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On the Direct Sum of Semirings

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감 사 의 글

이 논문이 완성되기까지 연구에 바쁘신 가운데 자상하고 친절하게 지도를 하여 주신 현진오 교수님께 감사드리며, 아울러 양성호 교수님과 수학교육과의 여러 교수님께 심심한 사의를 표합니다.

그리고 그동안 저에게 사랑과 격려를 주신 주위의 많은 분들께 감사드립니다.

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김 상 택

CONTENTS

1. Introduction

Semiring was first introduced by Vandiver in 1934. P.J.Allen defined Q-ideal and maximal homomorphism in a large class of semirings.

In this paper, we shall investigate the direct sum of semirings. This investigation is done by proving the followings.

Firstly, if I_i is a Q_i-ideal of the semiring R_i for $i=1,2,\cdots,n$, then $\Pi_{i=1}^{n} I_i$ is a $\Pi_{i=1}^{n} Q_i$-ideal of $\Pi_{i=1}^{n} R_i$.

Secondly, if the semiring R is the internal direct sum of ideals $I_1, I_2, \cdots, I_n$, then R is isomorphic to the external direct sum of $I_1, I_2, \cdots, I_n$.

Thirdly, if $R=\sum_{i=1}^{n} \oplus R_i$ is the direct sum of a finite number of semiring R_i, then every ideal I of R is of the form $I=\sum_{i=1}^{n} \oplus I_i$ where I_i are ideal of R_i.

Fourthly, if I_i is a Q_i-ideal of R for $i=1,2,\cdots,n$ and $R=\sum_{i=1}^{n} \oplus I_i$, then $R/I_j \cong \sum_{\substack{i=1 \\ i\neq j}}^{n} \oplus I_i$ for $j=1,2,\cdots,n$.

Lastly, if R is the direct sum of the semirings R_i and I_i a Q_i-ideal of R_i for $i=1,2,\cdots,n$ and $I=\sum_{i=1}^{n} \oplus I_i$, then $R/I \cong \sum_{i=1}^{n} \oplus (R_i/I_i)$.

2. Definitions and Preliminaries

There are many different definitions of a semiring appearing in the literature. Throughout this paper, a semiring will be defined as follows ;

DEFINITION(2-1). A set R together with two associative binary operations called addition and multiplication(denoted by + and $\cdot$, respectively) will be called a semiring provided ;

(i) addition is a commutative operation,

(ii) there exist $0 \in R$ such that $x + 0 = x$ and $x0 = 0x = 0$ for each $x \in R$, and

(iii) multiplication distributes over addition both from the left and from the right.

DEFINITION (2-2). A subset I of a semiring R will be called an ideal if $a, b \in I$ and $r \in R$ implies $a+b \in I$, $ra \in I$ and $ar \in I$.

DEFINITION(2-3). A mapping φ from the semiring R into the semiring R' will be called a homomorphism if $(a+b)\varphi = a\varphi + b\varphi$ and $(ab)\varphi = a\varphi b\varphi$ for each $a, b \in R$. An isomorphism is a one-to-one homomorphism. The semirings R and R' will be called isomorphic (denoted by $R \cong R'$) if there exists an isomorphism from R onto R'.

DEFINITION (2-4). An ideal I in the semiring R will be called a Q-ideal if there exists a subset Q of R satisfying the followings ;

(i) $\{q+I\}_{q\in Q}$ is a partition of R, and

(ii) if $q_1, q_2 \in Q$ such that $q_1 \neq q_2$, then $(q_1+I)\cap(q_2+I)=\phi$.

DEFINITION (2-5). A homomorphism φ from the semiring R onto the semiring R' is said to be maximal if for each $a\in R'$ there exists $c_a \in \varphi^{-1}(\{a\})$ such that $x+\ker(\varphi)\subset c_a+\ker(\varphi)$ for each $x\in\varphi^{-1}(\{a\})$, where $\ker(\varphi)=\{x\in R \mid x\varphi=0\}$.

THEOREM (2-6). Let I be a Q-ideal in the semiring R. If $x\in R$, then there exists a unique $q\in Q$ such that $x+I\subset q+I$.

proof. Refer to Lemma 7. in [1].

THEOREM (2-7). If I is a Q-ideal in the semiring R, then $(\{q+I\}_{q\in Q}, \oplus_Q, \odot_Q)$ is a semiring.

proof. Refer to Theorm 8. in [1].

In Theorem (2-7), we can define the binary operations $\oplus_Q$ and $\odot_Q$ on $\{q+I\}_{q\in Q}$ as follows ;

(i) $(q_1+I)\oplus_Q(q_2+I)=q_3+I$ where q_3 is the unique element in Q such that $q_1+q_2+I\subset q_3+I$, and

(ii) $(q_1+I)\odot_Q(q_2+I)=q_3+I$ where q_3 is the unique element in Q such that $q_1q_2+I\subset q_3+I$. The elements q_1+I and q_2+I in $\{q+I\}_{q\in Q}$ will be called equal (denoted by $q_1+I=q_2+I$) if and only if $q_1=q_2$.

THEOREM (2-8). Let f be a homomorphism from the semiring R onto the semiring R'. Then

(1) for each ideal I' of R', the subsemiring $f^{-1}(I')$ is an ideal of R, and

(2) for each ideal I of R, the subsemiring f(I) is an ideal of R'.

Proof. Refer to Proposition (2-2) in [2].

THEOREM (2-9). If I is a Q-ideal in the semiring R, then I is a zero element in R/I.

Proof. Refer to Proposition 13 in [3].

THEOREM (2-10). If φ is a maximal homomorphism from the semiring R onto the semiring R', then $R/\ker(\varphi) \cong R'$.

Proof. Refer to Theorem 16 in [1].

3. The direct sum of semirings

Let $R_1, R_2, \cdots, R_n$ be a finite number of semirings and consider their Cartesian product $R = \Pi_{i=1}^{n} R_i$ (or $R_1 \times R_2 \times \cdots \times R_n$) consisting of all ordered n-tuples $(a_1, a_2, \cdots, a_n)$ with $a_i \in R_i$. We can easily convert R into a semiring by performing the semiring operations componentwise ; in other words, if $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ are two elements of R, simply define

$$(a_1, a_2, \cdots, a_n) + (b_1, b_2, \cdots, b_n) = (a_1 + b_1,\ a_2 + b_2, \cdots, a_n + b_n)$$

and $(a_1, a_2, \cdots, a_n)(b_1, b_2, \cdots, b_n) = (a_1 b_1,\ a_2 b_2, \cdots, a_n b_n)$.

The semiring so obtained is called the external direct sum of $R_1, \ldots, R_n$.

LEMMA (3-1). Let I_1 be a Q_1-ideal of the semiring R_1 and I_2 a Q_2-ideal of the semiring R_2.

Then $I_1 \times I_2$ is a $Q_1 \times Q_2$-ideal of the semiring $R_1 \times R_2$.

Proof. It is clear that $I_1 \times I_2$ is an ideal of $R_1 \times R_2$.

(1) If $(r_1, r_2) \in R_1 \times R_2$, then $r_1 \in q_1 + I_1$ and $r_2 \in q_2 + I_2$ for some $q_1 \in Q_1$ and $q_2 \in Q_2$. Thus $r_1 = q_1 + i_1$ and $r_2 = q_2 + i_2$ for some $i_1 \in I_1$ and $i_2 \in I_2$. So, $(r_1, r_2) = (q_1 + i_1,\ q_2 + i_2)$

$= (q_1, q_2) + (i_1, i_2) \in (q_1, q_2) + (I_1 \times I_2) \subset \bigcup_{q \in Q_1 \times Q_2} [q + (I_1 \times I_2)]$.

Hence $\bigcup_{q \in Q_1 \times Q_2} [q + (I_1 \times I_2)] = R_1 \times R_2$.

(2) Suppose that $[(q_1, q_2) + (I_1 \times I_2)] \cap [(q_1', q_2') + (I_1 \times I_2)] \neq \phi$ for some $(q_1, q_2), (q_1', q_2') \in Q_1 \times Q_2$. Then $x = (q_1, q_2) + (i_1, i_2) = (q_1', q_2') + (i_1', i_2')$ for some $i_1, i_1' \in I_1$ and for some $i_2, i_2' \in I_2$. i.e. $x = (q_1 + i_1, q_2 + i_2) = (q_1' + i_1', q_2' + i_2')$.

Thus $q_1 + i_1 = q_1' + i_1' \in (q_1 + I_1) \cap (q_1' + I_1)$ and $q_2 + i_2 = q_2' + i_2' \in (q_2 + I_2) \cap (q_2' + I_2)$. So, $(q_1 + I_1) \cap (q_1' + I_1) \neq \phi$ and $(q_2 + I_2) \cap (q_2' + I_2) \neq \phi$. Thus $q_1 = q_1'$ and $q_2 = q_2'$. i.e. $(q_1, q_2) = (q_1', q_2')$.

Making successive use of the Lemma (3-1) immediately yields the following theorem.

THEOREM (3-2). If I_i is a Q_i-ideal of the semiring R_i for $i = 1, \cdots, n$, then $\Pi_{i=1}^{n} I_i$ is a $\Pi_{i=1}^{n} Q_i$-ideal of $\Pi_{i=1}^{n} R_i$.

Let us now describe certain binary operation on the set of ideals of the semiring R. Given a finite number of ideals $I_1, I_2, \cdots, I_n$ of the semiring R, one defines their sum in the natural way;

$$I_1 + I_2 + \cdots + I_n = \{ a_1 + a_2 + \cdots + a_n \mid a_i \in I_i \}.$$

Then $I_1 + I_2 + \cdots + I_n$ is likewise an ideal of R and is the smallest ideal of R which contains every I_i; phrased in other way, $I_1 + I_2 + \cdots + I_n$ is the ideal generated by $I_1 \cup I_2 \cup \cdots \cup I_n$.

DEFINITION (3-3). Let $I_1, I_2, \cdots, I_n$ be ideals of the semiring R. We call R the internal direct sum of $I_1, I_2, \cdots, I_n$, and write $R = \sum_{i=1}^{n} \oplus I_i$ (or $I_1 \oplus I_2 \oplus \cdots \oplus I_n$), if each element x of R is uniquely expressible in the form $x = a_1 + a_2 + \cdots + a_n$ where $a_i \in I_i$.

LEMMA (3-4). Let $I_1, I_2, \cdots, I_n$ be ideals of the semiring R. If $R = \sum_{i=1}^{n} \oplus I_i$, then $I_i \cap (I_1 + \cdots + I_{i-1} + I_{i+1} + \cdots + I_n) = \{0\}$ for each $i = 1, 2, \cdots, n$.

Proof. For each i, if $x \in I_i \cap (I_1 + \cdots + I_{i-1} + I_{i+1} + \cdots + I_n)$, then $x = a_i$ and $x = a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n$ where $a_j \in I_j$. Since $x \in R$, x is uniquely representable as a sum of elements from the ideals I_j. Thus $a_1 = a_2 = \cdots = a_n = 0$. i.e. $x = 0$.

LEMMA (3-5). Let $I_1, I_2, \cdots, I_n$ be ideals of the semiring R. If $R = \sum_{i=1}^{n} \oplus I_i$, then $I_i \cap I_j = \{0\}$ for each $i \neq j$.

Proof. For each $i \neq j$, $I_j \subset I_1 + \cdots + I_{i-1} + I_{i+1} + \cdots + I_n$. By Lemmma (3-4), $I_i \cap I_j = \{0\}$.

THEOREM (3-6). If the semiring R is the internal direct sum of ideals $I_1, I_2, \cdots, I_n$, then R is isomorphic to the external direct sum of $I_1, I_2, \cdots, I_n$.

Proof. Define the mapping $\varphi : R \rightarrow \Pi_{i=1}^{n} I_i$ by $(a_1 + a_2 + \cdots + a_n)\varphi =$

$(a_1, a_2, \cdots, a_n)$. Since every element of R is uniquely representable as a sum of elements from the ideals I_i, φ is well-defined.

It is clear that φ is an 1 - 1 and onto mapping.

Let $a_1 + a_2 + \cdots + a_n$ and $b_1 + b_2 + \cdots + b_n$ are elements in R. Then $(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$ by Lemma(3-5). Thus $((a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n))\varphi = (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)\varphi = (a_1 b_1, a_2 b_2, \cdots, a_n b_n) = (a_1, a_2, \cdots, a_n)(b_1, b_2, \cdots, b_n) = ((a_1 + a_2 + \cdots + a_n)\varphi)((b_1 + b_2 + \cdots + b_n)\varphi)$ and $((a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n))\varphi = (a_1 + a_2 + \cdots + a_n)\varphi + (b_1 + b_2 + \cdots + b_n)\varphi$. So, φ is a homomorphism. Hence $R \cong \Pi_{i=1}^{n} I_i$.

Because of the isomorphism proved in theorem (3-6) we shall henceforth refer to the semiring R as being a direct sum, not qualifying it with the adjective "internal" or "external", and rely exclusively on the $\oplus$ - notation.

THEOREM (3-7). Let $R = \sum_{i=1}^{n} \oplus R_i$ be the direct sum of a finite number of semirings R_i $(i = 1, 2, \cdots, n)$. Then every ideal I of R is of the form $I = \sum_{i=1}^{n} \oplus I_i$ where I_i an ideal of R_i.

Proof. For fixed i, define the mapping $\Pi_i : R \to R_i$ as follows ; if $a = (a_1, a_2, \cdots, a_n)$ where $a_i \in R_i$, then $a\Pi_i = a_i$. Then, for all i, Π_i is a homomorphism from R onto R_i. Let I be any ideal of R and

$I_i = (I)\Pi_i$ for $i = 1, 2, \cdots, n$. By Theorem (2-8), then I_i is an ideal of R_i for $i = 1, 2, \cdots, n$. We claim that $I = \sum_{i=1}^{n} \oplus I_i$. It is clear that $I \subset \sum_{i=1}^{n} \oplus I_i$. If $(b_1, b_2, \cdots, b_n) \in \sum_{i=1}^{n} \oplus I_i$, then $b_i \in I_i$ for each i. For each i, there exists $(x_1, \cdots, x_{i-1}, b_i, x_{i+1}, \cdots, x_n) \in I$ such that $(x_1, \cdots, x_{i-1}, b_i, x_{i+1}, \cdots, x_n)\Pi_i = b_i$. Thus $(0, \cdots, 0, b_i, 0, \cdots, 0) = (x_1, \cdots, x_{i-1}, b_i, x_{i+1}, \cdots, x_n)(0, \cdots, 0, 1, 0, \cdots, 0) \in I$ for each i. So, $(b_1, b_2, \cdots, b_n) = (b_1, 0, \cdots, 0) + (0, b_2, 0, \cdots, 0) + \cdots + (0, \cdots, 0, b_n) \in I$. Hence $I = \sum_{i=1}^{n} \oplus I_i$.

THEOREM (3-8). Let I_i be a Q_i-ideal of R for $i = 1, 2, \cdots, n$ and $R = \sum_{i=1}^{n} \oplus I_i$. Then $R/I_j \cong \sum_{\substack{i=1 \\ i \neq j}}^{n} \oplus I_i$ for each $j = 1, 2, \cdots, n$.

Proof. For fixed j, define $\varphi_j : R \rightarrow \sum_{\substack{i=1 \\ i \neq j}}^{n} \oplus I_i$ by $(a_1 + \cdots + a_n)\varphi_j = a_1 + \cdots + a_{j-1} + a_{j+1} + \cdots + a_n$. Then it is clear that φ_j is well-defined and a homomorphism from R onto $\sum_{\substack{i=1 \\ i \neq j}}^{n} \oplus I_i$. For each $a = a_1 + \cdots + a_{j-1} + a_{j+1} + \cdots + a_n \in \sum_{\substack{i=1 \\ i \neq j}}^{n} \oplus I_i$, let $c_a = a_1 + \cdots + a_{j-1} + 0 + a_{j+1} + \cdots + a_n$ where 0 is the zero element in I_j. Then $c_a \in \varphi_j^{-1}(\{a\})$. For any $x \in \varphi_j^{-1}(\{a\})$, $x = a_1 + \cdots + a_{j-1} + x_j + a_{j+1} + \cdots + a_n$ for some $x_j \in I_j$. Then $x + \ker \varphi_j = c_a + (0 + \cdots + 0 + x_j + 0 + \cdots + 0) + \ker \varphi_j \subset c_a + \ker \varphi_j$ since $(0 + \cdots + 0 + x_j + 0 + \cdots + 0) \in \ker \varphi_j$. Hence φ_j is a maximal homomorphism from R onto $\sum_{\substack{i=1 \\ i \neq j}}^{n} \oplus I_i$. By Theorem (2-10), $R/\ker \varphi_j \cong \sum_{\substack{i=1 \\ i \neq j}}^{n} \oplus I_i$. Since $\ker \varphi_j \cong I_j$,

$$R/I_j \cong \sum_{\substack{i=1 \\ i \neq j}}^{n} \oplus I_i .$$

THEOREM (3-9). Let R be the direct sum of the semirings R_i for $i = 1, 2, \cdots, n$. If I_i is a Q_i-ideal of R_i for $i = 1, 2, \cdots, n$ and $I = \sum_{i=1}^{n} \oplus I_i$, then $R/I \cong \sum_{i=1}^{n} \oplus (R_i/I_i)$; in other words,

$$(\sum_{i=1}^{n} \oplus R_i)/(\sum_{i=1}^{n} \oplus I_i) \cong \sum_{i=1}^{n} \oplus (R_i/I_i)$$

Proof. By Theorem (2-6), for any element $(a_1, a_2, \cdots, a_n) \in R$, there exists a unique $q_i \in Q_i$ such that $a_i + I_i \subset q_i + I_i$, for $i = 1, 2, \cdots, n$. Define $\varphi : R \to \sum_{i=1}^{n} \oplus (R_i/I_i)$ by $(a_1, a_2, \cdots, a_n)\varphi = (q_1 + I_1, q_2 + I_2, \cdots, q_n + I_n)$ where q_i is the unique element in Q_i such that $a_i + I_i \subset q_i + I_i$ for $i = 1, 2, \cdots, n$.

Clearly, φ is well-defined and onto.

Let $(a_1, \cdots, a_n)\varphi = (q_1 + I_1, \cdots, q_n + I_n)$ and $(b_1, \cdots, bn)\varphi = (q_1' + I_1, \cdots, q_n' + I_n)$. Then $a_i + I_i \subset q_i + I_i$ and $b_i + I_i \subset q_i' + I_i$ for all i. If $(a_1+b_1, \ldots, a_n+b_n)\varphi = (q_1'' + I_1, \cdots, q_n'' + I_n)$ and $(q_1 + I_1, \cdots, q_n + I_n) + (q_1' + I_1, \cdots, q_n' + I_n) = (q_1^* + I_1, \cdots, q_n^* + I_n)$, then $a_i + b_i + I_i \subset q_i'' + I_i$ and $q_1 + q_i' + I_i \subset q_i^* + I_i$ for all i. Since $a_i + I_i \subset q_i + I_i$ and $b_i + I_i \subset q_i' + I_i$, $a_i = q_i + j_i$ and $b_i = q_i' + j_i'$ for some j_i, $j_i' \in I_i$. Thus $a_i + b_i + I_i = q_i + q_i' + j_i + j_i' + I_i \subset q_i + q_i' + I_i \subset q_i^* + I_i$ for all i. Since $a_i + b_i + I_i \subset q_i'' + I_i$ and $a_i + b_i + I_i \subset q_i^* + I_i$ for all i, $q_i'' = q_i^*$ for all i. Thus $((a_1, \cdots, a_n) + (b_1, \cdots, b_n))\varphi = (a_1, \cdots, a_n)\varphi + (b_1, \cdots, b_n)\varphi$.

If $(a_1b_1, \cdots, a_nb_n)\varphi = (p_1 + I_1, \cdots, p_n + I_n)$ and $(q_1 + I_1, \cdots, q_n + I_n)$ $(q_1' + I_1, \cdots, q_n' + I_n) = (p_1' + I_1, \cdots, p_n' + I_n)$ where $p_i, p_i' \in Q_i$ for all i, then $a_ib_i + I_i \subset p_i + I_i$, $q_iq_i' + I_i \subset p_i' + I_i$ and $a_ib_i + I_i = (q_i + j_i)(q_i' + j_i') + I_i = q_iq_i' + q_ij_i' + q_i'j_i + j_ij_i' + I_i \subset q_iq_i' + I_i \subset p_i' + I_i$ for all i. By Theorem (2-6), $p_i = p_i'$ for all i. Thus $((a_1, \cdots, a_n)(b_1, \cdots, b_n))\varphi = (a_1, \cdots, a_n)\varphi(b_1, \cdots, b_n)\varphi$. We claim that $\ker\varphi = I (= \sum_{i=1}^{n} \oplus I_i)$.

If $(x_1, \cdots, x_n) \in \sum_{i=1}^{n} \oplus I_i$, then $x_i \in I_i$ for all i. Thus $x_i + I_i \subset I_i$ for all i. So, $(x_1, \cdots, x_n)\varphi = (I_1, \cdots, I_n)$ is the zero element in $\sum_{i=1}^{n} \oplus (R_i/I_i)$. i.e. $(x_1, \cdots, x_n) \in \ker\varphi$.

If $(y_1, \cdots, y_n) \in \ker\varphi$, then $y_i \in y_i + I_i \subset I_i$ for all i. Thus $(y_1, \cdots, y_n) \in \sum_{i=1}^{n} \oplus I_i$.

Lastly, we claim that φ is maximal. For any $z = (q_1 + I_1, \cdots, q_n + I_n) \in \sum_{i=1}^{n} \oplus (R_i/I_i)$, let $c_z = (q_1, \cdots, q_n)$. Then $c_z \in \varphi^{-1}(\{z\})$. For any $t = (t_1, \cdots, t_n) \in \varphi^{-1}(\{z\})$, $(t_1, \cdots, t_n)\varphi = (q_1 + I_1, \cdots, q_n + I_n)$. Then $t_i + I_i \subset q_i + I_i$ for all i. So, for all i, $t_i = q_i + m_i$ for some $m_i \in I_i$. Thus $t + \ker\varphi = (q_1 + m_1, \cdots, q_n + m_n) + \ker\varphi = c_z + (m_1, \cdots, m_n) + \ker\varphi \subset c_z + \ker\varphi$ for every $t \in \varphi^{-1}(\{z\})$. Hence φ is a maximal homomorphism from R onto $\sum_{i=1}^{n} \oplus (R_i/I_i)$ By Theorem (2-10), $R/I \simeq \sum_{i=1}^{n} \oplus (R_i/I_i)$.

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〈國 文 抄 錄〉

반환의 직합(直合)에 관해서

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이 논문의 중요한 목적은 반환 R이 반환 R_i $(i = 1, 2, \cdots, n)$ 들의 직합(直合)이며, I_i는 반환 R_i의 Q_i - 이데알이고, $I = \sum_{i=1}^{n} \oplus I_i$ 일때 R/I 와 $\sum_{i=1}^{n} \oplus (R_i/I_i)$은 서로 동형임을 증명하였다.