ON THE K-SPACES

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May, 1983

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이를 教育學碩士學位 論文으로 提出함

濟州大學校教育大學院數學教育專攻

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1983 年 5 月 日

張君守의 碩士學位 論文을 認准함

濟州大學校 敎育大學院

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1983年 5月 日

감 사 의 글

이 논문이 완성되기 까지 연구에 바쁘신 가운데도 친절하고 자상하게 지도하여 주신 한철순 교수님께 감사를 드리며, 그동안 많은 도움을 주신 수학교육과의 여러 교수님께 심심한 사의를 표합니다.

아울러 그동안 저에게 좋은 지도 조언의 말씀과 격려를 하여주신 주위의 많은 분들께 또한 감사를 드립니다.

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이 논문은 K-공간의 조건(條件)을 구하고 두 공간의 Cartesian Product 공간이 K-공간이 되는 조건(條件)을 구명(究明)함과 아울러 K-공간은 locally compact 공간과 거리공간(距離空間)을 포함하는 보다 확장된 공간임을 증명(證明)하였다.

1. INTRODUCTION

In this paper we will study the class of k-spaces which is larger than that of locally compact spaces and metric spaces, and prove also the condition that the cartesian product of two k-spaces is a k-space.

2. MAIN THEOREMS

PROPOSITION 2-1

Let X be locally compact.

An $A \subset X$ is open if and only if its intersection with each compact $C \subset X$ is open in C.

Proof

Let C be compact in a space X and $A \subset X$.

Assume $A \cap C$ open in C.

We claim that A is open in X.

Let $a \in A$. Then $a \in X$.

Since X is locally compact, then there exists a relatively compact nbd $V(a)$.

Then $\overline{V(a)}$ is compact and hence $A \cap \overline{V(a)}$ is open in $\overline{V(a)}$.

So, $A \cap V(a)$ is open in $V(a)$.

Thus, $A \cap V(a)$ is open in X.

Therefore A is open in X.

For the converse, let A be open in X, then $A \cap C$ is open in each compact $C \subset X$.

DEFINITION 2-2

Let X be a set, and let $\mathcal{U} = \{A_\alpha \mid \alpha \in \mathbf{A}\}$ be a family of subsets of X, with each A_α having a topology.

Assume that for each $(\alpha, \beta) \in \mathbf{A} \times \mathbf{A}$, both

(1) The topologies of A_α and A_β agree on $A_\alpha \cap A_\beta$.

(2) Either (a) each $A_\alpha \cap A_\beta$ is open in A_α and in A_β or (b) each $A_\alpha \cap A_\beta$ is closed in A_α and in A_β.

The weak topology in X determined (or induced) by $\mathcal{U}$

$\mathcal{T}(\mathcal{U}) = \{ U \subset X \mid \forall \alpha : U \cap A_\alpha$ is open in $A_\alpha \}$

PROPOSITION 2-3

If X is a space with weak topology determined by $\{A_\alpha \mid \alpha \in \Lambda\}$, then an $f : X \to Y$ is continuous if and only if each $f \mid A_\alpha : A_\alpha \to Y$ is continuous.

Proof

If $f : X \to Y$ is continuous, then the restriction f to A_α is evidently continuous.

Let $U \subset Y$ be open, then $f^{-1}(U) \cap A_\alpha = f^{-1}(U \cap A_\alpha) = f_\alpha^{-1}(U)$ is open in A_α for each $\alpha \in \Lambda$.

Since X has a weak topology induced by $\{A_\alpha \mid \alpha \in \Lambda\}$, then $f^{-1}(U)$ is open in X.

Therefore f is continuous.

DEFINITION 2-4

Let $\{Y_\alpha \mid \alpha \in \Lambda\}$ be any family of spaces.

For each $\alpha \in \Lambda$, let Y'_α be the space $\{\alpha\} \times Y_\alpha$, so that $Y'_\alpha \cong Y_\alpha$ and the family $\{ Y'_\alpha \mid \alpha \in \Lambda \}$ is pairwise disjoint.

The free union of the given family $\{Y_\alpha \mid \alpha \in \Lambda\}$ is the set $\bigcup Y'_\alpha$ with the weak topology determined by the spaces Y'_α.

This space is denoted by $\sum_\alpha Y'_\alpha$.

PROPOSITION 2-5

Let $(X, \mathcal{T})$ be a space with weak topology determined by the covering $\{A_\alpha : \alpha \in \Lambda\}$.

Let $A = \sum_\alpha A'_\alpha$ be the free union of $\{A_\alpha : \alpha \in \Lambda\}$, and for each α, let $h_\alpha : A'_\alpha \to A_\alpha \subset X$ be the homeomorphism $(\alpha, a) \to a$.

Define $h : \sum_\alpha A'_\alpha \to X$ by $h|A'_\alpha = h_\alpha$ for each $\alpha \in \Lambda$.

Then h is continuous and $A/K(h) \cong X$, where K(h) is a relation defined by $x \sim x'$ if $h(x) = h(x')$.

Note that K(h) is an equivalence relation in A.

Proof

If follows from proposition 2-3 that h is continuous.

Obviously, h is surjective.

To show the proposition 2-5, we need to show only that h is an identification.

To do this, let $U \subset X$ be such that $h^{-1}(U)$ is open in A.

Then $h^{-1}(U) \cap A_\alpha = h_\alpha^{-1}(U \cap A_\alpha)$ is open in A'_α for each $\alpha \in \Lambda$, and, since h_α is a homeomorphism, $U \cap A_\alpha$ is an open in A_α.

Thus U is open in X.

Therefore h is an identification.

This identification h turns out to be $A/K(h) \cong X$.

It follows from definition 2-2 that a locally compact space has the weak topology determined by the family of its compact subsets.

So we have the following Definition:

DEFINITION 2-6

A Hausdorff space X is called a k-space if and only if it has the weak topology determined by the family of its compact subspaces.

It follows from definition 2-2 and 2-6 that every locally compact space is a k-space.

PROPOSITION 2-7

Every 1st countable Hausdorff space is a k-space.

Proof

Let X be a 1st countable Hausdorff space and $A \subset X$ such that $A \cap C$ is closed in C for each compact C, then $A \cap C$ is closed in X. We claim that A is closed in X. Let $x \in \bar{A}$, then there is a sequence $\{a_n \mid n \in Z^+\} \subset A$ with $a_n \to x$, where Z^+ is the set of all natural numbers. Thus $\{a_n \mid n \in Z^+\} \cup \{x\}$ is compact and so also is the closed $A \cap (\{a_n \mid n \in Z^+\} \cup \{x\})$.

Thus this intersection being infinite subset of $\{a_n \mid n \in Z^+\} \cup \{x\}$, must contain x. so $x \in A$.

Therefore A is closed.

It follows from definition 2-2 that X is a k-space.

DEFINITION 2-8

Let X be a space, R an equivalence relation in X, X/R the quotient set, and p the cannonical projection of X onto X/R given by $p(x)=[x]$, where $[x]$ is an equivalence class of x. Then the set X/R with the identification topology determined by the projection $P:X\to X/R$ is called the quotient space of X by R.

THEOREM 2-9

Let X be Hausdorff.

Then X is a k-space if and only if it is a quotient space of a locally compact space.

Proof

Assume X to be a k-space.

It follows from proposition 2-5 that X is a quotient space of the free union of its compact subspaces, and since the free union of compact subspaces is clearly locally compact, a quotient space of a locally compact space.

For the converse, let $p:Y\to X$ be the identification map, where Y is locally compact, and let $U\subseteq X$ such that $U\cap C$ is open in C for each compact C.

We claim U is open in X. For each relatively compact open $V\subset Y$, we have $U\cap p(\overline{V})$ open in $p(\overline{V})$, that is, $U\cap p(\overline{V}) = p(\overline{V})\cap G$ for some open $G\subset X$.

Since $p^{-1}(U)\cap p^{-1}p(\overline{V})=p^{-1}p(\overline{V})\cap p^{-1}(G)$, we find by interecting with V, that $p^{-1}(U)\cap V= V\cap p^{-1}(G)$, therefore $p^{-1}(U)\cap V$ is open in Y. Since there is an open covering $Y=\bigcup_{\alpha} V_{\alpha}$ by relatively compact open sets, $p^{-1}(U)=\bigcup_{\alpha} p^{-1}(U)\cap V_{\alpha}$ shows $p^{-1}(U)$ open in Y, so U is open in X. Therefore, X is a k-space.

THEOREM 2-10

If X is a k-space and $p : X \to Z$ is an identification, then Z is also a k-space.

Proof

Let Y be locally compact and $g : Y \to X$ an identification. Then $p \circ g$ is an identification and so by theorem 2-9 and definition 2-8 Z is a k-space.

THOEREM 2-11

The cartesian product of two k-spaces is a k-space if either (1) both factors are 1st countable or (2) one factor is locally compact.

Proof

Consider (1)

We have known that the cartesian product of two 1st countable spaces is 1st countable. It follows from proposition 2-7 that the cartersian product of two 1st countable spaces is a k-space.

Next we consider (2)

Let X be a k-space and Y a locally compact space. We claim $X \times Y$ is a k-space. We first observe that if P is any k-space and R is any locally compact space, then $f : P \times R \to Z$ is continuous if and only if $f|C \times R$ is continuous for each compact $C \subset P$.

In fact, since R is locally compact, the continuity of f is equivalent to that of $\hat{f}: P \to Z^R$, since P has the weak topology determined by compact subsets, proposition 2-3 shows that $\hat{f}$ is continuous if and only if $\hat{f}|C$ is continuous for each compact C, where Z^R is the set of all continuous maps of R into Z. By definition 2-2, every open set in the cartesian product topology $\mathcal{J}(c)$ of $X \times Y$ is open in k-topology $\mathcal{J}(k)$ of $X \times Y$, so we need prove only that $1 : (X \times Y, \mathcal{J}(c)) \to (X \times Y, \mathcal{J}(k))$ is continuous.

For compact $C \subseteq X$, $C' \subseteq Y$, the compactness of $C \times C'$ and proposition 2-1 assumes that $1 | C \times C'$ is continuous.

Keeping any compact C fixed and recalling proposition 2-7 that Y is a k-space, our observation above shows that $1 | C \times Y$ is continuous.

Since X is a k-space, 1 is continuous.

So $X \times Y$ is a k-space.

3. PRODUCTS OF K-SPACES

In this section, we have that if a product of nonempty pace is a k-space then for each infinite cardinal n, some product of all but n of the factors has each n-fold subproduct n-$\aleph_0$-compact.

DEFINITION 3-1

A subset F of a topological space X is k-closed if $F \cap K$ is closed in K for each compact subset K of X. A space in which each k-closed subset is closed is called a k-space.

DEFINITION 3-2

A space is n-$\aleph_0$-compact of each n-fold open cover contains a finite subcover.

DEFINITION 3-3

A space is n-determined if a subset is closed whenever it meets each subset S having n or fewer elements in a set which is closed in S.

A space is n-bounded if each subset with n or fewer elements is contained in a compact set.

LEMMA 3-4

If a product of nonempty spaces is a k-space then, for each infinite cardinal n, some product of all but n of its factors has each n-fold subproduct n-$\aleph_0$-compact.

Proof

To see Reference (5) p. 160.

LEMMA 3-5

For n on infinite cardinal, an m-fold Product of n-determined spaces is n-determined if and only if all but at most n of the factors are indiscrete.

Proof

To see Reference (5) p. 611.

DEFINITION 3-6

A space is called strong n-bounded if each subset with fewer than n— elements is contained in a compact set. And a space is called strong n-determined if a subset is closed whenever it meets each subest S having fewer than n elements in a set which is closed in S.

PROPOSITION 3-7

Let $X = \prod_{\alpha \in n} X_\alpha$. If each X_α is strong n-bounded and strong n-determined, then X is a k-space

Proof

Let $A \subseteq X$ be k-closed and let x be any point in the closure of A. We will produce a subset A′ of such that x is in the closure of A′ and such that, for each α an n, $\Pi_\alpha A'$ has cardinality less than n. Since each X_α is strong n-bounded, each $\Pi_\alpha A'$, and hence A′ itself, is contained in a compact set. It follows that x must be in A and hence that X is k-space, as desired. Let Π^α denote the projection from X to $X^\alpha = \prod_{\beta<\alpha} X_\beta$ and note that, since n is regular, we have that X^α is strong n-determined. We first show that, for each α, $\Pi^\alpha(x)$ is in $\Pi^\alpha(A)$. Certainly $\Pi^\alpha(x)$ is in the closure of $\Pi^\alpha(A)$ and hence, since X^α is strong n-determined, $\Pi^\alpha(x)$ is in the closure of $\Pi^\alpha(B)$ for some subset B of A having fewer than n-elements. Since X is strong n-bounded, B is contained in some compact set K. Let K_1 be the projection of K onto $\prod_{\beta\geq\alpha} X_\beta$, and let $K_2 = \Pi^\alpha(K) \cup \{\Pi^\alpha(x)\}$. Since A is k-closed, $A \cap K_1 \times K_2$ is closed in $K_1 \times K_2$ and therefore its projection onto K_2, which is just $\Pi^\alpha(A) \cap K_2$, is closed in K_2. Since $\Pi^\alpha(B) \subseteq \Pi^\alpha(A) \cap K_2$ and $\Pi^\alpha(x)$ is in the intersection of the closure of $\Pi^\alpha(B)$ with K_2, it follows that $\Pi^\alpha(x)$ is in $\Pi^\alpha(A)$, as desired. To construct the set A′, choose, for each α, a point x^α in A such that $\Pi^\alpha(x^\alpha) = \Pi^\alpha(x)$ and let $A' = \{x^\alpha : \alpha \in n\}$. It is clear that A′ has the desired properties, so the proof is complete.

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