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ON THE CLASS OF GENERALIZED *-PARANORMAL OPERATORS

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일반화된 *-PARANORMAL 작용소들의 집합에 관한 연구

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<abstract>

ON THE CLASS OF GENERALIZED $*$-PARANORMAL OPERATORS

In this paper, we shall study the various characteristics of the M^*-paranormal operators which generalizes $*$-paranormal operators, and those of kth roots of $*$-paranormal on a Hilbert space. The main results are as follows:

(1) Let N be any closed linear subspace invariant under the operator T. If T is a M^*-paranormal operator, then (a) $T|_N$ and λT is M^*-paranormal for every complex number λ. Also (b) if S is unitarily equivalent to a M^*-paranormal operator T, then S is M^*-paranormal. And (c) if T is a M^*-paranormal operator, then $\ker T \subseteq \ker T^*$ and $\ker T = \ker T^2$.

(2) The sum of M^*-paranormal operators even commuting or double commuting may not be M^*-paranormal. Also the product of two M^*-paranormal operators, in general, may not be M^*-paranormal.

(3) If a M^*-paranormal operator T is double commutative with a hyponormal operator S, then TS is M^*-paranormal. Also if a M^*-paranormal operator T commutes with an unitary operator S, then TS is M^*-paranormal.

(4) Let T and S are doubly commuting M^*-paranormal operators. Then the product TS is M^*-paranormal if one of the following holds ;

(a) $\|T^*Sx\|\,\|x\| \geq \sqrt{M}\|T^*x\|\,\|Sx\|$ for any x in H.

(b) $\|T^*S^2x\|\,\|x\| \geq M\|T^*x\|\,\|S^2x\|$ for any x in H.

(5) Let T be any M^*-paranormal operator. Then $\|T^nx\|^3 \leq M\|T^{n+2}x\|\|T^{n-1}x\|^2$ for any unit vector x in H. In particular $\|Tx\|^3 \leq M\|T^3x\|$ for any unit vector x in H.

(6) Let T be a weighted shift with nonzero weights $\{\alpha_n\}$ ($n=0,1,2,\ldots$). Then T is a kth root of M^*-paranormal operator if and only if

$$|\alpha_{n-1}|^2|\alpha_{n-2}|^2\cdots|\alpha_{n-k}|^2 \leq M|\alpha_n||\alpha_{n+1}|\cdots|\alpha_{n+2k-1}|$$

for $n=k, k+1, k+2, \ldots$.

(7) Let T be a kth root of $*$-paranormal operator and let $N \in \mathrm{Lat}(T)$ be any closed linear subspace invariant under the operator T. Then $T|_N$ is a kth root of $*$-paranormal operator, and λT is a kth root of $*$-paranormal operator for all scalar λ. Also if S is a kth root of $*$-paranormal operator and S is unitarily equivalent to T, then T is a kth root of $*$-paranormal operator.

(8) Let T be a kth root of M^*-paranormal operator and commute with an unitary operator S, then TS is also a kth root of M^*-paranormal operator.

(9) The set of all the kth root of $*$-paranormal operator is closed in the norm topology.

(10) Let T_x be the weighted shift with nonzero weights $\alpha_0 = x$, $\alpha_1 = \sqrt{\frac{2}{3}}$, $\alpha_2 = \sqrt{\frac{3}{4}}$, $\cdots$. Then we give a necessary and sufficient condition that T_x is a kth root of *-paranormal operator. Also we obtain a condition which T_x is *-paranormal but not paranormal. Similarly we obtain a condition which T_x is not *-paranormal and not paranormal.

1. Introduction

In the theory of non-normal operators on Hilbert spaces, it is important to seek ways to reduce the problem to the normal operator case. Many mathematicians have tried to extend the significant properties of normal operators to the case of non-normal operators in various way since early 1960. Some classes of non-normal operators are closely related to normal operators, and the analogy and the difference between such non-normal operators and normal operators have been discussed.

Let H be a Hilbert space and let $L(H)$ be the set of all bounded linear operators on H. We denote the *kernel* of T and the *range* of T by $\ker T (= N(T))$ and $R(T)$ respectively. Write $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not invertible$\}$ for the *spectrum* of T, $\rho(T) = \sigma(T)^c$ for the *resolvent* of T, $\sigma_p(T) = \pi_o(T) = \{\lambda \in \mathbb{C} : \ker(T-\lambda) \neq \{0\}\}$ for the set of eigenvalues of T, $\pi_{0f}(T)$ for the points of $\sigma(T)$ that are eigenvalues of finite multiplicity, and $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity. A complex number $\lambda \in \mathbb{C}$ is an *approximate eigenvalue* of T if there exists a sequence $\{x_n\}$ with $\|x_n\| = 1$ such that $Tx_n - \lambda x_n \to 0$, i.e., $(T-\lambda)x_n \to 0$. Let

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} \ : \ \lambda \text{ is an approximate eigenvalue of } T\}.$$

Then $\sigma_{ap}(T)$ is the *approximate point spectrum* of T. The *spectral radius* $r(T)$ of T is defined by

$$\lim_{n\to\infty} \|T^n\|^{1/n} = \sup\{|\lambda| \ : \lambda \in \sigma(T)\}.$$

A closed linear subspace M of H is *invariant* under the operator T if $T(M) \subseteq M$. A closed linear subspace M *reduces* the operator T if both M and $M^{\perp}$ are invariant under T. Clearly, $\{0\}$ and H are invariant under every operator T.

If K is a subset of $\mathbb{C}$, we write iso K for the set of isolated points of K and Lat T for the lattice of the operator T, i.e. the set of all closed linear subspaces which are invariant under T.

The well-known results on the spectra are as follows;

Lemma 1.1.([11],[14]) *For any operator* $T \in L(H)$,

(1) $\ker T^* = R(T)^{\perp}$.

(2) $\sigma(T)$ *is a nonempty compact subset of* $\mathbb{C}$.

(3) $\sigma_p(T) \subset \sigma_{ap}(T) \subset \sigma(T)$.

(4) $\sigma_{ap}(T)$ *is a closed subset of* $\sigma(T)$.

(5) $\partial\sigma(T) \subset \sigma_{ap}(T)$.

Lemma 1.2.([7]) *If* $T \in L(H)$ *and* M *is any closed linear subspace of* H*, the following conditions are equivalent ;*

(1) M *reduces* T.

(2) $M^{\perp}$ *reduces* T.

(3) M *reduces* T^*.

(4) M is invariant under both T and T^*.

By T. Saito, T. Furuta, etc., the following non-normal operators have been defined as follows; An operator $T \in L(H)$ is called *normal* if $T^*T = TT^*$, *quasinormal* if T commutes T^*T, i.e., $T(T^*T) = (T^*T)T$, *subnormal* if T has a normal extension(i.e., there exist a Hilbert space K containing H as a subspace and a normal operator B on K such that $Tx = Bx$ for all $x \in H$), *hyponormal* if $T^*T - TT^* = D \geq 0$, or equivalently $\|Tx\| \geq \|T^*x\|$ for $x \in H$, *seminormal* if $T^*T - TT^* = D$, $D \geq 0$ or $D \leq 0$ (or equivalently T or T^* is hyponormal), and *normaloid* if $\|T\| = r(T)$ or equivalently $\|T^n\| = \|T\|^n$ for any positive integer n. An operator T is called $*$-*paranormal* if $\|T^*x\|^2 \leq \|T^2x\|\|x\|$ for every $x \in H$.

We have the following implication, but the converse of the implication are not reversible([23], [42]).

$$\text{Normal} \subset \text{Quasinormal} \subset \text{Subnormal} \subset \text{Hyponormal}$$
$$\subset *\text{-paranormal} \subset \text{Normaloid}.$$

B. L. Wadhwa([53]) introduced the class of M-hyponormal operators and V. Istratescu([29]) has studies some structure theorem for a subclass of M-hyponormal operator. An operator T is called *M-hyponormal* if there exists a real number $M > 0$ such that $M\|(T - \lambda I)x\| \geq \|(T - \lambda I)^*x\|$ for any unit vector x in H and for any complex number λ. Every hyponormal operator is M-hyponormal, but the converse is not true in general : For example, consider the weighted shift S on l_2 given by

$$S(x_1, x_2, \ldots) = (0, 2x_1, x_2, x_3, \ldots).$$

Then S is M-hyponormal, but not hyponormal.

On the other hand, an operator T is called *M^*-paranormal* if $\|T^*x\|^2 \leq M\|T^2x\|$ for any unit vector x in H. In paticular if $M = 1$, the class of M^*-paranormal operators becomes the class of $*$-paranormal operators as studied by S. C. Arora and S. M. Patel([6], [38]).

S. M. Patel has characterized the $*$-paranormal operator as follows: An operator T is $*$-paranormal if and only if

$$T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I \geq 0$$

for all $\lambda > 0$.

Theorem 1.3.([16]) *(The Spectral Mapping Theorem) If $T \in L(H)$ and f is analytic in a neighborhood of $\sigma(T)$, then $\sigma(f(T)) = f(\sigma(T))$.*

The organization of this thesis is as follows:

In section 1, we introduce basic properties of various spectra(spectrum, point spectrum, approximate point spectrum etc.) of a bounded linear operator and the spectral mapping theorem.

In section 2, we give the well-known results of hyponormal operators and $*$-paranormal operators on a Hilbert space H.

In section 3, we shall study certain properties of M^*-paranormal operators. In particular, we shall give an essentially characterization of M^*-paranormal operators in the following way; An operator T is M^*-paranormal if and only if

$$M^2 T^{*2} T^2 + 2\lambda T T^* + \lambda^2 I \geq 0$$

for all real number λ.

In section 4, we shall study a new class of operators called a kth root of G-operator : An operator $T \in L(H)$ is a *kth root of a G-operator* if T^k is a G-operator. In particular, if a G-operator is $*$-paranormal, then T is called the *kth root of a $*$-paranormal operator*. We shall show the following results:

(1) Let T be a weighted shift with non-zero weights $\{\alpha_n\}$ $(n = 0, 1, 2, \ldots)$. Then T is a kth root of M^*-paranormal operator if and only if

$$|\alpha_{n-1}|^2 |\alpha_{n-2}|^2 \cdots |\alpha_{n-k}|^2 \leq M |\alpha_n||\alpha_{n+1}| \cdots |\alpha_{n+2k-1}|$$

for $n = k, k+1, k+2, \ldots$

(2) Let T be a kth root of $*$-paranormal operator and $N \in$ Lat T. Then $T|_N$ is a kth root of $*$-paranormal operator.

(3) If $S \in L(H)$ is a kth root of $*$-paranormal operator and S is unitarily equivalent to T, then T is a kth root of $*$-paranormal operator.

(4) Let T be any kth root of M^*-paranormal operator and commute with an unitary operator S. Then TS is also a kth root of M^*-paranormal operator.

(5) The set of all the kth roots of $*$-paranormal operator is a proper closed subset of $L(H)$ with the norm topology.

2. Hyponormal operators and $*$-paranormal operators

Lemma 2.1. *Let T be a hyponormal operator on a Hilbert space H. Then*

(1) $T - \lambda I$ and T^{-1} are hyponormal operators for all $\lambda \in \mathbb{C}$.

(2) $Tx = \lambda x$ implies $T^*x = \bar{\lambda}x$ for all $x \in H$ and $\lambda \in \mathbb{C}$.

(3) $Tx = \lambda x$, $Ty = \mu y$ and $\lambda \neq \mu$ for all $x, y \in H$, $\lambda, \mu \in \mathbb{C}$ imply that x and y are orthogonal.

Proof. (1) Since T is a hyponormal oprator, we have

$$\begin{aligned}(T - \lambda I)(T^* - \bar{\lambda} I) &= TT^* - \lambda T^* - \bar{\lambda} T + |\lambda|^2 I \\ &\leq T^*T - \lambda T^* - \bar{\lambda} T + |\lambda|^2 I \\ &= (T^* - \bar{\lambda} I)(T - \lambda I)\end{aligned}$$

for all $\lambda \in \mathbb{C}$. Hence $T - \lambda I$ is a hyponormal operator.

If T is invertible and $T^*T - TT^* \geq 0$, then

$$\begin{aligned}0 &\leq T^{-1}(T^*T - TT^*)T^{*-1} \\ &= T^{-1}T^*TT^{*-1} - I.\end{aligned}$$

Since $A \geq I$ implies $A^{-1} \leq I$, we have $I - T^*T^{-1}T^{*-1}T \geq 0$ and hence

$$T^{*-1}T^{-1} - T^{-1}T^{*-1} = T^{*-1}(I - T^*T^{-1}T^{*-1}T)T^{-1} \geq 0.$$

(2) Since $T-\lambda I$ is a hyponormal operator and $Tx=\lambda x$

$$0 \leq \|(T-\lambda I)^* x\| \leq \|(T-\lambda I)x\| = 0.$$

Thus $\|(T-\lambda I)^* x\| = 0$, and so $T^* x = \bar{\lambda} x$.

(3) Since

$$\lambda(x,\, y) = (\lambda x,\, y) = (Tx,\, y) = (x,\, T^* y) = (x, \bar{\mu} y) = \mu(x,\, y),$$

$(\lambda-\mu)(x,\, y) = 0$ implies $(x,\, y) = 0$ $(\lambda \neq \mu)$. Hence x, y are orthogonal. □

Definition 2.2. An operator $T \in L(H)$ is said to be *nilpotent* if $T^n = 0$ for some positive integer $n \in \mathbb{N}$, and *quasinilpotent* if $\|T^n\|^{\frac{1}{n}} \longrightarrow 0$ as $n \longrightarrow \infty$.

Evidently, if T is nilpotent then T is also quasinilpotent and since the spectral radius $r(T)$ can be expressed as

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}},$$

it follows that $r(T) = 0$ if T is quasinilpotent.

Lemma 2.3. *Let T be a hyponormal operator on H. Then*

(1) For any vector $x \in H$,

$$\|Tx\| = \|T^* x\| \quad \textit{if and only if} \quad T^*Tx = TT^*x.$$

(2) The set $N = \{x \in H : \|Tx\| = \|T^* x\|\}$ is a closed subspace of H.

(3) The restriction $T|_N$ of T to an invariant subspace N is hyponormal.

(4) For every positive integer n, $\|T^n\| = \|T\|^n$ and so T is normaloid.

(5) The only quasinilpotent hyponormal operator is a zero operator.

Proof. (1) The proof of the sufficiency is obvious. If $\|Tx\| = \|T^*x\|$ for each vector $x \in H$, then $((T^*T - TT^*)x, x) = 0$ and hence for each vector $y \in H$,

$$|((T^*T - TT^*)x, y)|^2 \leq |((T^*T - TT^*)x, x)|\,|((T^*T - TT^*)y, y)| = 0$$

by the generalized Schwarz inequality for positive operators. Since y is arbitrary, we have $T^*Tx = TT^*x$ for each $x \in H$.

(2) By (1),

$$\begin{aligned} N &= \{x \in H : \|Tx\| = \|T^*x\|\} \\ &= \{x \in H : (T^*T - TT^*)x = 0\} = \ker(T^*T - TT^*) \end{aligned}$$

is clearly closed.

(3) Since N is invariant under T, $PTP = TP$ where P is the projection on N. Since T is hyponormal,

$$\|PT^*Px\| \leq \|T^*Px\| \leq \|TPx\| = \|PTPx\|$$

for each vector $x \in H$, and so PTP is a hyponormal operator. Hence $T|_N$ is a hyponormal operator.

(Another method) Let x be any vector in N. Since $(T|_N)x = Tx$ and T is a hyponormal operator, we have

$$\|(T|_N)x\| = \|Tx\| \geq \|T^*x\| = \|(T|_N)^*x\|.$$

Hence $T|_N$ is a hyponormal operator.

(4) For $n = 1$, the equality is trivial. Assume that $\|T^n\| = \|T\|^n$ for $1 \leq k \leq n$. We shall prove it for $n+1$. Then

$$\begin{aligned} \|T^n x\|^2 &= (T^n x,\, T^n x) = (T^* T^n x,\, T^{n-1}x) \\ &\leq \|T^* T^n x\|\,\|T^{n-1}x\| \\ &\leq \|T^{n+1}x\|\,\|T^{n-1}x\|. \end{aligned}$$

Since $\|T^{n-1}\| = \|T\|^{n-1}$, we get $\|T^{n+1}\| \geq \|T\|^{n+1}$. The converse inequality being obvious, the proof is complete.

(5) By hypothesis $\sigma(T) = \{0\}$ and so $\|T\| = r(T) = 0$. Hence T is a zero operator. □

Corollary 2.4. *Every nonzero hyponormal operator has a nonzero element in its spectrum.*

Proof. $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \|T\| > 0$.

Theorem 2.5.([5]) *The class of all hyponormal operators on a Hilbert space H is closed in the norm topology of operators.*

Theorem 2.6.([42]) *Let T be the weighted shift operator defined by $Te_n = \alpha_n e_{n+1}\,(n \geq 1)$ with weights $\{\alpha_n\}_{n=0}^{\infty}$. Then T is a hyponormal operator if and only if the weight sequence $\{\alpha_n\}$ is monotonically increasing.*

S. L. Campbell([13]) and Peng Fan([17]) showed the following examples.

Example 2.1. Let T be the unilateral weighted shift with weight sequence $\{1, \frac{1}{2}, 1, 1, \ldots\}$. Then T is not a hyponormal operator by Theorem 2.6.

Example 2.2. Let T be a bilateral shift defined by

$$Te_n = \begin{cases} e_{n-1} & \text{for } n \leq 2 \\ 2e_{n-1} & \text{for } n \geq 3. \end{cases}$$

Then T is a hyponormal operator.

Remark. ([21]) T^2 may not be hyponormal when T is a hyponormal operator. For example, if U is the unilateral shift on l_2 and $T = U^* + 2U$, then

$$T^*T - TT^* = 3(I - UU^*) > 0.$$

Therefore T is hyponormal, However, if we take $x = (1,\, 0,\, -2,\, 0,\, 0,\, \ldots)$, then $T^2x = (0,\, 0,\, -4,\, 0,\, -8,\, 0, \ldots)$, $(T^*)^2x = (-6,\, 0,\, -7,\, -2,\, 0, \ldots)$ and so

$$\|T^2x\|^2 = 80 < 89 = \|(T^*)^2x\|^2.$$

Hence T^2 is not a hyponormal operator.

A hyponormal operator T does not imply that T^2 is hyponormal. This can be seen from the following another example due to M. Putinar[34].

Example 2.3. Let H denote an arbitrary Hilbert space and let Λ denote the set of all function $x = x(n)$ defined on integers with values in H and satisfying $\sum_{-\infty}^{\infty} \|x(n)\|^2 < \infty$. Then Λ become a Hilbert space with inner product $(x, y) = \sum (x(n), y(n))$. Next, let $\{P_n\}$ be a bounded sequence of nonnegative operators on H, so that $0 \leq P_n \leq (constant) \cdot I$, and define the operators U on Λ by

$$Ux(n) = x(n+1) \quad \text{and} \quad Px(n) = P_n x(n).$$

It is clear that U is unitary and that P is a nonnegative bounded operator. Furthermore, if $T = UP$ then

$$Tx(n) = P_{n+1}x(n+1) \quad \text{and} \quad T^*x(n) = P_n x(n-1),$$

and hence $T^*Tx(n) = P_n^2 x(n)$ and $TT^*x(n) = P_{n+1}^2 x(n)$.

Consequently, $T^*T - TT^* \geq 0$ if and only if

$$P_n^2 \geq P_{n+1}^2 \quad \text{for} \quad n = 0\,,\, \pm 1\,,\, \pm 2, \ldots. \qquad (1)$$

An easy calculation shows that

$$T^2 x(n) = P_{n+1}P_{n+2}\, x(n+2)$$

and $T^{*2}x(n) = P_n P_{n-1}\, x(n-2)$, and hence

$$T^{*2}T^2\, x(n) = P_n P_{n-1}^2 P_n\, x(n)$$

and

$$T^2T^{*2}\,x(n) = P_{n+1}P_{n+2}^2P_{n+1}\,x(n).$$

Thus T^2 is a hyponormal operator if and only if

$$P_nP_{n-1}^2P_n \geq P_{n+1}P_{n+2}^2P_{n+1} \text{ for all } n \qquad (2)$$

It will be shown that (1) does not imply (2).

Let H be two-dimensional, so that operators on H can be regarded as 2×2 matrices and let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $A \geq 0$, $B \geq 0$ and $A - B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geq 0$ but $A^2 - B^2 = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$ is not positive definite. Let P_n be the nonnegative square root of A for $n \leq 0$ and nonnegative square root of B for $n \geq 0$. Then $P_n^2 \geq P_{n+1}^2$, so that (1) holds and T is a hyponormal operator. But

$$P_0P_{-1}^2P_0 = A^2 \quad \text{and} \quad P_1P_2^2P_1 = B^2$$

so that (2) fails to hold for $n = 0$. Hence T^2 is not a hyponormal operator.

□

Definition 2.7. An operator T is said to be *unitarily equivalent* to an operator S if $S = U^*TU$ for a unitary operator U.

Theorem 2.8. *An operator unitarily equivalent to a hyponormal operator is a hyponormal operator.*

Proof. Suppose $S = U^*TU$, T is hyponormal and U is unitary. Now for every $x \in H$,

$$\|S^*x\| = \|U^*T^*Ux\| = \|T^*Ux\| \leq \|TUx\| = \|U^*TUx\| = \|Sx\|$$

and so S is hyponormal. □

In [19], T. Furuta and R. Nakamoto have proved the following theorem.

Theorem 2.9. *A hyponormal operator unitarily equivalent to its adjoint is normal.*

Proof. Suppose $T^* = U^*TU$, U is unitary and T is a hyponormal operator. Now for any vector x in H,

$$\|Tx\| = \|U^*T^*Ux\| = \|T^*Ux\| \leq \|TUx\| = \|U^*TUx\| = \|T^*x\|.$$

Thus $\|Tx\| \leq \|T^*x\|$ and $\|Tx\| \geq \|T^*x\|$. Therefore $\|Tx\| = \|T^*x\|$. □

Definition 2.10. Two bounded linear operators S and T are *doubly commutative(resp. weakly doubly commutative)* if $TS = ST$ and $TS^* = S^*T$(resp. $TS \neq ST$ but $TS^* = S^*T$).

Theorem 2.11. *Let T be a hyponormal operator such that T^*T commutes with TT^*. Then T^2 is a hyponormal operator.*

Proof. By hypothesis, we have

$$\begin{aligned} T^{*2}T^2 - T^2T^{*2} &= T^*(T^*T)T - T(TT^*)T^* \\ &\geq T^*(TT^*)T - T(T^*T)T^* \\ &= (T^*T)^2 - (TT^*)^2 \text{ (because } \; T^*T \geq TT^*) \end{aligned}$$

$$\geq \quad 0.$$

Thus T^2 is a hyponormal operator. $\square$

In the following Lemma, we show that if two operators are weakly doubly commutative, then the sum and product of two hyponormal operators are hyponormal.

Lemma 2.12. *If T and S are hyponormal operators such that $T^*S = ST^*$, then $T+S$ is a hyponormal operator.*

Proof. By hypothesis, we have

$$\begin{aligned}(T+S)^*(T+S) &= T^*T + T^*S + S^*T + S^*S \\ &\geq TT^* + T^*S + S^*T + S^*S \\ &\geq TT^* + TS^* + ST^* + SS^* \\ &= (T+S)(T+S)^*.\end{aligned}$$

Thus $T+S$ is a hyponormal operator. $\square$

Lemma 2.13. *If T and S are hyponormal operators such that $T^*S = ST^*$, then TS is a hyponormal operator.*

Proof. For all $x \in H$, we have

$$\|(TS)^*x\|^2 = \|S^*T^*x\|^2 \leq \|ST^*x\|^2 = \|T^*Sx\|^2 \leq \|(TS)x\|^2.$$

Thus TS is a hyponormal operator.

(Another method) By the hyponormality and the hypothesis, we have

$$(TS)^*(TS) \quad = \quad S^*(T^*T)S \geq S^*(TT^*)S$$

$$= \quad T(S^*S)T^* \geq T(SS^*)T^* = (TS)(TS)^*.$$

Thus TS is a hyponormal operator. □

From Lemma 2.12 and Lemma 2.13, the sum and the product of two weakly double commuting hyponormal operators are hyponormal.

The sum and product of two double commuting hyponormal operators are easily shown to be a hyponormal operator. But the sum and product of two commuting hyponormal operators are not necessarily hyponormal. We attempt to find conditions under which the product of two hyponormal operators is also hyponormal.

If we replace one of the hyponormal operators by an isometric operator in Lemma 2.13, then the condition of commutativity is sufficient to ensure the hyponormality of their product.

Theorem 2.14. *If a hyponormal operator T commutes with an isometric operator S, then TS is hyponormal.*

Proof. For any $x \in H$, we have

$$\begin{aligned} \|(TS)^*x\| &= \|S^*T^*x\| = \|ST^*x\| \\ &= \|T^*x\| \leq \|Tx\| = \|STx\| = \|(TS)x\|. \end{aligned}$$

Thus TS is a hyponormal operator. □

Lemma 2.15. *If a hyponormal operator S is unitarily equivalent to T such that T commute with S^*, then ST is hyponormal.*

Proof. Let $T = U^*SU$ for a unitary operator U. Then, for each $x \in H$,

$$\|T^*x\| = \|U^*S^*Ux\| = \|S^*Ux\| \leq \|SUx\| = \|U^*SUx\| = \|Tx\|.$$

Thus T is hyponormal. Since T and S^* commute,

$$\|(ST)^*x\| = \|T^*S^*x\| \leq \|TS^*x\| = \|S^*Tx\| \leq \|STx\|.$$

Therefore ST is a hyponormal operator. $\square$

Let H be a separable dimensional Hilbert space. Recall that an operator $T \in L(H)$ is *hyponormal* if $TT^* \leq T^*T$, or equivalently, $\|T^*x\| \leq \|Tx\|$ for every $x \in H$. In general, T^2 can be hyponormal without T being hyponormal. An operator $T \in L(H)$ is said to be *Fredholm* if its range $R(T)$ is closed and both the null space $\ker T$ and $\ker T^*$ are finite dimensional. The index of a Fredholm operator T, denoted by ind T or $i(T)$, is defined by

$$\text{ind}(T) = \dim\, \ker T - \dim\, \ker T^*.$$

The essential spectrum of T, denoted by $\sigma_e(T)$, is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \ \text{ is not } \ \text{Fredholm}\}.$$

An operator $T \in L(H)$ is called *Browder* if it is Fredholm of finite ascent and descent, or equivalently if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. A Fredholm operator of index zero is called a *Weyl operator*. The Weyl spectrum of T, denoted by $w(T)$, is defined by

$$w(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \text{ is not Weyl}\}.$$

For any operator T, $\sigma_e(T) \subset w(T) \subset \sigma(T)$ and $w(T)$ is a nonempty compact subset of $\mathbb{C}$.

H. Weyl([54]) asserted that if T is a self-adjoint operator acting on a Hilbert space H, then $w(T)$ consists precisely of all points of $\sigma(T)$ except the isolated eigenvalues of finite multiplicity, that is,

$$w(T) = \sigma(T) - \pi_{00}(T).$$

Following L. A. Coburn([14]), we say that Weyl's theorem holds for T if $w(T) = \sigma(T) - \pi_{00}(T)$, or equivalently, if $\sigma(T) - w(T) = \pi_{00}(T)$.

There are several classes of operators for which Weyl's theorem holds :

(1) H. Weyl([54]) showed that Weyl's theorem holds for any self adjoint operator.

(2) L.A. Courn([14]) showed that Weyl's theorem holds for any hyponormal operator and any Toeplitz operator.

(3) S.K. Berberian([8],[9]) showed that Weyl's theorem holds for any seminormal operator.

(4) K.K. Oberai([37]) showed that if N is nilpotent operator commuting with T and if Weyl's theorem holds for T, then it also holds for $T+N$.

Theorem 2.16.([14]) *Weyl's theorem holds for hyponormal operators.*

Proof. If T is hyponormal, then $T - \lambda I$ is hyponormal. Thus it suffices to show that $0 \in \sigma(T) - w(T)$ if and only if $0 \in \pi_{00}(T)$.

($\Longrightarrow$) Let $0 \in \sigma(T) - w(T)$. Then T is Weyl but not invertible. Then $R(T)$ is closed, $\dim \ker T = \dim R(T)^\perp < \infty$ and $\ker T \neq \{0\}$, so that $R(T)^\perp \neq \{0\}$. Since T is hyponormal, $\|Tx\| \geq \|T^*x\|$. In particular, $\ker T \subset \ker T^* = R(T)^\perp$. Thus $T = 0 \oplus B$, where B is invertible. Hence $\sigma(T) = \{0\} \cup \sigma(B)$. Since $0 \notin \sigma(B)$, $0 \in \operatorname{iso} \sigma(T)$. Thus $0 \in \pi_{00}(T)$.

($\Longleftarrow$) Let $0 \in \pi_{00}(T)$. Then $0 \in \operatorname{iso} \sigma(T)$ and $0 < \dim \ker T < \infty$. By hyponormality, $\ker T \subset R(T)^\perp$. So $T = 0 \oplus B$, where B is injective and hyponormal. Also, B is invertible. Since

$$H = \ker T \oplus (\ker T)^\perp = \ker T \oplus R(T),$$

$\ker T = R(T)$ and $(\ker T)^\perp = R(T)$. Thus $\dim \ker T = \dim R(T)^\perp < \infty$ and ind $(T) = 0$. Since $0 \in \operatorname{iso} \sigma(T)$, T is not invertible. Hence $0 \in \sigma(T) - w(T)$. □

Definition 2.17. An operator $T \in L(H)$ is said to be *isoloid* if isolated points of $\sigma(T)$ are eigenvalues of T.

Theorem 2.18. *Every hyponormal operator T is isoloid.*

Proof. It suffices to show that if $0 \in \operatorname{iso} \sigma(T)$, then $0 \in \sigma_p(T)$. Choose $R > 0$ sufficiently enough that 0 is the only point of $\sigma(T)$ contained in or on the circle $|\lambda| = R$. Define

$$P = \int_{|\lambda|=R} (\lambda I - T)^{-1} d\lambda.$$

Then P is the Riesz projection corresponding to 0. So PH is an invariant subspace for T. Moreover, $PH \neq \{0\}$ and $\sigma(T|_{PH}) = \{0\}$. $T|_{PH}$ is hyponormal, since P be a projection of H onto PH. By Lemma 2.3(5), it follows that $T|_{PH} = 0$, so that T is not one-to-one. Therefore $0 \in \sigma_p(T)$. □

Lemma 2.19. *Every hyponormal operator is $*$-paranormal.*

Proof. If T is a hyponormal operator, then

$$\begin{aligned} \|T^*x\|^2 = (T^*x, T^*x) = (TT^*x, x) &\leq (T^*Tx, x) \leq \|T^*(Tx)\|\|x\| \\ &\leq \|T(Tx)\|\|x\| = \|T^2x\|\|x\|. \end{aligned}$$

Thus T is a $*$-paranormal operator. □

Every hyponormal operator is a $*$-paranormal operator, but the converse is not true([5]).

Example 2.4. Suppose H is a 2-dimensional Hilbert space. Let K be the direct sum of denumerable copies of H. Let A and B be any two positive operators on H. Let n be any fixed positive integer. Define an operator $T = T_{A,B,n}$ on K as

$$T(x_1, x_2, x_3, \ldots) = (0, Ax_1, Ax_2, \ldots Bx_{n+1},\ Bx_{n+2}, \ldots),$$

where A and B are positive operators on H satisfying this time $A^2 = C$ and $B^4 = D$, where C and D are positive operators on H defined as

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}.$$

By the computations, T is a $*$-paranormal operator if and only if

$$T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I \;=\; B^4 - 2\lambda A^2 + \lambda^2 I \;\geq\; 0$$

for each $\lambda \;>\; 0$. Now

$$B^4 - 2\lambda A^2 + \lambda^2 I = D - 2\lambda C + \lambda^2 = \begin{pmatrix} (1-\lambda)^2 & 2(1-\lambda) \\ 2(1-\lambda) & (2-\lambda)^2 + 4 \end{pmatrix}$$

which is a positive operator for each $\lambda \;>\; 0$. Therefore T is a $*$-paranormal operator. However,

$$T^*T - TT^* = B^4 - A^4 = D - C^2 = \begin{pmatrix} -1 & -1 \\ -1 & 3 \end{pmatrix}$$

which is not positive. Hence T is not a hyponormal operator. $\square$

Lemma. 2.20.([5]) *Every $*$-paranormal operator is normaloid.*

Proof. Let T be any $*$-paranormal operator. We prove that $\|T^n\| = \|T\|^n$ by mathematical induction on positive integer n. For any unit vector $x \in H$, we have $\|T^*x\|^2 \leq \|T^2x\|$ and so $\|T^*\|^2 = \|T\|^2 \leq \|T^2\| \leq \|T\|^2$. That means $\|T^2\| = \|T\|^2$.

Assume that the result is true for all positive integers $k \leq n$. To prove the result for $n+1$, we prove the following inequality : $\|T^n x\|^3 \leq \|T^{n+2}x\|\|T^{n-1}x\|^2$.

We assume that $T^n x \neq 0$. Now we have

$$\begin{aligned} \|T^n x\|^4 &= (\|T^n x\|^2)^2 = (T^n x, T^n x)^2 \\ &= (T^*T^n x, T^{n-1}x)^2 \leq \|T^*T^n x\|^2\|T^{n-1}x\|^2 \end{aligned}$$

by Schwarz's inequality. This gives $\|T^n x\|^4 \leq \|T^{n+2}x\|\|T^{n-1}x\|^2\|T^n x\|$ and so $\|T^n x\|^3 \leq \|T^{n+2}x\| \, \|T^{n-1}x\|^2$. Therefore

$$\|T^n\|^3 \leq \|T^{n+2}\|\|T^{n-1}\|^2 \leq \|T\|\|T^{n+1}\|\|T^{n-1}\|^2.$$

Since $\|T^k\| = \|T\|^k$ for all $k \leq n$, we obtain $\|T\|^{n+1} \leq \|T^{n+1}\|$. Therefore T is normaloid. □

The inclusion relation of the classes of non-normal operators listed above is as follows :

$$\text{Normal} \subset \text{Quasinormal} \subset \text{Subnormal} \subset$$

$$\text{Hyponormal} \subset *\text{-paranormal(or} \quad \text{paranormal)} \subset \text{Normaloid.}$$

Every $*$-paranormal operator is normaloid, but the converse is not true([5]).

Example 2.5. Suppose H is a 2-dimensional Hilbert space. Let K be the direct sum of denumerable copies of H. Let A and B be any two positive operators on H. Let n be any fixed positive integer. Define an operator $T = T_{A,B,n}$ on K as

$$T(x_1, x_2, x_3, \ldots) = (0, Ax_1, Ax_2, \ldots Bx_{n+1}, Bx_{n+2}, \ldots).$$

It can be computed to see that T is a hyponormal operator if and only if $B^2 \geq A^2$. Let C and D be defined on H as

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then C and D are positive operators on H satisfying

$$D - C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geq 0.$$

Choose A and B to be positive operators so as to satisfy $A^2 = C$ and $B^2 = D$. With this choice T is a hyponormal operator and hence normaloid and therefore T^2 is also normaloid. We claim that T^2 is not a $*$-paranormal operator. By the simple computations, we show that T^2 is a $*$-paranormal operator if and only if $B^8 - 2\lambda A^4 + \lambda^2 I > 0$ for each $\lambda > 0$. Putting $\lambda = \frac{1}{2}$, we obtain

$$B^8 - 2\lambda A^4 + \lambda^2 I = \begin{pmatrix} \frac{133}{4} & 21 \\ 21 & \frac{53}{4} \end{pmatrix}.$$

Now if $x = (1, -\frac{84}{53})$, then $(\begin{pmatrix} \frac{133}{4} & 21 \\ 21 & \frac{53}{4} \end{pmatrix} x, x) < 0$. Hence T^2 is not a $*$-paranormal operator. $\square$

Lemma 2.21.([21]) *The classes of $*$-paranormal operators and paranormal operators are independent by using example given by Halmos.*

We give an example of a paranormal operator which is not a $*$-paranormal operator.

Example 2.6. Let H be a Hilbert space, and let Λ be the set of all functions $x = x(n)$ defined on integers with values H, such that $\sum \|x_n\|^2 < \infty$. Then Λ becomes a Hilbert space with inner product $(x, y) = \sum (x_n, y_n)$.

Let $\{S_n\}$ be a sequence of positive operators on Z such that $\{\|S_n\|\}$ is bounded, and define, for every x in H, $(Ux)_n = x_{n+1}$, $(Sx)_n = S_n x_n$. It is easy to verify that U and S are operators on H. Since U is unitary, $(U^* x)_n = (U^{-1}x)_n = x_{n-1}$. Let $T = US$. Then $(T^*Tx)_n = S_n^2 x_n$, $(TT^*x)_n = S_{n+1}^2 x_n$,

$$(T^{*2}T^2x)_n = S_n S_{n-1}^2 S_n \, x_n, \quad (T^2T^{*2}x)_n = S_{n+1} S_{n+2}^2 S_{n+1} \, x_n.$$

Now let H be a two-dimensional Hilbert space and let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (\text{action on } H).$$

Then $A - B \geq 0$, and $A^2 - B^2$ is not positive. Let $S_n = \begin{cases} \sqrt{A} & (n \leq 0) \\ \sqrt{B} & (n > 0). \end{cases}$

Then $T^*T - TT^* = S_n^2 - S_{n+1}^2 \geq 0$. So T is a hyponormal operator. Thus T^2 is a paranormal operator.

Since $(T^{*4}T^4x)_n = S_n S_{n-1} S_{n-2} S_{n-3}^2 S_{n-2} S_{n-1} S_n x_n$, T^2 is a $*$-paranormal operator if and only if

$$
\begin{aligned}
&((T^{*4}T^4 - 2\lambda T^2T^{*2} + \lambda^2 I)x)_n \\
= \; &((S_nS_{n-1}S_{n-2}S_{n-3}^2S_{n-2}S_{n-1}S_n - 2\lambda S_{n+1}S_{n+2}^2S_{n+1} + \lambda^2 I)x)_n \geq 0
\end{aligned}
$$

for each $\lambda > 0$.

But, if $n = 0$, then

$$
\begin{aligned}
&S_0S_{-1}S_{-2}S_{-3}^2S_{-2}S_{-1}S_0 - 2\lambda S_1S_2^2S_1 + \lambda^2 I \\
= \; &A^4 - 2\lambda B^2 + \lambda^2 I \\
= \; &\begin{pmatrix} 34 - 2\lambda + \lambda^2 & 21 \\ 21 & 13 + \lambda^2 \end{pmatrix} \quad \text{for each } \lambda > 0.
\end{aligned}
$$

Putting $\lambda = \frac{1}{2}$, we obtain that $A^4 - 2\lambda B^2 + \lambda^2 I = \begin{pmatrix} \frac{133}{4} & 21 \\ 21 & \frac{53}{4} \end{pmatrix}$ is not positive, and so T^2 is not a $*$-paranormal operator. Therefore, T^2 is a paranormal operator, but T^2 is not a $*$-paranormal operator. $\square$

The above example shows that there exists a paranormal operator which is not $*-$paranormal. On the other hand, the example 4.8(later) shows that there exists a $*$-paranormal operator which is not paranormal.

3. Properties of M^*-paranormal operators

Definition 3.1. An operator $T \in L(H)$ is said to be *M-hyponormal* if there exists a real number $M > 0$ such that

$$\|(T - \lambda I)^* x\| \leq M \|(T - \lambda I)x\|$$

for all x in H and for all $\lambda \in \mathbb{C}$. An operator T is said to be *M^*-paranormal* if there exists a real number $M > 0$ such that $\|T^* x\|^2 \leq M\|T^2 x\|$ for any unit vector x in H.

Every hyponormal operator is M-hyponormal, but the converse is not true in general : for example, consider the weighted shift S on l_2 given by

$$S(x_1, x_2, \ldots) = (0, 2x_1, x_2, x_3, \ldots).$$

The examples of M-hyponormal non-hyponormal operators seem to be scarce from the literature. B.L. Wadhwa([53]) gave an example of M-hyponormal non-hyponormal weighted shift T on l_2 :

$$T = \begin{pmatrix} 0 & & & & & & \\ 1 & 0 & & & & & \\ & 2 & 0 & & & & \\ & & 1 & 0 & & & \\ & & & 1 & 0 & & \\ & & & & \cdot & \cdot & \\ & & & & & \cdot & \cdot \\ & & & & & & \cdot \quad \cdot \end{pmatrix}.$$

The notion of a M-hyponormal operator is due to J. Stampfli and B. L. Wadhwa([48]). The M-hyponormality of operators has been studied by many authors ([2], [3], [18], [27], [29], [48], [52], [53]).

If T is a $*$-paranormal operator, then T is a M^*-paranormal operator for each real number $M \geq 1$. But the converse is not true.

Example 3.1. Let H be a separable Hilbert space and let $\{a_n\}_{n=1}^{\infty}$ be an orthonormal basis of H. Define a weighted shift T on H as follows :

$$Te_1 = e_2,\ Te_2 = \sqrt{2}e_3,\ Te_n = e_{n+1} \quad \text{for all } n \geq 3.$$

Then $T^*e_1 = 0$, $T^*e_2 = e_1$, $T^*e_3 = \sqrt{2}e_2$ and $T^*e_{n+1} = e_n$ for all $n \geq 3$. Therefore T is M^*-paranormal for $M \geq 2$ and T is not $*$-paranormal.

Theorem 3.2. *T is a M^*-paranormal operator if and only if*

$$M^2T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I \geq 0$$

for each $\lambda > 0$.

Proof. If T is M^*-paranormal, then $\|T^*x\|^2 \leq M\|T^2x\|$ for any unit vector x, and so $(\|T^*x\|^2)^2 \leq M^2\|T^2x\|^2$, i.e., $(\|T^*x\|^2)^2 - M^2\|T^2x\|^2 \leq 0$. By the elementary properties of real quadratic forms, this gives

$$\lambda^2 I - 2\lambda\|T^*x\|^2 + M^2\|T^2x\|^2 \geq 0$$

for each $\lambda > 0$. Hence $M^2T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I \geq 0$ for each $\lambda > 0$.

Conversely, suppose $M^2T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I \geq 0$ for each $\lambda > 0$. Then for each unit vector $x \in H$,

$$
\begin{aligned}
& ((M^2T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I)x,\, x) \geq 0 \\
\Longrightarrow \quad & \lambda^2 I - 2\lambda(TT^*x,\, x) + M^2(T^{*2}T^2x,\, x) \geq 0 \\
\Longrightarrow \quad & \lambda^2 I - 2\lambda\|T^*x\|^2 + M^2\|T^2x\|^2 \geq 0 \\
\Longrightarrow \quad & (\|T^*\|^2)^2 - M^2\|T^2x\|^2 \leq 0 \\
\Longrightarrow \quad & \|T^*x\|^2 \leq M\|T^2x\|.
\end{aligned}
$$

Therefore T is a M^*-paranormal operator. □

Patel([38]) has characterized $*$-paranormal operator as follow :

Corollary 3.3.([38]) *An operator T is a $*$-paranormal operator if and only if*

$$T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I \geq 0 \quad \textit{for all } \lambda > 0.$$

Corollary 3.4. *Let T be a weighted shift with weights $\{\alpha_n\}_{n=1}^{\infty}$. Then T is a M^*-paranormal operator if and only if*

$$|\alpha_{n-1}|^2 \leq M|\alpha_n||\alpha_{n+1}|$$

for each $n = 2, 3, 4, \ldots$.

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of the Hilbert space H. Suppose T is a M^*-paranormal operator. Since $Te_n = \alpha_n e_{n+1}$,

$$\|T^2e_n\| = \|T(\alpha_n e_{n+1})\| = |\alpha_n|\|\alpha_{n+1}e_{n+2}\| = |\alpha_n||\alpha_{n+1}|$$

and $\|T^*e_n\| = |\alpha_{n-1}|$ for each $n = 2, 3, 4, \ldots$. Since T is a M^*-paranormal operator,

$$\|T^*e_n\|^2 \leq M\|T^2e_n\|$$

and so $|\alpha_{n-1}|^2 \leq M|\alpha_n||\alpha_{n+1}|$ for each $n = 2, 3, 4, \ldots$.

Conversely, suppose $|\alpha_{n-1}|^2 \leq M|\alpha_n||\alpha_{n+1}|$ for each $n = 2, 3, 4, \ldots$. Then for each $n = 2, 3, 4, \ldots$, we have

$$M\|T^2e_n\| - \|T^*e_n\|^2 = M|\alpha_n||\alpha_{n+1}| - |\alpha_{n-1}|^2 \geq 0.$$

Therefore $M\|T^2e_n\| \geq \|T^*e_n\|^2$ for each $n = 2, 3, 4, \ldots$, and so T is a M^*-paranormal operator. $\square$

Corollary 3.5.([6]) *Let T be a non-singular weighted shift with weights $\{\alpha_n\}$. Then T^{-1} is a M^*-paranormal operator if and only if*

$$|\alpha_{n-1}||\alpha_{n-2}| \leq M|\alpha_n|^2$$

for each $n = 3, 4, 5, \ldots$.

Theorem 3.6. *An operator T is a M^*-paranormal operator if and only if*

$$M^2T^{*2}T^2 + 2\lambda TT^* + \lambda^2 I \geq 0$$

for all real number λ.

Proof. Let x be any unit vector in H. Then

$$M^2T^{*2}T^2 + 2\lambda TT^* + \lambda^2 I \geq 0 \quad \text{for all real number } \lambda$$

$$\iff ((M^2T^{*2}T^2 + 2\lambda TT^* + \lambda^2 I)x,\, x) \geq 0 \quad \text{for all real number } \lambda$$

$$\iff M^2\|T^2x\|^2 + 2\lambda\|T^*x\|^2 + \lambda^2\|x\|^2 \geq 0 \quad \text{for all real number } \lambda$$

$$\iff (\|T^*x\|^2)^2 \leq M^2\|T^2x\|^2$$

$$\iff \|T^*x\|^2 \leq M\|T^2x\|$$

$$\iff T \text{ is a } M^*\text{-paranormal operator.}$$

□

Corollary 3.7. *An operator T is a $*$-paranormal operator if and only if*

$$T^{*2}T^2 + 2\lambda TT^* + \lambda^2 I \geq 0$$

for all real number λ.

We establish that the classes of M^*-paranormal operators and M-paranormal operators are independent.

Example 3.2. Let $\{e_n\}_{n=-\infty}^{\infty}$ be an orthonormal basis of the Hilbert space H. Define a bilateral weighted shift T on H with weights $\{\alpha_n\}$ given by

$$\alpha_n = \begin{cases} \frac{3}{7} & \text{if } n \leq -1 \\ \sqrt{\frac{2}{7}} & \text{if } n = 0 \\ \frac{n}{n+1} \cdot \frac{6}{7} & \text{if } n > 0. \end{cases}$$

Then it can be easily seen that the weights $\{\alpha_n\}$ satisfy

$$|\alpha_{n-1}|^2 \leq \frac{7}{6}|\alpha_n||\alpha_{n+1}|$$

for each n. Hence T is a $\frac{7}{6}^*$-paranormal operator by Corollary 3.4. But $\frac{7}{6}|\alpha_1| = \frac{1}{2} \not\geq \sqrt{\frac{2}{7}} = |\alpha_0|$. By the fact that $M|\alpha_{n+1}| \geq |\alpha_n| \Leftrightarrow T$ is a M-paranormal operator. Thus T is not a $\frac{7}{6}$-paranormal operator.

Example 3.3. Let T be a bilateral weighted shift defined as

$$Te_n = \frac{1}{2^{|n|}} e_{n+1}$$

for each n. Then T is M-paranormal for $M \geq 2$. However, by Corollary 3.4, T is M^*-paranormal provided $M \geq 8$. Thus T is not M^*-paranormal, for $2 \leq M \leq 8$, although T is M-paranormal.

Theorem 3.8. *Let T be any M^*-paranormal operator and let N be any invariant subspace under T. Then*

(1) $T|_N$ is M^*-paranormal .

(2) λT is M^*-paranormal for every complex number λ.

(3) If T is unitarily equivalent to an operator S, then S is M^*-paranormal.

(4) $\ker T \subseteq \ker T^*$

(5) $\ker T = \ker T^2$

Proof. (1) Let x be any unit vector in N. Since T is M^*-paranormal, $\|T^*x\|^2 \leq M\|T^2x\|$ and so

$$\|(T|_N)^*x\|^2 = \|T^*x\|^2 \leq M\|T^2x\| = M\|(T|_N)^2x\|.$$

Therefore $T|_N$ is M^*-paranormal.

(2) Let x be any unit vector in H. Then

$$\begin{aligned}\|(\lambda T)^* x\|^2 &= \|\bar{\lambda} T^* x\|^2 = |\lambda|^2 \|T^* x\|^2 \\ &\leq |\lambda|^2 M \|T^2 x\| = M\|(\lambda T)^2 x\|.\end{aligned}$$

Therefore λT is M^*-paranormal.

(3) We must show that $M^2 S^{*2} S^2 + 2\lambda S S^* + \lambda^2 I \geq 0$ for all real number λ. Since T is unitarily equivalent to S, there is a unitary operator U such that $S = U^* T U$. For any real number $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}& M^2 S^{*2} S^2 + 2\lambda S S^* + \lambda^2 I \\ = \;& M^2 (U^* T^* U)(U^* T^* U)(U^* T U)(U^* T U) + 2\lambda (U^* T U)(U^* T^* U) + \lambda^2 U^* U \\ = \;& U^* (M^2 T^{*2} T^2 + 2\lambda T T^* + \lambda^2 I) U \geq 0.\end{aligned}$$

Therefore S is M^*-paranormal.

(4) Since $\|T^* x\|^2 \leq M\|T^2 x\|$ for any unit vector $x \in H$. Let $x \in \ker T$, i.e., $Tx = 0$. Then $T(Tx) = 0$, and so $T^2 x = 0$. By definition, $T^* x = 0$. Hence $x \in \ker T^*$ and so $\ker T \subseteq \ker T^*$.

(5) Since $\ker T \subset \ker T^2$, it suffices to show that $\ker T^2 \subset \ker T$. If $x \in \ker T^2$, then $Tx \in \ker T$. Since T is a M^*-paranormal operator, $\ker T \subset \ker T^*$. Therefore $T^* T x = 0$, and so

$$\|Tx\|^2 = (T^* T x, x) \leq \|T^* T x\| \|x\| = 0.$$

Thus $Tx = 0$, i.e., $x \in \ker T$. $\square$

Corollary 3.9. *Let T be any $*$-paranormal operator and let N be any invariant subspace under T. Then*

(1) $T|_N$ *is $*$-paranormal .*

(2) λT *is $*$-paranormal for every complex number λ.*

(3) ([19]) If T is unitarily equivalent to an operator S, then S is $*$-paranormal.

(4) $\ker T \subseteq \ker T^*$.

(5) $\ker T = \ker T^2$.

Corollary 3.10. *Let N be any closed linear reducing subspace under the operator T. Then T is a M^*-paranormal operator if and only if both $T|_N$ and $T|_{N^\perp}$ are M^*-paranormal operators.*

The inverse of a M^*-paranormal operator may not be M^*-paranormal.

Example 3.4. Let T be a bilateral weighted shift with weights $\{\alpha_n\}$ defined as

$$\alpha_n = \begin{cases} 1 & \text{if } n \leq 0 \\ \frac{n}{n+1} & \text{if } n \geq 1. \end{cases}$$

Then T is a 3^*-paranormal operator by Corollary 3.4, but by Corollary 3.5, T^{-1}, is not 3^*-paranormal since $|\alpha_0||\alpha_{-1}| = 1 \not\leq 3|\alpha_1|^2 = \frac{3}{4}$.

The sum of M^*-paranormal operators even commuting or double commuting may not be M^*-paranormal.

Example 3.5. Let $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ be operators on 2-dimensional space. Then T and S are 4^*-paranormal operator while $T + S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is not so.

The product of two M^*-paranormal operators, in general, may not be M^*-paranormal.

Example 3.6. Suppose that H is a 2-dimensional Hilbert space. Let K be the direct sum of denumerably many copies of H. Let A and B be any two positive operators on H. Let n be a fixed positive integer. Define an operator $T = T_{A,B,n}$ on K as

$$T(x_1, x_2, \ldots) = (0, Ax_1, \ldots, Ax_n, Bx_{n+1}, \ldots).$$

Then T is a M^*-paranormal operator if and only if

$$M^2B^4 - 2\lambda A^2 + \lambda^2 I \geq 0$$

for each $\lambda > 0$.

$$\text{Set } C = \begin{pmatrix} M & M \\ M & 2M \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}.$$

Then both C and D are positive and for each $\lambda > 0$

$$M^2D - 2\lambda C + \lambda^2 I = \begin{pmatrix} (M-\lambda)^2 & 2M(M-\lambda) \\ 2M(M-\lambda) & (2M-\lambda)^2 + 4M^2 \end{pmatrix}$$

is a positive operator. Now choose $A = C^{\frac{1}{2}}$ and $B = D^{\frac{1}{4}}$. With this choice, $T = T_{A,B,n}$ is a M^*-paranormal operator.

Now we show that $T \otimes T$ is not a M^*-paranormal operator. In fact for $\lambda = 1$,

$$\begin{aligned}
& M^2(T\otimes T)^{*2}(T\otimes T)^2 - 2(T\otimes T)(T\otimes T)^* + I\otimes I \\
= \; & [M^2(A^4\otimes A^4) + I\otimes I] \oplus [M^2(A^4\otimes A^4) - 2(A^2\otimes A^2) + I\otimes I] \\
\oplus \; & \cdots \oplus [M^2(A^4\otimes A^4) - 2(A^2\otimes A^2) + I\otimes I] \\
\oplus \; & [M^2(AB^2A)\otimes AB^2A) - 2(A^2\otimes A^2) + I\otimes I] \\
\oplus \; & [M^2(B^4\otimes B^4) - 2(A^2\otimes A^2) + I\otimes I] \oplus \cdots
\end{aligned}$$

which is not positive.

From the above examples, we can summarized as follows :

Theorem 3.11.([38]) *We have the following properties;*

(1) The power and the inverse(if exists) of M^*-paranormal operators are not necessarily M^*-paranormal.

(2) The sum, the direct sum, the product and the tensor product of M^*-paranormal operators are not necessarily M^*-paranormal.

(3) The class of $*$-paranormal operators is closed in the norm topology of operators.

(4) The class of $*$-paranormal operators is not translation invariant.

Theorem 3.12. *Let T be any M^*-paranormal operator. Then we have the following properties:*

(1) $\|T^n x\|^3 \leq M\|T^{n+2}x\|\|T^{n-1}x\|^2$ for any unit vector x in H and positive integer n.

(2) $\|Tx\|^3 \leq M\|T^3 x\|$ for any unit vector x in H.

Proof. (1) Assume that $T^n x \neq 0$. We have

$$\begin{aligned}\|T^n x\|^4 &= (T^n x, T^n x)^2 = (T^*T^n x, T^{n-1}x)^2 \leq \|T^*T^n x\|^2\|T^{n-1}x\|^2\\ &= \|T^*\frac{T^n x}{\|T^n x\|}\|^2\|T^{n-1}x\|^2\|T^n x\|^2\\ &\leq M\|T^{n+2}x\|\|T^{n-1}x\|^2\|T^n x\|.\end{aligned}$$

Thus $\|T^n x\|^3 \leq M\|T^{n+2}x\|\|T^{n-1}x\|^2$.

(2) It follows from (1).

(Another method) Since T satisfies $M^2T^{*2}T^2 + 2\lambda TT^* + \lambda^2 I \geq 0$ for all real number λ. This implies that

$$T^*(M^2T^{*2}T^2 + 2\lambda TT^* + \lambda^2 I)T \geq 0$$

for all real number λ. Let x be any unit vector in H. Then

$$\begin{aligned}&((M^2T^{*3}T^3 + 2\lambda T^*TT^*T + \lambda^2 T^*T)x, x) \geq 0 \quad \text{for all real number } \lambda\\ \Longrightarrow\ & M^2\|T^3x\|^2 + 2\lambda\|T^*Tx\|^2 + \lambda^2\|Tx\|^2 \geq 0 \quad \text{for all real number } \lambda\\ \Longrightarrow\ & (\|T^*Tx\|^2)^2 - M^2\|T^3x\|^2\|Tx\|^2 \leq 0\\ \Longrightarrow\ & (\|Tx\|^2)^2 \leq M\|T^3x\|\|Tx\|.\end{aligned}$$

Hence $\|Tx\|^3 \leq M\|T^3x\|$ for any unit vector x in H. $\square$

Corollary 3.13. *Let T be any $*$-paranormal operator. Then we have the following properties:*

(1) $\|T^n x\|^3 \leq \|T^{n+2}x\|\|T^{n-1}x\|^2$ for any unit vector x in H and positive integer n.

(2) $\|Tx\|^3 \leq \|T^3x\|$ for any unit vector x in H.

Recall that an operator T is *quasinormal* if T commutes with T^*T, or equivalently $(T^*T)T = T(T^*T)$

Theorem 3.14. *If a partial isometry T is M^*-paranormal, then T is quasinormal.*

Proof. Let T be a M^*-paranormal partial isometry. We claim that $R(T)$, the range space of T, is contained in $\ker(T)^\perp$, the initial space of T. Since T is M^*-paranormal, we have $\|T^*x\|^2 \leq M\|T^2x\|$ for any unit vector x in H. Hence

$$\ker(T) \subseteq \ker(T^*) = \overline{R(T)}^\perp.$$

This implies that

$$R(T) \subseteq \overline{R(T)} = \overline{R(T)}^{\perp\perp} \subseteq \ker(T)^\perp.$$

From this it follows that $\ker(T)^\perp$ reduces T. Since T is a partial isometry, T is of the form $A \oplus O$, where A is an isometry. Thus T commutes T^*T and hence T is quasinormal. $\square$

Theorem 3.15. *If a M^*-paranormal operator T commutes with an isometric and surjective operator S, then TS is M^*-paranormal.*

Proof. Let $A = TS$. We must show that

$$M^2 A^{*2} A^2 + 2\lambda A A^* + \lambda^2 I \geq 0$$

for all real number λ. Since $(S^*Sx, x) = (Sx, Sx) = (x, x), S^*S = I$ and

$$SS^* = SS^*(SS^{-1}) = S(S^*S)S^{-1} = SIS^{-1} = I.$$

Thus

$$\begin{aligned} M^2 A^{*2} A^2 + 2\lambda A A^* + \lambda^2 I &= M^2 S^* T^* S^* T^* TSTS + 2\lambda TSS^*T^* + \lambda^2 I \\ &= M^2 T^{*2} T^2 + 2\lambda TT^* + \lambda^2 I \geq 0 \end{aligned}$$

for all real number λ. Hence TS is M^*-paranormal. □

Corollary 3.16. *If a $*$-paranormal operator T commutes with an isometric operator S, then TS is $*$-paranormal.*

Corollary 3.17. *If a M^*-paranormal operator T commutes with a unitary operator S, then TS is M^*-paranormal.*

Theorem 3.18. *Let T and S be doubly commuting M^*-paranormal operators.*

(1) If $\|T^*Sx\|\|x\| \geq \sqrt{M}\|T^*x\|\|Sx\|$ for all x in H, then TS is M^*-paranormal.

(2) If $\|T^*S^2x\|\|x\| \geq M\|T^*x\|\|S^2x\|$ for all x in H, then TS is M^*-paranormal.

Proof. (1) Assume that $\|T^*Sx\|\|x\| \geq \sqrt{M}\|T^*x\|\|Sx\|$ for all x in H. Since T and S are doubly commuting M^*-paranormal operators, we have

$$\begin{aligned}
M^2\|T^2S^2x\|\|S^2x\|\|Sx\|^2\|T^*x\|\|x\|^2 &\geq M\|T^*S^2x\|^2\|Sx\|^2\|T^*x\|\|x\|^2 \\
&= \|S^2T^*x\|\|T^*x\|\|S^2T^*x\|\|Sx\|^2\|x\|^2 \\
&\geq \|S^*T^*x\|^2\|S^2T^*x\|\|Sx\|^2\|x\|^2 \\
&= \|S^*T^*x\|^2\|T^*S^2x\|\|Sx\|^2\|x\|^2 \\
&\geq M\|S^*T^*x\|^2\|T^*x\|\|S^2x\|\|Sx\|^2\|x\|.
\end{aligned}$$

Hence $M\|(TS)^2x\|\|x\| \geq \|(TS)^*x\|^2$. Thus TS is a M^*-paranormal operator.

(2) Assume that $\|T^*S^2x\|\|x\| \geq M\|T^*x\|\|S^2x\|$ for all x in H. Since T and S are doubly commuting M^*-paranormal operators, we have

$$\begin{aligned}
M^2\|T^2S^2x\|\|S^2x\|\|S^*x\|\|T^*x\|\|x\| &\geq M\|T^*S^2x\|^2\|S^*x\|\|T^*x\|\|x\| \\
&= M\|S^2T^*x\|\|T^*x\|\|S^*x\|\|x\|\|S^2T^*x\| \\
&\geq \|S^*T^*x\|^2\|S^*x\|\|x\|\|T^*S^2x\| \\
&\geq M\|S^*T^*x\|^2\|T^*x\|\|S^2x\|\|S^*x\|.
\end{aligned}$$

Hence $M\|(TS)^2x\|\,\|x\| \geq \|(TS)^*x\|^2$. Thus TS is M^*-paranormal. □

An operator may be M^{3*}-paranormal operator but may not be M-hyponormal.

Example 3.7. If T is an operator on H with basis $\{e_n\}_{n=1}^{\infty}$ defined as

$$Te_1 = e_2,\ Te_2 = \frac{1}{3}e_3,\ Te_n = e_{n+1}$$

for each $n \geq 3$. Then T is 8^*-paranormal, but is not 2-hyponormal.

Theorem 3.19. *If a M^*-paranormal operator T double commutes with a N-hyponormal operator S, then the product TS is $(MN^3)^*$-paranormal.*

Proof. Let $\{E(t)\}$ be the resolution of the identity for the self adjoint operator S^*S. By hypothesis both TT^* and $T^{*2}T^2$ commute with every projection $E(t)$. Since by the N-hyponormality of S

$$SS^* \leq N^2S^*S \text{ and } N^2S^{*2}S^2 \geq (S^*S)^2,$$

it follows from the double commutativity that for each $\lambda > 0$.

$$\begin{aligned}
& M^2N^6(TS)^*(TS)^2 - 2\lambda(TS)(TS)^* + \lambda^2 \\
= \;& M^2N^6(T^{*2}T^2)(S^{*2}S^2) - 2\lambda(TT^*)(SS^*) + \lambda^2 \\
\geq \;& M^2N^4(T^{*2}T^2)(S^*S)^2 - 2\lambda N^2(TT^*)(S^*S) + \lambda^2 \\
= \;& N^4\int_0^{\infty}[M^2T^{*2}T^2 - 2\frac{\lambda}{N^2t}TT^* + (\frac{\lambda}{N^2t})^2]t^2dE(t) \\
\geq \;& 0.
\end{aligned}$$

Hence TS is $(MN^3)^*$-paranormal. □

Corollary 3.20. *If a M^*-paranormal operator T is double commutative with a hyponormal operator S, then TS is M^*-paranormal.*

Corollary 3.21. *If a $*$-paranormal operator T is double commutative with a hyponormal operator S, then TS is $*$-paranormal.*

Definition 3.22. An operator T is said to be *co-isometry* if $TT^* = I$ but $T^*T \neq I$. T is called *M-quasihyponormal* if there exists a real number $M > 0$ such that $\|T^*Tx\| \leq M\|T^2x\|$ for any unit vector x in H.

Theorem 3.23. *Let $T \in L(H)$ be a contraction M^*-paranormal operator. Let N be a closed invariant subspace for T. If $T|_N$ is a co-isometry, then $T|_{N^\perp}$ is M^*-paranormal.*

Proof. Let $S = T|_N$ be a co-isometry. Then $S^* = (T|_N)^*$ is isometry.

$$\begin{aligned}
\|S^*x - T^*x\|^2 &= \|T^*x\|^2 - (S^*x, T^*x) - (T^*x, S^*x) + \|S^*x\|^2 \\
&\leq \|x\|^2 - (TS^*x, x) - (x, TS^*x) + \|x\|^2 \\
&= 2\|x\|^2 - 2\|S^*x\|^2 \\
&= 0.
\end{aligned}$$

Thus $T^*x = (T|_N)^*x \in N$ for all $x \in N$, which implies that N is invariant under T^*. Hence N reduces T. By Corollary 3.10, $T|_{N^\perp}$ is M^*-paranormal. □

Theorem 3.24. *Let $T = VP = PV$ where $P \geq 0$, V is isometry and surjective. Then T is M-paranormal if and only if T is M^*-paranormal.*

Proof. We must show that $\|Tx\|^2 = \|T^*x\|^2$ for unit vector x in H.

$$\|Tx\|^2 = (Tx,\, Tx) \;=\; (x,\, T^*Tx) = (x,\, TT^*x)$$

$$= \quad (T^*x,\, T^*x) = \|T^*x\|^2.$$

Therefore $M\|T^2x\| \geq \|Tx\|^2 = \|T^*x\|^2$. $\square$

Theorem 3.25. *If T is a M^*-paranormal partial isometry and it is co-isometry, then T is M-quasihyponormal.*

Proof. Let $A = M^2T^{*2}T^2 + 2\lambda TT^* + \lambda^2 I \geq 0$ for any real number λ. If $B = T^*T$, then $B = B^2 \geq 0$, and so $AB \geq 0$. Thus for all real number λ,

$$\begin{aligned}
& (M^2T^{*2}T^2 + 2\lambda TT^* + \lambda^2 I)(T^*T) \geq 0 \\
\Longrightarrow\quad & M^2T^*T^*TTT^*T + 2\lambda TT^*T^*T + \lambda^2 T^*T \geq 0 \\
\Longrightarrow\quad & M^2T^{*2}T^2 + 2\lambda TT^*T^*T + \lambda^2 T^*T \geq 0 \\
\Longrightarrow\quad & M^2T^{*2}T^2 + 2\lambda T^*T + \lambda^2 T^*T \geq 0 \\
\Longrightarrow\quad & M^2T^{*2}T^2 + 2\lambda (T^*T)^2 + \lambda^2 (T^*T)^2 \geq 0
\end{aligned}$$

and so for any unit vector x, we have

$$\begin{aligned}
& ((M^2T^{*2}T^2 + 2\lambda(T^*T)^2 + \lambda^2(T^*T)^2)x,\, x) \geq 0 \text{ for all real number } \lambda \\
\Longrightarrow\quad & M^2\|T^2x\|^2 + 2\lambda\|T^*Tx\|^2 + \lambda^2\|T^*Tx\|^2 \geq 0 \text{ for all real number } \lambda \\
\Longrightarrow\quad & \|T^*Tx\|^4 \leq M^2\|T^2x\|^2\|T^*Tx\|^2 \\
\Longrightarrow\quad & \|T^*Tx\| \leq M\|T^2x\|.
\end{aligned}$$

$\square$

Theorem 3.26. *Let T, S and $W \in L(H)$, where W has a dense range. Assume that $TW = WS$ and $T^*W = WS^*$. Then*

(1) If S is a partial isometry, then T is a partial isometry.

(2) If S is M^*-paranormal, then T is M^*-paranormal.

Proof. Let $W^* = V^*B$ be the polar decomposition of W^*, where $WW^* = B^2$. Since W has a dense range, W^* is injective. Thus $B^2 = WW^*$ is injective, and V is co-isometry. From $TW = WS$ and $T^*W = WS^*$, we have $TWW^* = WSW^* = WW^*T$. Thus WW^* commutes with T, and so B commutes with T. Hence we have $BTV = TBV = TW = WS = BVS$, which implies that $TV = VS$ because B is injective. Since V is co-isometry, we have $T = TVV^* = VSV^*$ and

$$V^*TB = V^*BT = W^*T = SW^* = SV^*B.$$

Hence $V^*T = SV^*$ and so $V^*VS = V^*TV = SV^*V$.

(1) Assume that S is a partial isometry. Then

$$\begin{aligned} T = VSV^* &= VV^*VSS^*SV^*VV^* \\ &= VSV^*VS^*V^*VSV^* \\ &= VSV^*(VSV^*)^*VSV^* \\ &= TT^*T. \end{aligned}$$

Therefore T is a partial isometry.

(2) If S is a M^*-paranormal operator. Then $M^2S^{*2}S^2 + 2\lambda SS^* + \lambda^2 I \geq 0$ for all real number λ.

$$M^2T^{*2}T^2 + 2\lambda TT^* + \lambda^2 I$$

$$= \quad M^2(VSV^*)^*(VSV^*)^*(VSV^*)(VSV^*) + 2\lambda(VSV^*)(VSV^*)^* + \lambda^2 I$$

$$= M^2VS^*V^*VS^*V^*VSV^*VSV^* + 2\lambda VSV^*VS^*V^* + \lambda^2 I$$
$$= V(M^2S^{*2}S^2 + 2\lambda SS^* + \lambda^2 I)V^* \geq 0.$$

Therefore $M^2T^{*2}T^2 + 2\lambda TT^* + \lambda^2 I \geq 0$ for all real number λ. $\square$

Corollary 3.27. *Let S, T and $W \in L(H)$, where W has a dense range, Assume that $TW = WS$ and $T^*W = WS^*$*

(1) If S is a normal operator, then every operator T is M-paranormal if and only if T is M^*-paranormal.

(2) If S is a hyponormal operator and T is a M-paranormal operator, then T is M^*-paranormal.

(3) If S is co-hyponormal operator and T is a M^*-paranormal operator, then T is M-paranormal.

4. kth roots of $*$-paranormal operators

Definition 4.1. An operator $T \in L(H)$ is a *kth root of a G-operator* if T^k is a G-operator. In particular, if a G-operator is a $*$-paranormal operator, then T is called a *kth root of a $*$-paranormal operator.* We denote these classes by $(\sqrt[k]{H})$, $(\sqrt[k]{P^*})$ and $(\sqrt[k]{P})$ respectively. In particular, if $k = 1$, the class $(\sqrt[k]{P^*})$ becomes the class of $*$-paranormal operators and the class $(\sqrt{P^*})(= (\sqrt[2]{P^*}))$ consists of square roots of $*$-paranormal operators.

Lemma 4.2. *(1) Every hyponormal operator T on finite dimension Hilbert space is a normal operator.*

(2) If T is a $*$-paranormal and quasinilpotent, then T is zero.

An operator T is a kth root of a $*$-paranormal operator, but it is not necessarily a $*$-paranormal operator.

Example 4.1. (1) Let H be a k-dimensional Hilbert space. Define T on H as

$$T = (a_{ij}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

where $a_{ij} = 0$ if $i \geq j$ and $a_{ij} = 1$ if $i < j$. Then $T^k = 0$ $(k \geq 2)$ is a hyponormal operator and so T is a kth root of a hyponormal operator. But $TT^* \not\leq T^*T$. Therefore T is not hyponormal.

(2) Let T be an operator on a two-dimensional Hilbert space defined by $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then T is the square root of a $*$-paranormal operator since $T^2 = 0$ is $*$-paranormal. But T is not $*$-paranormal since $\|T^* x\|^2 = 1 > 0 = \|T^2 x\|$ for some $x = (1, 0)$.

(3) If U is the unilateral shift on l_2 and $T = U^* + 2U$, then

$$T^*T - TT^* = 3 - 3UU^* = 3(I - UU^*) > 0.$$

Therefore T is hyponormal(and so $*-$paranormal). However, if we take $x = (1, 0, -2, 0, \cdots)$, then $\|T^2 x\|^2 = 80 < 89 = \|(T^*)^2 x\|^2$. Hence T^2 is not hyponormal. Also we see that T^2 is not $*$-paranormal by the direct calculation. This ia an example of a $*$-paranormal operator which is not the square root of a $*$-paranormal operator.

From the above Example 4.1(1), we can deduce that if T is any nilpotent operator of order k, i.e., $T^k = 0$, then T is a kth root of a $*$-paranormal operator, but it is not necessarily a $*$-paranormal operator.

If T is hyponormal, then T is normaloid, i.e., $\|T^n\| = \|T\|^n$ for each natural number n. This is not true in the case of a square root of a hyponormal operator. This can be seen as follow ; Let T be the operator on k-dimensional Hilbert space H in Example 4.1. Then T^k is hyponormal and so T is kth root of a hyponormal operator. Also $\|T^k\| = 0$. However, it is easy to show that $\|T\|^k = 1$. Hence $\|T\|^k \neq \|T^k\|$. Thus T is not normaloid.

If T is a $*$-paranormal operator, then T is normaloid, i.e., $\|T^k\| = \|T\|^k$ for each natural number k, but the converse is not true. This is not true in the case of the kth root of a $*$-paranormal operator by Example 4.1.

Theorem 4.3. *Let $T \in (\sqrt[k]{H})$ be any kth root of a hyponormal operator. Then*

(1) λT is a kth root of hyponormal operator for all scalar λ.

(2) If N is invariant subspace of T, then $T|_N$ is a kth root of a hyponormal operator.

(3) If T is quasinilpotent, then T is nilpotent.

(4) If T is unitarily equivalent to S, then S is a kth root of a hyponormal operator.

(5) If T is invertible, then T^{-1} is a kth root of hyponormal operator.

(6) If $\ker T^*$ is equal to $\ker(T^*)^k$, then $\ker T$ reduces for T.

Proof. (1) For each vector x in H,

$$\|(\lambda T)^k x\| = |\lambda|^k \|T^k x\| \geq |\bar{\lambda}|^k \|(T^k)^* x\| = \|(\lambda T)^{k*} x\|.$$

Thus $(\lambda T)^k$ is a hyponormal operator. Therefore λT is a kth root of a hyponormal operator, for all scalar λ.

(2) If N is invariant subspace of T, then $(T|_N)^k = T^k|_N = T^k$. Since $T^k|_N$ is hyponormal, $T|_N$ is a kth root of a hyponormal operator.

(3) Since T is quasinilpotent, $\sigma(T) = \{0\}$. By the spectral mapping theorem, we get that $\sigma(T^k) = \{\sigma(T)\}^k = \{0\}$. Hence T^k is quasinilpotent. Since T^k is hyponormal and quasinilpotent, T^k is a zero operator(Lemma 2.3). Therefore T is nilpotent.

(4) Since T is unitarily equivalent to S, there exists a unitary operator U such that $S = U^*TU$. Thus $S^k = (U^*TU)^k = U^*T^kU$ and so S^k is unitarily equivalent to T^k. Since T^k is hyponormal by hypothesis, S^k is hyponormal and hence S is a kth root of a hyponormal operator.

(5) If T is invertible, then T^k is invertible and hyponormal. Hence $T^{-k} = (T^{-1})^k$ is hyponormal. Thus T^{-1} is a kth root of a hyponormal operator.

(6) Let x be any point in $\ker T$. Then $T(Tx) = 0$ and so $Tx \in \ker T$. Hence $T(\ker T) \subset \ker T$. We need to show that $T^*(\ker T) \subset \ker T$. Since T^k is hyponormal,

$$\|(T^*)^k x\| \leq \|T^k x\|$$

for all $x \in H$, and hence $\ker T^k \subset \ker (T^*)^k$. Since $\ker T^* = \ker (T^*)^k$ and $\ker T \subset \ker T^k$,

$$\ker T \subset \ker T^k \subset \ker (T^*)^k \subset \ker T^*.$$

Therefore $T^*x = 0$ for all $x \in \ker T$. Hence $T(T^*x) = 0$ for all $x \in \ker T$ i.e., $T^*x \in \ker T$ for all $x \in \ker T$, and so $T^*(\ker T) \subset \ker T$. $\square$

The set of operators on H has three useful topologies(weak, strong and norm). The corresponding concepts of convergence can be described by the following ; $A_n \longrightarrow A$ in *norm* if and only if $\|A_n - A\| \longrightarrow 0$, $A_n \longrightarrow A$ *strongly* if and only if $\|(A_n - A)x\| \longrightarrow 0$ for every $x \in H$, and $A_n \longrightarrow A$ *weakly* if and only if $(A_n x,\, y) \longrightarrow (Ax,\, y)$ for every x and y.

By [5], we know that the class of all hyponormal operators on H is closed in the norm topology.

Theorem 4.4. *The set of all the kth roots of hyponormal operators is a proper closed subclass of $L(H)$ with the norm topology.*

Proof. Since T^k is hyponormal, $\ker T^k = \ker T^{2k}$. Hence $\ker T^k = \ker T^{k+1}$. Let U^* be any unilateral backward shift on l_2. Since $\ker (U^*)^k \neq \ker (U^*)^{k+1}$ for any $k \in N$, U^* is not a kth roots of hyponormal operator.

Finally we show that the class $(\sqrt[k]{H})$ is closed in $L(H)$. Let T_n be a kth root of a hyponormal operator for each positive integer n and let $\{T_n\}$ converge to an operator T in norm. Then $\{T_n^k\}$ converge to an operator T^k in norm. Since the set of all hyponormal operators is closed in the norm topology and T_n^k are hyponormal, T^k is hyponormal and hence $T \in (\sqrt[k]{H})$ is a kth roots of hyponormal operator. $\square$

Example 4.2. If $T \in L(H)$ is any nilpotent operator of order $k-1$, then by Halmos characterization T is unitarily equivalent to the following operator matrix

$$A = \begin{pmatrix} 0 & A_{12} & \cdots & A_{1k} \\ & 0 & \cdots & A_{2k} \\ & & \cdots & . \\ & & & 0 \end{pmatrix}.$$

Since A is a kth roots of a hyponormal operator and a kth roots of hyponormal operators are unitarily invariant, T is a kth roots of hyponormal operator.

The following are the straightway conclusions about shift.

Theorem 4.5. *Let T be a weighted shift with nonzero weights $\{\alpha_n\}_{n=0}^{\infty}$. Then T is a kth roots of hyponormal operator if and only if $|\alpha_{n-k}|\cdots|\alpha_{n-1}| \leq |\alpha_n|\cdots|\alpha_{n+k-1}|$ for $n = k, k+1, \ldots$.*

Proof. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of a Hilbert space H. Since $T^k e_n = \alpha_n \cdots \alpha_{n+k-1} e_{n+k}$ and $T^{*k} e_n = \bar{\alpha}_{n-1} \cdots \bar{\alpha}_{n-k} e_{n-k}$, it is easy to calculate that T^k is a hyponormal if and only if $|\alpha_{n-k}|\cdots|\alpha_{n-1}| \leq |\alpha_n|\cdots|\alpha_{n+k-1}|$ for $n = k, k+1, \ldots$. $\square$

Corollary 4.6. *Let T be a weighted shift with non-zero weights $\{\alpha_n\}_{n=0}^{\infty}$. If T is hyponormal, then T is a kth roots of hyponormal operator for every $k \in N$.*

Next we give another example of kth roots of hyponormal operators.

Example 4.3. Let T_x be the weighted shift with nonzero weights $\alpha_0 = x$, $\alpha_1 = \sqrt{\frac{2}{3}}$, $\alpha_2 = \sqrt{\frac{3}{4}}, \ldots$. Then it is an easy calculation from Theorem 4.5 that T_x is a kth root of hyponormal operator if and only if $0 < x \leq \sqrt{\frac{(k+1)^2}{4k+2}}$.

Proof. Let T_x be the weighted shift with non-zero weights $\alpha_0 = x$, $\alpha_1 = \sqrt{\frac{2}{3}}$, $\alpha_2 = \sqrt{\frac{3}{4}}, \ldots$. Then T_x is a kth root of hyponormal operator if and only if $|\alpha_{n-k}| \cdots |\alpha_{n-1}| \leq |\alpha_n| \cdots |\alpha_{n+k-1}| \quad \text{for } n = k, k+1, \ldots$.

Case 1. If $n = k$, then $|\alpha_0||\alpha_1| \cdots |\alpha_{k-1}| \leq |\alpha_k||\alpha_{k+1}| \cdots |\alpha_{2k-1}|$

$$\Leftrightarrow \quad x\sqrt{\frac{2}{3}}\sqrt{\frac{3}{4}} \cdots \sqrt{\frac{k}{k+1}} \leq \sqrt{\frac{k+1}{k+2}}\sqrt{\frac{k+2}{k+3}} \cdots \sqrt{\frac{2k}{2k+1}}$$

$$\Leftrightarrow \quad 0 < x \leq \sqrt{\frac{k+1}{2k+1}}\sqrt{\frac{k+1}{2}} = \sqrt{\frac{(k+1)^2}{4k+2}}.$$

Case 2. If $n \geq k+1$, then it is always true since $\{\alpha_n\}_{n=1}^{\infty}$ is increasing sequence. $\square$

Example 4.4. If $k = 1$ in Example 4.3, then T_x is hyponormal if and only if $0 < x \leq \sqrt{\frac{2}{3}}$. And T_x is a kth root of a hyponormal, but is not hyponormal if and only if

$$\sqrt{\frac{2}{3}} < x \leq \sqrt{\frac{(k+1)^2}{4k+2}} \qquad (k > 1).$$

Next we characterize a matrix on 2-dimensional complex Hilbert space which is in $(\sqrt[k]{H})$. Since every matrix on a finite dimensional complex

Hilbert space is unitarily equivalent to a upper triangular matrix and kth root of a hyponormal operator is invariant, it suffices to characterize a upper triangular matrix T. From the direct calculation, we get the following characterization. The class of the kth root of a hyponormal operators denoted by $(\sqrt[k]{H})$.

Lemma 4.7. *For $k \geq 2$ we have*

$$T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in (\sqrt[k]{H}) \Leftrightarrow b(a^{k-1} + a^{k-2}c + \cdots + c^{k-1}) = 0.$$

Proof. $(\Longrightarrow)$ Let $T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. Then

$$T^k = \begin{pmatrix} a^k & b(a^{k-1} + a^{k-2}c + \cdots + c^{k-1}) \\ 0 & c^k \end{pmatrix}$$

and T^k is hyponormal.

Case 1. If $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then clearly $\|T^k x\| \geq \|T^{*k} x\|$.

Case 2. If $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $b(a^{k-1} + a^{k-2}c + \cdots + c^{k-1})$ is zero so that $\|T^k x\| \geq \|T^{*k} x\|$.

$(\Longleftarrow)$ If $b(a^{k-1} + a^{k-2}c + \cdots + c^{k-1}) = 0$, then $T^k = \begin{pmatrix} a^k & 0 \\ 0 & c^k \end{pmatrix}$ is normal and so hyponormal. Thus $T \in (\sqrt[k]{H})$. $\square$

We remark here that Lemma 4.7 offers the convenient criterion to find some examples of operators in $(\sqrt[k]{H})$. Also we observe that $(\sqrt[k]{H})$ is not necessarily normal on a finite dimensional space.

Example 4.5. If $k = 3$ in Lemma 4.7, then $T \in (\sqrt[3]{H})$ if and only if $b(a^2 + ac + c^2) = 0$. Take $a = 2$, $b = 1$ and $c = -1 + \sqrt{3}i$. Then

$$T = \begin{pmatrix} 2 & 1 \\ 0 & -1 + \sqrt{3}i \end{pmatrix} \in (\sqrt[3]{H}),$$

but T is not a normal operator.

Theorem 4.8. *Let $T \in (\sqrt[k]{P^*})$ be any kth root of a $*$-paranormal operator. Then*

(1) $\lambda T \in (\sqrt[k]{P^*})$ for any complex number $\lambda \in \mathbb{C}$.

(2) If $N \in Lat\ T$ is invariant subspace of T, then $T|_N$ is a kth root of $*$-paranormal operator.

(3) If T is is quasinilpotent, then T is nilpotent.

(4) If T is unitarily equivalent to S, then S is a kth root of $*$-paranormal operator.

Proof. (1) For each vector x in H,

$$\begin{aligned} \|(\lambda T)^{k*}x\|^2 &= \|(\bar{\lambda})^k T^{k*}x\|^2 = |\bar{\lambda}|^{2k}\|T^{k*}x\|^2 \\ &\leq |\bar{\lambda}|^{2k}\|T^{2k}x\| = |\lambda|^{2k}\|T^{2k}x\| = \|(\lambda T)^{2k}x\|. \end{aligned}$$

Thus $(\lambda T)^k$ is $*$-paranormal and so λT is a kth root of a $*$-paranormal operator.

(2) If $N \in$ Lat T is invariant subspace of T, then $(T|_N)^k = T^k|_N$. Since T^k is $*$-paranormal, $T^k|_N$ is $*$-paranormal. Thus $T|_N$ is a kth root of $*$-paranormal.

(3) Since T is quasinilpotent, $\sigma(T) = \{0\}$. By the spectral mapping theorem, we get that $\sigma(T^k) = \{\sigma(T)\}^k = \{0\}$. Hence T^k is quasinilpotent. Since T^k is $*$-paranormal and quasinilpotent, T^k is a zero operator(Lemma 4.2). Therefore T is nilpotent.

(4) Since T is unitarily equivalent to S, there exists a unitary operator U such that $S = U^*TU$. Thus $T^k = (U^*SU)^k = U^*S^kU$ and so S^k is unitarily equivalent to T^k. Since T^k is $*$-paranormal by hypothesis, S^k is $*$-paranormal and hence S is a kth root of $*$-paranormal operator. □

Theorem 4.9. *Let T be a kth root of a M^*-paranormal operator. Then*

(1) If T commutes with an unitary operator S, then TS is also a kth root of a M^*-paranormal operator.

(2) If $\ker T^*$ is equal to $\ker (T^*)^k$, then $\ker T$ reduces for T.

Proof. (1) If $A = (TS)^k$, then we have for any real number λ, there exists $M > 0$ such that

$$\begin{aligned} & M^2A^{*2}A^2 + 2\lambda AA^* + \lambda^2 I \\ = \; & M^2S^{k*}T^{k*}S^{k*}T^{k*}T^kS^kT^kS^k + 2\lambda T^kS^kT^{k*}S^{k*} + \lambda^2 I. \end{aligned}$$

Since $TS = ST, T^*S^* = S^*T^*$ and $S^*S = I$, we get

$$M^2A^{*2}A^2 + 2\lambda AA^* + \lambda^2 I = M^2T^{k*2}T^{k2} + 2\lambda T^kT^{k*} + \lambda^2 I \geq 0,$$

so that $A = (TS)^k$ is a M^*-paranormal operator. Hence TS is a kth root of M^*-paranormal operator.

(2) Let x be any point in $\ker T$. Then $T(Tx) = 0$ and so $Tx \in \ker T$. Hence $T(\ker T) \subset \ker T$. We need to show that $T^*(\ker T) \subset \ker T$. Since T^k is $*$-paranormal, $\|(T^k)^*x\|^2 \leq M\|T^{2k}x\|$ for unit vector x in H, and hence $\ker T^{2k} \subset \ker (T^k)^*$. Since $\ker T^* = \ker (T^*)^k$ and $\ker T \subset \ker T^{2k}$,

$$\ker T \subset \ker T^{2k} \subset \ker (T^*)^k = \ker T^*.$$

Therefore $T^*x = 0$ for all $x \in \ker T$. Hence $T(T^*x) = 0$ for all $x \in \ker T$. Thus $T^*(\ker T) \subset \ker T$. □

Theorem 4.10. *Let T be a weighted shift with non-zero weights $\{\alpha_n\}$ $(n = 0, 1, 2, \ldots)$. Then T is a kth root of M^*-paranormal operator if and only if*

$$|\alpha_{n-1}|^2|\alpha_{n-2}|^2 \cdots |\alpha_{n-k}|^2 \leq M|\alpha_n||\alpha_{n+1}| \cdots |\alpha_{n+2k-1}|$$

for $n = k, k+1, k+2, \ldots$.

Proof. Since T is a kth root of M^*-paranormal operator, T^k is a M^*-paranormal operator. Therefore $\|(T^k)^*e_n\|^2 \leq M\|T^{2k}e_n\|$ $(n = 1, 2, \ldots)$. Here $T^{2k}e_n = \alpha_n\alpha_{n+1}\cdots\alpha_{n+(2k-1)}e_{n+2k}$ and

$$(T^k)^*e_n = \alpha_{n-1}\alpha_{n-2}\cdots\alpha_{n-k}e_{n-k}$$

for $k = 1, 2, \ldots$. Since T^k is M^*-paranormal,

$$\|(T^k)^* e_n\|^2 \leq M \|T^{2k} e_n\| (n = 1, 2 \ldots).$$

$$\Leftrightarrow |\alpha_{n-1}|^2 |\alpha_{n-2}|^2 \cdots |\alpha_{n-k}|^2 \leq M |\alpha_n||\alpha_{n+1}| \cdots |\alpha_{n+2k-1}|$$

for $n = k, k+1 \ldots$. $\square$

Corollary 4.11. *Let T be a weighted shift with non-zero weights $\{\alpha_n\}_{n=0}^{\infty}$. Then T is a kth root of $*$-paranormal operator if and only if $|\alpha_{n-k}|^2 \cdots |\alpha_{n-1}|^2 \leq |\alpha_n| \cdots |\alpha_{n+2k-1}|$ for $n = k, k+1, \ldots$.*

Example 4.6. Let T_x be the weighted shift with nonzero weights $\alpha_0 = x$, $\alpha_1 = \sqrt{\frac{2}{3}}$, $\alpha_2 = \sqrt{\frac{3}{4}}, \ldots$. Then T_x is a kth root of $*$-paranormal operator if and only if

$$0 < x \leq \frac{1}{\sqrt{2}} \cdot \sqrt[4]{\frac{(k+1)^3}{3k+1}}.$$

Proof. From Corollary 4.11, T_x is a kth root of $*$-paranormal operator if and only if

$$|\alpha_{n-k}|^2 \cdots |\alpha_{n-1}|^2 \leq |\alpha_n| \cdots |\alpha_{n+2k-1}|$$

for $n = k, k+1, \ldots$.

Case 1. If $n = k$, $|\alpha_0|^2 |\alpha_1|^2 \cdots |\alpha_{k-1}|^2 \leq |\alpha_k| \cdots |\alpha_{3k-1}|$

$$\Leftrightarrow \quad x^2 \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{k}{k+1}$$

$$\leq \quad \sqrt{\frac{k+1}{k+2}} \cdot \sqrt{\frac{k+2}{k+3}} \cdots \sqrt{\frac{3k}{3k+1}}$$

$$\Leftrightarrow \quad x^2 \leq \frac{k+1}{2} \cdot \sqrt{\frac{k+1}{3k+1}}$$

$$\Leftrightarrow \quad 0 < x \ \leq \ \frac{1}{\sqrt{2}} \cdot \sqrt[4]{\frac{(k+1)^3}{3k+1}}.$$

Case 2. If $n \geq k+1$, then it is always true. $\square$

Example 4.7. If $k = 1$ in Example 4.6, then T_x is $*$-paranormal if and only if $0 < x \ \leq \frac{1}{\sqrt[4]{2}}$. And T_x is a kth root of $*$-paranormal, but not $*$-paranormal if and only if

$$\frac{1}{\sqrt[4]{2}} < x \leq \frac{1}{\sqrt{2}} \cdot \sqrt[4]{\frac{(k+1)^3}{3k+1}} \quad (k > 1).$$

Example 4.8. If $0 < x \ \leq \sqrt{\frac{2}{3}}$, then T_x is $*$-paranormal and paranormal. If $\sqrt{\frac{2}{3}} < x \ \leq \frac{1}{\sqrt[4]{2}}$, then T_x is $*$-paranormal but not paranormal. If $x > \frac{1}{\sqrt[4]{2}}$, then T_x is not $*$-paranormal and not paranormal.

Theorem 4.12. *The set* $(\sqrt[k]{P^*})$ *of all the* k*th roots of* $*$*-paranormal operator is a proper closed subclass of* $L(H)$ *with the norm topology.*

Proof. Since T^k is $*$-paranormal, $\ker T^k = \ker T^{2k}$. Hence $\ker T^k = \ker T^{k+1}$. Let U^* be any unilateral backward shift on l_2. Since $\ker (U^*)^k \neq \ker (U^*)^{k+1}$ for any $k \in N$, U^* is not a kth roots of $*$-paranormal operator.

Finally we show that the class $(\sqrt[k]{P^*})$ is closed in $L(H)$. Let T_n be a kth root of a $*$-paranormal operator for each positive integer n and let $\{T_n\}$

converge to an operator T in norm. Then $\{T_n^k\}$ converge to an operator T^k in norm. Since the set of all $*$-paranormal operators is closed in the norm topology and T_n^k are $*$-paranormal, T^k is $*$-paranormal and hence $T \in (\sqrt[k]{P^*})$ is a kth roots of $*$-paranormal operator. $\square$

It is known that hyponormal operators have translation invariant property. On the other hand, the class of square roots of hyponormal operators may not have the translation invariant property.

Example 4.9. Let $T \in L(H \oplus H)$ is defined as

$$T = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}.$$

Then T is a square roots of hyponormal operator. But $[(T-\lambda I)^{*2}, (T-\lambda I)^2] = \begin{pmatrix} -4|\lambda|^2 AA^* & 0 \\ 0 & 4|\lambda|^2 A^*A \end{pmatrix}$, which is not positive. Hence $(T-\lambda I)^2$ is not necessarily hyponormal.

Theorem 4.13. *If $T - \lambda I$ is a kth root of hyponormal operator for every $\lambda \in \mathbb{C}$, then T is hyponormal.*

Proof. If $(T-\lambda I)^k$ is a hyponormal for every $\lambda \in \mathbb{C}$, then

$$[(T^* - \bar{\lambda})^k, (T-\lambda I)^k] \geq 0.$$

Therefore, we have

$$0 \;\leq\; ((T^* - \bar{\lambda})^k (T-\lambda)^k - (T-\lambda)^k (T^* - \bar{\lambda})^k$$

$$
\begin{aligned}
&= (T^* - \bar{\lambda})^k (T-\lambda)^k - (T-\lambda)^k (T^* - \bar{\lambda})^k, \\
(*) \quad &= \sum_{r=0}^{k} \binom{k}{r} (T^*)^{k-r} (-\bar{\lambda})^r \cdot \sum_{s=0}^{k} \binom{k}{s} T^{k-s} (-\lambda)^s \\
&- \sum_{s=0}^{k} \binom{k}{s} T^{k-s} (-\lambda)^s \cdot \sum_{r=0}^{k} \binom{k}{r} (T^*)^{k-s} (-\bar{\lambda})^r.
\end{aligned}
$$

Set $\lambda = \rho e^{i\theta}$ for every $0 \le \theta < 2\pi$ and $\rho > 0$. Then we get

$$
\begin{aligned}
(*) \quad &= \sum_{r=0}^{k} \sum_{s=0}^{k} (-1)^{r+s} \binom{k}{r} \binom{k}{s} \rho^{r+s} e^{i(s-r)\theta} (T^*)^{k-r} T^{k-s} \\
&- \sum_{r=0}^{k} \sum_{s=0}^{k} (-1)^{r+s} \binom{k}{r} \binom{k}{s} \rho^{r+s} e^{i(s-r)\theta} T^{k-s} (T^*)^{k-r}.
\end{aligned}
$$

Since terms in $(*)$ are eliminated when $r = s = k$, we do eliminate these terms and then divide by ρ^{2k-2}. Then we obtain

$$
0 \le \binom{k}{k-1} \binom{k}{k-1} [T^*T - TT^*] + \frac{1}{\rho}(\text{the other terms}).
$$

Letting $\rho \longrightarrow \infty$, we get $T^*T \ge TT^*$. $\square$

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< 국 문 초 록 >

일반화된 *-Paranormal 작용소들의 집합에 관한 연구

본 논문에서는 힐버트 공간(Hilbeert space) H 위에서 비정규(nonnormal) 유계선형작용소로서 *-paranormal 작용소를 일반화하는 M^*-paranormal 작용소들과 *-paranormal 작용소의 k제곱근 작용소들의 여러 가지 특성을 조사하였는 데, 그 특성 중 주요 결과들은 다음과 같다.

(1) N이 M^*-paranormal 작용소 T 하에서 불변인 폐부분공간이면, $T|_N$ 과 λT도 M^*-paranormal 작용소이다. 또한 M^*-paranormal 작용소 T와 유니터리 동치(unitarily equivalent)가 되는 작용소 S도 M^*-paranormal 작용소이다. 그리고 만약 T가 M^*-paranormal 작용소이면, ker $T \subseteq$ ker T^*이고 ker T=ker T^2이다.

(2) M^*-paranormal 작용소 S, T가 가환이거나 이중 가환일지라도 이들 작용소의 합 $S+T$은 M^*-paranormal 작용소가 되지 않는다. 마찬가지로 M^*-paranormal 작용소들의 곱도 일반적으로 M^*-paranormal 작용소가 되지 않는다.

(3) M^*-paranormal 작용소가 hyponormal 작용소와 이중 가환이면, 그 곱은 M^*-paranormal 작용소이다. 또한 M^*-paranormal 작용소가 유니터리(unitary) 작용소와 교환가능이면, 이들 곱은 M^*-paranormal 작용소이다.

(4) 일반적으로 가환인 두 M^*-paranormal 작용소의 곱 TS는 M^*-paranormal 작용소가 되지 않지만, 다음 성질 중 하나가 성립하면 그 곱 TS도 M^*-paranormal 작용소이다.

임의의 $x \in H$에 대하여

(a) $\|T^*Sx\|\|x\| \geq \sqrt{M}\|T^*x\|\|Sx\|$

(b) $\|T^*S^2x\|\|x\| \geq M\|T^*x\|\|S^2x\|$

(5) T가 M^*-paranormal 작용소이고 $x \in H$는 임의의 단위벡터이면 $\|T^nx\|^3 \leq M\|T^{n+2}x\|\|T^{n-1}x\|^2$과 $\|Tx\|^3 \leq M\|T^3x\|$이 성립한다.

(6) 영이 아닌 가중값 $\{\alpha_n\}$($n=1,2,3\ldots$)을 갖는 가중인 밀림 작용소(weighted shift) T가 M^*-paranormal 작용소의 k제곱근 작용소가 되기 위한 필요충분조건은 $n=k, k+1, k+2, \ldots$에 대하여

$$|\alpha_{n-1}|^2|\alpha_{n-2}|^2\cdots|\alpha_{n-k}|^2 \leq M|\alpha_n||\alpha_{n+1}|\cdots|\alpha_{n+2k-1}|.$$

(7) N이 $*$-paranormal 작용소의 k제곱근 작용소 T 하에서 불변인 폐부분공간이면 $T|_N$와 λT 작용소도 $*$-paranormal 작용소의 k제곱근 작용소이다. 또한 $*$-paranormal 작용소의 k제곱근 작용소와 유니터리동치(unitary equivalent)가 되는 작용소도 $*$-paranormal 작용소의 k제곱근 작용소이다.

(8) M^*-paranormal 작용소 T의 k제곱근 작용소가 유니터리(unitary)작용소 S와 교환가능이면, 그 곱 ST도 M^*-paranormal 작용소의 k제곱근 작용소이다.

(9) $*$-paranormal 작용소의 k제곱근 작용소들의 집합은 $L(H)$의 노름위상(norm topology)에 대하여 폐집합이다.

(10) T_x는 영이 아닌 가중값 $\alpha_0 = x$, $\alpha_1 = \sqrt{\frac{2}{3}}$, $\alpha_2 = \sqrt{\frac{3}{4}}$, … 를 갖는 가중인 밀림 작용소(weighted shift)라 하자. 이때 T_x가 *-paranormal 작용소의 k제곱근 작용소이 되기 위한 필요충분조건을 제시한다. 또한 T_x가 *-paranormal이지만 paranormal 작용소가 안되는 조건을 제시한다. 마찬가지로 T_x가 *-paranormal 작용소도 아니고 paranormal 작용소가 안되는 조건을 얻는다.