On the monotonically normal spaces

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감 사 의 글

이 논문이 완성되기 까지 연구에 바쁘신 가운데도 자상한 마음으로 친절하게 지도를 하여 주신 한철순 교수님께 감사드리며, 아울러 그동안 많은 도움을 주신 수학교육과의 여러 교수님께 심심한 사의를 표합니다.

그리고 그동안 저에게 사랑과 격려를 하여주신 주위의 많은 분들께 또한 감사를 드립니다.

1983년 5월 일

이 동 근

CONTENTS

국 문 초 록

단조정규공간(單調正規空間)에 관해서

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이 논문은 단조정규공간(單調正規空間)이 될 완전조건(完全條件)을 구명(究明)하고 단조정규작용소(單調正規作用素) D를 이용하여 Stratifiable 공간이 Paracompact 임을 증명하였다.

1. INTRODUCTION

The notion of paracompactness is a relatively recent one as topological ideas go, and was first introduced by Dieudonné[1944]. In the hierarchy of topological spaces, Paracompactness lies between normality and metrizability.

In this paper we see that stratifiability lies between paracompactness and metrizability in the hierarchy of topological spaces. And also, we can find the exact condition of a monotonically normal space introduced in this paper.

This paper is organized into three sections.

§ 2. is a preliminary section containing some useful properties for a stratifiable space and a monotonically normal space.

§ 3. contains the main theorem. In this section we see the exact condition of a monotonically normal space and also that every stratifiable space is paracompact.

2. PRELIMINARY

In this section we collect some basic definition and some useful properties for a stratifiable space and a monotonically normal space.

DEFINITION 1. A T_1-space X is stratifiable iff to each closed set $A \subset X$, one can assign a sequence $G_1(A)$, $G_2(A)$, $\cdots$ of open sets such that

(1) $A = \bigcap_{n=1}^{\infty} G_n(A) = \bigcap_{n=1}^{\infty} \overline{G_n(A)}$, and

(2) if $A \subset B$ then $G_n(A) \subset G_n(B)$ for every $n \in Z^+$.

In the above definition, a T_1-space X which holds the condition $A = \bigcap_{n=1}^{\infty} G_n(A)$ instead of (1), is called a semi-stratifiable space. Thus every stratifiable space is semi-stratifiable.

Note that every metric space obviousely is stratifiable.

DEFINITION 2. A space X is called a monotonically normal space if to each ordered pair(H, K) of disjoint closed subsets H and K of X, one can assign an open set D(H, K) such that

(1) $H \subset D(H, K) \subset \overline{D(H, K)} \subset X - K$, and

(2) if $H \subset H'$ and $K \supset K'$ then $D(H, K) \subset D(H', K')$.

Moreover D is called a monotone normality operator.

Using the above difinitions, we have the following result;

PROPOSITION 1. Stratifiable spaces are monotonically normal.

Proof). Let X be stratifiable. Then for each closed $A \subset X$, we can find a decreasing sequence $\{G_n(A)\}$ of open sets such that

(1) $A = \bigcap_{n=1}^{\infty} G_n(A) = \bigcap_{n=1}^{\infty} \overline{G_n(A)}$, and

(2) if $A \subset B$ then $G_n(A) \subset G_n(B)$ for every $n \in Z^+$.

For any ordered pair (H, K) of disjoint closed sets, let $D(H, K) = \bigcup_{n=1}^{\infty} [X - \overline{G_n(K)} - \overline{(X - \overline{G_n(H)})}]$. Then D(H,K) is clearly open.

To show that $H \subset D(H,K)$, let $p \in H$. Then $p \notin K = \bigcap_{n=1}^{\infty} \overline{G_n(K)}$ since $H \cap K = \phi$. Thus there exists an $n \in Z^+$ such that $p \notin \overline{G_n(K)}$, and so $p \in X - \overline{G_n(K)}$. Since $p \in G_n(H)$ for every $n \in Z^+$, we have $p \notin X - G_n(H)$ for every $n \in Z^+$. Since $\overline{X - \overline{G_n(H)}} \subset \overline{X - G_n(H)} = X - G_n(H)$. We have $p \notin \overline{X - \overline{G_n(H)}}$ for every $n \in Z^+$. Thus $p \in D(H,K)$; so $H \subset D(H,K)$.

Next, to show that $\overline{D(H,K)} \subset X - K$, let $p \in \overline{D(H,K)}$. Suppose $p \in K$. Then $p \in D(K,H) = \bigcup_{m=1}^{\infty} [X - \overline{G_m(H)} - \overline{(X - \overline{G_m(K)})}]$. Since $p \in \overline{D(H,K)}$ and $p \in D(K,H)$, there is a point $q \in D(H,K) \cap D(K,H)$ by the definition of closure. Thus for some $m, n \in Z^+$,

$$q \in X - \overline{G_m(K)} - \overline{(X - \overline{G_m(H)})} \text{ and}$$

$$q \in X - \overline{G_n(H)} - \overline{(X - \overline{G_n(K)})}.$$

If $n<m$, then $q\in X-\overline{G_n(K)}\subset X-\overline{G_m(K)}$.

If $m<n$, then $q\in X-\overline{G_m(H)}\subset X-\overline{G_n(H)}$. In either case we have a contradiction. Thus $p\notin K$, that is, $p\in X-K$, and so $\overline{D(H,K)}\subset X-K$.

Finally, we must show that $D(H,K)\subset D(H',K')$ if $H\subset H'$ and $K\supset K'$. Let $p\in D(H,K)$. Then there exists an $n\in Z^+$ such that

$$p\in X-\overline{G_n(K)}-\overline{(X-\overline{G_n(H)})}$$

or

$$p\in X-\overline{G_n(K)} \text{ but } p\notin\overline{X-\overline{G_n(H)}}.$$

Since X is stratifiable, $K\supset K'$ and $H\subset H'$, then

$$\overline{G_n(K)}\supset\overline{G_n(K')} \text{ and } \overline{G_n(H)}\subset\overline{G_n(H')}.$$

Thus $X-\overline{G_n(K)}\subset X-\overline{G_n(K')}$ and $X-\overline{G_n(H)}\supset X-\overline{G_n(H')}$.

Hence $p\in X-\overline{G_n(K')}$, but $p\notin\overline{X-\overline{G_n(H')}}$ and so

$$p\in X-\overline{G_n(K')}-\overline{(X-\overline{G_n(H')})}=D(H',K').$$

Accordingly, $D(H,K)\subset D(H',K')$, and the proof is complete.

Recall that a family of subsets $\{H_\alpha \mid \alpha\in\Lambda\}$ of a space X is discrete iff $\{\overline{H}_\alpha \mid \alpha\in\Lambda\}$ is nbd-finite and $\overline{H}_\alpha$'s are mutually disjoint.

DEFINITION 3. A topological space X is collectionwise normal iff for each discrete family of subsets $\{H_\alpha \mid \alpha\in\Lambda\}$, there are mutually disjoint open subsets $\{G_\alpha \mid \alpha\in\Lambda\}$ such that $H_\alpha\subset G_\alpha$ for each $\alpha\in\Lambda$.

In above definition, we can make use of a discrete family of closed subsets $\{ C_\alpha \mid \alpha \in \Lambda \}$ instead of a discrete family of subsets $\{ H_\alpha \mid \alpha \in \Lambda \}$. For, $C_\alpha = \overline{H}_\alpha$ is closed and, $H_\alpha \subset \overline{H}_\alpha = C_\alpha$ for every $\alpha \in \Lambda$.

PROPOSITION 2. Monotonically normal spaces are collectionwise normal.

Proof). Let X be a monotonically normal space with the monotone normality operator D, and $C = \{ C_\alpha \mid \alpha \in \Lambda \}$ a discrete family of closed subsets of X where Λ is well-ordered, say $\Lambda = \{ \alpha_0, \alpha_1, \alpha_2 \cdots \}$ Then $\cup \{ C_\alpha \mid \alpha \in \Lambda \}$ is closed, since $\{ C_\alpha \mid \alpha \in \Lambda \}$ is a nbd-finite family of closed subsets of X.

Let $G_\alpha = D(\bigcup_{\beta \leq \alpha} C_\beta, \bigcup_{\beta > \alpha} C_\beta)$ for every $\alpha \in \Lambda$. Then

$C_\alpha \subset \bigcup_{\beta \leq \alpha} C_\beta \subset D(\bigcup_{\beta \leq \alpha} C_\beta, \bigcup_{\beta > \alpha} C_\beta) = G_\alpha$ for $\alpha \in \Lambda$. Thus $C_\alpha \subset G_\alpha$ and G_α is open for each $\alpha \in \Lambda$.

Since, for $\gamma < \alpha$, $\bigcup_{\beta \leq \gamma} C_\beta \subset \bigcup_{\beta < \alpha} C_\beta \subset \bigcup_{\beta \leq \alpha} C_\beta$ and

$\bigcup_{\beta > \gamma} C_\beta \supset \bigcup_{\beta \geq \alpha} C_\beta \supset \bigcup_{\beta > \alpha} C_\beta$, we have

$$\begin{aligned} G_\gamma &= D(\bigcup_{\beta \leq \gamma} C_\beta, \bigcup_{\beta > \gamma} C_\beta) \\ &\subset D(\bigcup_{\beta < \alpha} C_\beta, \bigcup_{\beta \geq \gamma} C_\beta) \\ &\subset D(\bigcup_{\beta \leq \alpha} C_\beta, \bigcup_{\beta > \alpha} C_\beta) \\ &= G_\alpha . \end{aligned}$$

Thus $G_{\alpha_0} \subset G_{\alpha_1} \subset G_{\alpha_2} \subset \cdots$.

Define $O_{\alpha_0} = G_{\alpha_0}$;

$O_{\alpha_i} = G_{\alpha_i} - \overline{G}_{\alpha_{i-1}}$ for $i \geq 1$.

Then $\{ O_\alpha \mid \alpha \in \Lambda \}$ is a family of mutually disjoint open set.

Obviously, $C_{\alpha_0} \subset O_{\alpha_0}$. Moreover,

$$
\begin{aligned}
O_{\alpha_i} &= G_{\alpha_i} - \overline{G}_{\alpha_{i-1}} \\
&= D\Big(\bigcup_{\beta \le \alpha_i} C_\beta ,\ \bigcup_{\beta > \alpha_i} C_\beta \Big) - \overline{D\Big(\bigcup_{\beta \le \alpha_{i-1}} C_\beta ,\ \bigcup_{\beta > \alpha_{i-1}} C_\beta \Big)} \\
&\supset \bigcup_{\beta \le \alpha_i} C_\beta - \Big(X - \bigcup_{\beta > \alpha_{i-1}} C_\beta \Big) \\
&= \Big(\bigcup_{\beta \le \alpha_i} C_\beta \Big) \cap \Big(\bigcup_{\beta > \alpha_{i-1}} C_\beta \Big) \\
&= (C_\beta \cup C_{\alpha_i}) \cap (C_{\alpha_i} \cup C_\beta') \\
&= C_{\alpha_i} \cup (C_\beta \cup C_\beta') \\
&\supset C_{\alpha_i} .
\end{aligned}
$$

for $i \geq 1$ since C_α's are disjoint.

Therefore, $C_\alpha \subset O_\alpha$ for each $\alpha \in \Lambda$, and the proof is complete.

Note that a family F is σ-discrete iff $F = \bigcup_{n=1}^{\infty} F_n$ where F_n is a discrete family.

DEFINITION 4. A space X is subparacompact iff every open cover of X has a σ-discrete closed refinement.

Note that a family N is discrete iff each point of the space has a nbd which intersects at most one member of N [proof : (6) p.6].

PROPOSITION 3. Every semi - stratifiable space is subpara-compact.

Proof). Let X be a semi - stratifiable space. Then $\{ x \}$ is closed for every $x \in X$ since X is a T_1-space. Thus for each $x \in X$ there is sequence $\{g_n(x)\}$ of open nbds of x such that

$$\bigcap_{n=1}^{\infty} g_n(x) = \{x\}.$$

Let $\mathcal{O} = \{ O_\alpha \mid \alpha \in \Lambda \}$ be an open cover of X where Λ is well-ordered, say $\Lambda = \{ \alpha_0, \alpha_1, \alpha_2 \cdots \}$. Let

$$H_{\alpha_0, n} = X - \bigcup_{x \notin O_\alpha} g_n(x);$$

$$H_{\alpha, n} = X - (\bigcup_{x \notin O_\alpha} g_n(x)) \cup (\bigcup_{\beta < \alpha} O_\beta) \text{ for each } \alpha > \alpha_0$$

and $n \in Z^+$. Then $H_{\alpha,n}$ is closed and $H_{\alpha,n} \subset O_\alpha$.

To show that $N_n = \{ H_{\alpha,n} \mid \alpha \in \Lambda \}$ is discrete, let $y \in X$. Then there is the least element $\alpha \in \Lambda$ such that $y \in O_\alpha$.

Consider an open nbd $g_n(y) \cap O_\alpha$ of y. suppose $z \in g_n(y) \cap O_\alpha \cap H_{\beta,n}$. If $\beta < \alpha$ then $y \notin O_\beta$ since $\alpha \in \Lambda$ is the least element such that $y \in O_\alpha$. Hence $z \in g_n(y) \subset \bigcup_{y \notin O_\beta} g_n(y)$ and so $z \notin H_{\beta,n}$.

If $\alpha < \beta$, then $z \in O_\alpha \subset \bigcup_{\alpha < \beta} O_\alpha$ and so $z \notin H_{\beta,n}$. Both are contrary to $z \in H_{\beta,n}$. Thus $\alpha = \beta$. In other words, each point of X has an open nbd which intersects at most one element of N_n. Accordingly, N_n is discrete.

Finally, we must show that $N = \bigcup_{n=1}^{\infty} N_n$ is a cover of X.

Let $y \in X$ and $\alpha \in \Lambda$ the least element such that $y \in O_\alpha$. Then $y \notin \bigcup_{\alpha < \beta} O_\beta$.

Now, there is an $n \in Z^+$ such that $y \notin \bigcup_{x \notin O_\alpha} g_n(x)$. If it is not so, for every $n \in Z^+$ there is an $x_n \notin X - O_\alpha$ such that

$y \in g_n(x_n)$. Since X is semi-stratifiable, we have

$$y \in g_1(x_1) = G_1(\{x_1\}) \subset G_1(X - O_\alpha),$$

$$y \in g_2(x_2) = G_2(\{x_2\}) \subset G_2(X - O_\alpha),$$

$$\vdots$$

$$y \in g_n(x_n) = G_n(\{x_n\}) \subset G_n(X - O_\alpha),$$

$$\vdots$$

Thus, $y \in \bigcap_{n=1}^{\infty} g_n(x_n) \subset \bigcap_{n=1}^{\infty} G_n(X - O_\alpha) = X - O_\alpha$, a contradiction.

Accordingly, $y \in X - (\bigcup_{x \notin O_\alpha} g_n(x)) \cup (\bigcup_{\beta < \alpha} O_\beta) = H_{\alpha,n}$, and so N is a cover of X.

Therefore $N = \bigcup_{n=1}^{\infty} N_n$ is a σ-discrete closed refinement of O; and hence X is subparacompact.

3. MAIN THEOREM

Recall that a Hausdorff space X is paracompact iff every open cover of X has a nbd- finite open refinement.

LEMMA 4. A Hausdorff space X is paracompact if it is subparacompact and collectionwise normal.

Proof) If X is subparacompact and collectionwise normal then every open cover X has a σ-discrete open refinement [proof;(6) p. 8].

The space X is paracompact iff each open cover of X has an open σ-discrete refinement [proof; (5) pp. 156 ~ 160].

Using the above lemma, we have the following main theorem an our paper.

PROPOSITION 5. Every stratifiable space is paracompact.

Proof) By proposition 1 and 2, every stratifiable space is collectionwise normal. Since every stratifiable space is semi-stratifiable, it is subparacompact by proposition 3.

Therefore, every stratifiable space is paracompact by lemma 4.

Now let us find the exact condition of a monotonically normal space.

PROPOSITION 6. A T_1-space X is montonically normal iff for each

$x \in X$ and open set U containing x, one can assign an open set U_x containing x such that if $U_x \cap V_y \neq \phi$ then either $x \in V$ or $y \in U$.

Proof). Let X be a monotonically normal space. For each $x \in X$ and open set U containing x, define

$$U_x = D(\{x\}, X - U) - \overline{D(X - U, \{x\})}.$$

Then for an open set V containing y,

$$V_y = D(\{y\}, X - V) - \overline{D(X - V, \{y\})}.$$

If $x \notin V$ and $y \notin U$ then $\{x\} \subset X - V$ and $\{y\} \subset X - U$.
Thus $D(\{x\}, X - U) \subset D(X - V, \{y\}) \subset \overline{D(X - V, \{y\})}$, and so $U_x \cap V_y = \phi$.

Therefore, $U_x \cap V_y \neq \phi$ implies either $x \in V$ or $y \in U$.

Conversely, let X be a T_1 – space and for each $x \in X$ and open set U containing x, one can assign an open set U_x containing x such that $U_x \cap V_y \neq \phi$ implies either $x \in V$ or $y \in U$.

For each $x \in X$ and open set U containing x, define
$V_{U_x} = \bigcup_{x \in W \subset U} W_x$ where W is open.

Suppose $p \in V_{U_x} \cap V_{R_y}$. Let $W \subset U$ such that $p \in W_x$, and $S \subset R$ such that $p \in S_y$. Then $W_x \cap S_y \neq \phi$ implies either $x \in S$ or $y \in W$.

Let (H,K) be any pair of disjoint closed sets, and
$D(H,K) = \bigcup_{x \in H} V_{(X-K)_x}$. Then $H \subset D(H,K)$.

Suppose $p \in \overline{D(H,K)} \cap K$. Let U be an open set containing p and contain no point of H. Then $V_{U_p} \cap D(H,K) \neq \phi$ and so there is a a point $q \in V_{U_p} \cap D(H,K)$. Since there is a point $x \in H$ such that $q \in V_{U_p} \cap V_{(X-K)_x}$, we have either $p \in X - K$ or $x \in U$, a contradic–

tion. Hence $\overline{D(H,K)} \subset X - K$.

Finally, let $H \subset H'$ and $K \supset K'$, and $p \in D(H,K)$. Then there is a point $x \in H$ such that $p \in V(X-K)_x$, and so there is an open set W such that $x \in W \subset X-K$ and $p \in W_x$. Since $X-K \subset X-K'$, we have $W_x \subset V(X-K')_x$. Thus $p \in W_x \subset V(X-K')_x \subset \bigcup_{x \in H'} V(X-K')_x = D(H',K')$, and so $D(H,K) \subset D(H',K')$.

Therefore, X is monotonically normal.

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ENGLISH ABSTRACT

On the monotonically normal spaces

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We have proved the exact condition of a monotonically normal space and also that every stratifiable space is paracompact by using the monotone normality operator D.