots |\alpha_{n+k-1}|^2 \leq M|\alpha_n||\alpha_{n+1}| \cdots |\alpha_{n+k-1}||\alpha_{n+k}| \cdots |\alpha_{n+2k-1}|.$$

Therefore for $n = 1, 2, 3, \cdots,$

$$|\alpha_n||\alpha_{n+1}| \cdots |\alpha_{n+k-1}| \leq M|\alpha_{n+k}||\alpha_{n+k+1}| \cdots |\alpha_{n+2k-1}|.$$

($\Longleftarrow$) Suppose

$$|\alpha_n||\alpha_{n+1}| \cdots |\alpha_{n+k-1}| \leq M|\alpha_{n+k}||\alpha_{n+k+1}| \cdots |\alpha_{n+2k-1}|$$

for $n = 1, 2, 3, \cdots$. Then we have

$$\begin{aligned}
&M\|T^{2k}e_n\| - \|T^k e_n\|^2 \\
&= M\|\alpha_n \alpha_{n+1} \cdots \alpha_{n+k-1} \alpha_{n+k} \cdots \alpha_{n+2k-1} e_{n+k}\| \\
&\quad - \|\alpha_n \alpha_{n+1} \cdots \alpha_{n+k-1} e_{n+k}\|^2 \\
&= |\alpha_n||\alpha_{n+1}| \cdots |\alpha_{n+k-1}| \left(M|\alpha_{n+k}| \cdots |\alpha_{n+2k-1}| - |\alpha_n| \cdots |\alpha_{n+k-1}|\right) \\
&\geq 0.
\end{aligned}$$

Therefore $\|T^k e_n\|^2 \leq M\|T^{2k}e_n\|$ $(n = 1, 2, \cdots)$ and so T is a kth root of $M-$paranormal operator. □

Theorem 4.9. *Let T be a kth root of a $M-$ paranormal operator. If T commutes with an isometric operator S, then TS is also a kth root of $M-$ paranormal.*

Proof. If $A = (TS)^k = T^k S^k$, then we have for any real λ, there exists $M > 0$ such that

$$\begin{aligned} &M^2 A^{*2} A^2 + 2\lambda A^* A + \lambda^2 I \\ &= M^2 S^{k*} T^{k*} S^{k*} T^{k*} T^k S^k T^k S^k + 2\lambda T^{k*} S^{k*} T^k S^k + \lambda^2 I. \end{aligned}$$

Using $TS = ST, T^*S^* = S^*T^*$ and $S^*S = I$, we get

$$M^2 A^{*2} A^2 + 2\lambda A^* A + \lambda^2 I = M^2 (T^k)^{*2} T^{k^2} + 2\lambda (T^k)^* T^k + \lambda^2 I \geq 0$$

and so $A = (TS)^k$ is $M-$ paranormal operator. Hence TS is a kth roots of $M-$ paranormal. □

The set of operators on H has three useful topologies (weak, strong, and norm). The corresponding concepts of convergence can be described by the following ; $A_n \to A$ *in norm* if and only if $\|A_n - A\| \to 0$, $A_n \to A$ *strongly* if and only if $\|(A_n - A)x\| \to 0$ for every $x \in H$, and $A_n \to A$ *weakly* if and only if $(A_n x, y) \to (Ax, y)$ for every x and y.

By ([34]), we know that the set of all hyponormal operators on H is closed in the norm topology.

Theorem 4.10. *The set of all the kth roots of hyponormal operators is closed in the norm topology.*

Proof. Let T_p be a kth root of a hyponormal operator for each positive integer p and let $\{T_p\}$ converge to an operator T in norm. Then T_p^k is hyponormal for each $p = 1, 2, 3, \cdots$ and we have

$$\begin{aligned} \|T_p^k - T^k\| &= \|T_p^k - T_p^{k-1}T + T_p^{k-1}T - T_p^{k-2}T^2 + T_p^{k-2}T^2 \\ &\quad - \cdots + T_p T^{k-1} - T^k\| \\ &\leq \|T_p^{k-1}\|\|T_p - T\| + \|T_p^{k-2}\|\|T\|\|T_p - T\| \\ &\quad + \cdots + \|T^{k-1}\|\|T_p - T\|. \end{aligned}$$

Since $T_p \to T$ in norm and $\{T_p\}$ is bounded, $T_p^k \to T^n$. Therefore T^k is hyponormal by the above remark. Hence T is the kth root of a hyponormal operator. □

We recall that if $T \in B(H)$ and M is a closed subspace of H, then M is an *invariant subspace for* T if $TM \subset M$, and *a reducing space* if, in addition, $T^*M \subset M$.

Theorem 4.11. *Let $T \in B(H)$ be a kth root of a hyponormal operator. If* $\ker T^*$ *is equal to* $\ker (T^*)^n$*, then* $\ker T$ *reduces for* T.

Proof. Let x be any point in $\ker T$. Then $T(Tx) = 0$ and so $Tx \in \ker T$. Hence $T(\ker T) \subset \ker T$. We need to show that $T^*(\ker T) \subset \ker T$. Since T^k is hyponormal, $\|(T^*)^k x\| \leq \|T^k x\|$ for all $x \in H$, and hence $\ker T^k \subset \ker (T^*)^k$. Since $\ker T^* = \ker (T^*)^k$ and $\ker T \subset \ker T^k$,

$$\ker T \subset \ker T^n \subset \ker (T^*)^k = \ker T^*.$$

Therefore $T^*x = 0$ for all $x \in \ker T$. Hence $T(T^*x) = 0$ for all $x \in \ker T$ i.e., $T^*x \in \ker T$ for all $x \in \ker T$, and so $T^*(\ker T) \subset \ker T$. □

Lemma 4.12. *(Index Continuity Theorem) If $T \in B(H)$ is Fredholm and if $S \in B(H)$ with $\|T - S\| < \epsilon$ for sufficiently small $\epsilon > 0$, then S is Fredholm and ind $S =$ ind T.*

Lemma 4.13.

(1) *([28]) If $ST = TS$ and ST is a Fredholm operator, then both S and T are Fredholm operators.*

(2) *If S and T are in $B(H)$ and ST is Fredholm, then S is Fredholm if and only if T is Fredholm.*

(3) *(Index Product Theorem) If both S and T are Fredholm, then ST is Fredholm and ind$(ST) =$ ind $S +$ ind T.*

Theorem 4.14. *If T is a kth root of a paranormal operator with $0 \in \pi_{00}(T^n)$, then T is a Weyl operator.*

Proof. By (1), T^k is Fredholm if and only if T is Fredholm. Since T^k is a paranormal operator, Weyl's theorem holds for T^k by ([54]). Therefore $\sigma(T^k) - \omega(T^k) = \pi_{00}(T^k)$. Since $0 \in \pi_{00}(T^k) = \sigma(T^k) - \omega(T^k)$, T^k is Fredholm of index 0, i.e., is Weyl. Therefore, T is Fredholm. By Lemma 4.13(3), $0 = \operatorname{ind}(T^k) = k \operatorname{ind} T$, and so $\operatorname{ind} T = 0$. Hence T is Fredholm of index 0, i.e., is Weyl. □

Corollary 4.15. *If T is a kth root of a hyponormal operator with $0 \in \pi_{00}(T^n)$, then T is a Weyl operator.*

Theorem 4.16. *Let both S and T in $B(H)$ be commuting kth roots of hyponormal operators respectively, Then ST is Weyl if and only if both S and T are Weyl.*

Proof. Assume that ST is Weyl. Then ST is Fredholm of index 0. By Lemma 4.13, both S and T are Fredholm and $\text{ind}\,(ST) = 0$. Now we must show that $\text{ind}\, S = 0$, and $\text{ind}\, T = 0$. Since both S^k and T^k are hyponormal, $\ker S^k \subset \ker (S^k)^*$ and $\ker T^k \subset \ker (T^k)^*$. Therefore, $\text{ind}\,(S^k) \leq 0$ and $\text{ind}\,(T^k) \leq 0$. And so $\text{ind}\, S \leq 0$ and $\text{ind}\, T \leq 0$. Since ST is Weyl,

$$\text{ind}\, ST = \text{ind}\, S + \text{ind}\, T = 0,$$

and so $\text{ind}\, S = 0$, and $\text{ind}\, T = 0$. Thus both S and T are Weyl.

Conversely, if both S and T are Weyl, then S and T are Fredholm of index 0, respectively. By Lemma 4.13, ST is Fredholm and

$$\text{ind}\, ST = \text{ind}\, S + \text{ind}\, T = 0.$$

Therefore, ST is Weyl. □

Theorem 4.17. *Let $T \in B(H)$ be a kth root of a hyponormal operator with $0 \in \pi_{00}(T^n)$. If S is similar to T, then S is Weyl.*

Proof. If S is similar to T, there exists an operator X such that $S = X^{-1}TX$. Since X is invertible, X is Fredholm of index 0 by [2]. Similarly, X^{-1} is Fredholm of index 0. Since T is Fredholm of index 0 by Corollary 4.15, Lemma 4.13(3) implies that $S = X^{-1}TX$ is Fredholm and

$$\text{ind}\, S = \text{ind}\, X^{-1} + \text{ind}\, T + \text{ind}\, X = 0.$$

Hence S is Weyl. □

Corollary 4.18. *Let $T \in B(H)$ be a kth root of a paranormal operator with $0 \in \pi_{00}(T^n)$. If S is similar to T, then S is Weyl.*

Theorem 4.19. *Let $T \in B(H)$ be a kth roots of a paranormal operator with $0 \in \pi_{00}(T^n)$. If $S \in B(H)$ with $\|T - S\| < \epsilon$ for sufficiently small $\epsilon > 0$, then S is Weyl operator.*

Proof. By Theorem 4.14, T is a Weyl operator. i.e., Fredholm of index 0. By the above Lemma 4.12, S is Fredholm and index S = index T. Therefore S is Weyl operator. □

Lemma 4.20. *If T is a Weyl operator and K is compact, then $T + K$ is Weyl.*

Corollary 4.21. *Let $T \in B(H)$ be a kth roots of a paranormal operator with $0 \in \pi_{00}(T^n)$ and K is compact. Then $T + K$ is Weyl operator.*

Proof. By Theorem 4.14, T is a Weyl operator. i.e., Fredholm of index 0. Thus $T + K$ is a Weyl operator by the above Lemma 4.20. □

Recall that an operator $T \in B(H)$ is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T (i.e., iso $\sigma(T) \subset \sigma_p(T)$). Every hyponormal operator is isoloid. Every paranormal operator is isoloid.

Theorem 4.22. *If T is the square root of an isoloid operater with $\sigma(T) \cap [-\sigma(T)] = \emptyset$, then T is isoloid.*

Proof. If $\lambda \in$ iso $\sigma(T)$, then $\lambda^2 \in$ iso $\sigma(T)^2 =$ iso $\sigma(T^2)$ by the spectral mapping theorem. Since T^2 is isoloid, $\lambda^2 \in \sigma_p(T^2) = \sigma_p(T)^2$. Therefore, $\lambda \in \sigma_p(T)$ or $-\lambda \in \sigma_p(T)$. By hypothesis, $-\lambda \in \rho(T)$ since $\lambda \in$ iso $\sigma_p(T)$. Hence $\lambda \in \sigma_p(T)$ and so T is isoloid. □

Corollary 4.23.

(1) *If T is the square root of a paranormal operater with $\sigma(T) \cap [-\sigma(T)] = \emptyset$, then T is isoloid.*

(2) *If T is the square root of a hyponormal operater with $\sigma(T) \cap [-\sigma(T)] = \emptyset$, then T is isoloid.*

Proof. (1) This follows from Theorem 4.22 and the fact that every paranormal operator is isoloid.

(2) This follows from Theorem 4.22 and the fact that every hyponormal operator is isoloid. □

Lemma 4.24. *Let $T \in B(H)$ be isoloid. Then for any polynominal $p(t)$ we have*

$$\sigma(p(T)) - \pi_{00}(p(T)) = p(\sigma(T) - \pi_{00}(T)).$$

Corollary 4.25. *If T is a square root of a paranormal operator with $\sigma(T) \cap [-\sigma(T)] = \emptyset$, then for any polynomial $p(t)$,*

$$p(\sigma(T) - \pi_{00}(T)) = \sigma(p(T)) - \pi_{00}(p(T)).$$

Proof. By Corollary 4.23(1) and Lemma 4.24, it is proved imediately. □

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< 국 문 초 록 >

일반화된 Paranormal 작용소들의 집합에 관한 연구

본 논문에서는 힐버트 공간(Hilbeert space) H 위에서 비정규(nonnormal) 유계 선형작용소로서 paranormal 작용소를 일반화하는 M-paranormal 작용소와 paranormal 작용소의 k제곱근 작용소의 여러 가지 특성을 조사하였는 데, 그 특성 중 주요 결과들은 다음과 같다.

(1) M-paranormal 작용소와 유니터리동치(unitary equivalent)가 되는 작용소도 역시 M-paranormal 작용소이다.

(2) 일반적으로 가환인 두 M-paranormal 작용소 S, T의 곱 ST는 M-paranormal 작용소가 되지 않지만, 다음 성질 중 하나가 성립하면 그 곱 ST는 M-paranormal 작용소이다.

임의의 $x \in H$에 대하여

(a) $\|TSx\|\|x\| \geq \sqrt{M}\|Tx\|\|Sx\|$.

(b) $\|T^2Sx\|\|x\| \geq M\|T^2x\|\|Sx\|$.

(3) 이중가환(doubly commuting)인 두 M-paranormal 작용소 S, T의 곱 ST는 M-paranormal 작용소가 되지 않지만, $M > \frac{1}{2}$ 이고

$$(2M-1)\|T^2S^2x\|\,\|x\| \geq \|T^2x\|\,\|S^2x\|$$

이면, 그 곱 ST는 M-paranormal 작용소가 된다.

(4) M-paranormal 작용소와 등거리변환(isometry)이 가환이면, 그 두 곱 역시 M-paranormal 작용소이다.

(5) λ, μ가 M-paranormal 작용소 T의 서로 다른 고유치이고 $0 < M \leq 1$일 때,

$$\ker(T-\lambda) \perp \ker(T-\mu)$$

이 성립한다.

(6) paranormal 작용소의 k제곱근 작용소와 paranormal 작용소, 새롭게 정의한 대수적 paranormal (algebraically paranormal) 작용소 사이의 포함관계는 다음과 같다.

paranormal 작용소 $\subseteq$ paranormal 작용소의 k제곱근 작용소
$\subseteq$ 대수적 paranormal 작용소.

(7) paranormal 작용소의 k제곱근 작용소들의 집합은 $B(H)$의 노름위상 (norm topology)에 대하여 폐집합이고 또한 $B(H)$의 진부분집합이다.

(8) paranormal 작용소의 k제곱근 작용소와 등거리변환이 가환이면, 그 두 곱도 paranormal 작용소의 k제곱근 작용소이다.

(9) 영이 아닌 가중값 $\{\alpha_n\}$ $(n=1, 2, \cdots)$를 갖는 가중인 밀림 작용소 (weighted shift) T가 paranormal 작용소의 k제곱근 작용소가 되기 위한 필요충분조건은 $n=1, 2, 3, \cdots$에 대하여

$$|\alpha_n||\alpha_{n+1}|\cdots|\alpha_{n+k-1}| \le M|\alpha_{n+k}||\alpha_{n+k+1}|\cdots|\alpha_{n+2k-1}|$$

이다.

(10) $0 \in \pi_{00}(T^n)$을 만족하는 paranormal 작용소의 k제곱근 작용소는 Weyl 작용소이다.

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