碩士學位論文

On the Properties of Regular Convergence Spaces

濟州大學校 大學院

數 學 科

咸 亮 奎

1998年 12月

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이 論文을 理學 碩士學位 論文으로 提出함

1998年 12月

咸亮奎의 理學 碩士學位 論文을 認准함

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濟州大學校 大學院

1998年 12月

On the Properties of Regular Convergence Spaces

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(Supervised by professor Jin-Won Park)

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL
CHEJU NATIONAL UNIVERSITY

1998. 12.

CONTENTS

< Abstract >

On the Properties of Regular Convergence Spaces

In this thesis, we study the regular convergence spaces and the fibrewise regular convergence spaces and some properties of those spaces. First, we introduce the notion of regular convergence spaces and investigate some properties of regular convergence spaces, including the function space $C(X, Y)$. And we define the fibrewise regular convergence spaces which can be regarded as a generalization of the regular convergence spaces. And we generalize the properties of regular convergence spaces as a fibrewise version.

1. Introduction.

The fibrewise viewpoint is standard in the theory of fibre bundles. However, it has been recognized only recently that the same viewpoint is also great value in other theories, such as general topology. I. M. James has been promoting the fibrewise viewpoint systematically in topology [3 , 4]. Many of the familiar definitions and theorems of ordinary topology can be generalized, in a natural way, so that one can develop a theory of topology over a base. On the other hand, as a convenient category, the category of convergence spaces was introduced which contains the category of topological spaces as a bireflective subcategory. So many familiar definitions of topological spaces were introduced in the convergence spaces. In this point of view, K. C. Min, S. J. Lee and J. W. Park developed a general fibrewise theory in the category of convergence spaces, including the fibrewise notion of Hausdorffness [6 , 7].

In this thesis, we study the regular convergence spaces and the fibrewise regular convergence spaces and some properties of those spaces. In section 2, we recall some basic definitions and some known results, which we shall need in later sections. In section 3, we introduce the notion of regular convergence spaces and investigate some properties of regular convergence spaces. And we find a condition for which the function space $C(X, Y)$ is regular. In section 4, we define the fibrewise regular convergence spaces which can be regarded as a generalization of the regular convergence spaces. And we generalize the results obtained in section 3 as a fibrewise version.

2. Preliminaries.

In this section, we collect some definitions and well known results, about the convergence spaces over a base, which we shall need in later.

For any set X, we denote by $\mathcal{F}(X)$ the set of all filters on X, and by $\mathcal{P}(\mathcal{F}(X))$ the power set of $\mathcal{F}(X)$.

Definition 2.1. [1] Let X be a set. A map $c : X \to \mathcal{P}(\mathcal{F}(X))$ is said to be a *convergence structure* if the following properties hold for any point $x \in X$:

(1) $\dot{x} \in c(x)$.

(2) $\mathcal{F} \in c(x)$ and $\mathcal{F} \subset \mathcal{G}$, then $\mathcal{G} \in c(x)$.

(3) $\mathcal{F}, \mathcal{G} \in c(x)$, then $\mathcal{F} \cap \mathcal{G} \in c(x)$.

Here $\dot{x}$ stands for the ultrafilter on X generated by $\{x\}$. The pair (X, c) is a *convergence space*. The filters in $c(x)$ are said to *converge to* x.

If $f : X \to Y$ is a map and $\mathcal{F} \in \mathcal{F}(X)$ then $f(\mathcal{F})$ is a filter base. In general, $f(\mathcal{F})$ is not a filter but the filter generated by $f(\mathcal{F})$ is also denoted by $f(\mathcal{F})$.

Definition 2.2. [1] Let (X, c) and (Y, c') be convergence spaces and $f : X \to Y$ a map. Then f is said to be *continuous at* $x \in X$ if for any $\mathcal{F} \in c(x)$, $f(\mathcal{F}) \in c'(f(x))$. And f is said to be *continuous* if f is continuous at each point $x \in X$.

Now we introduce the notions of initial and final convergence structures and some concepts based on them.

Let $\{f_i : X \to X_i|\ i \in I\}$ be a family of maps from a set X into a family of $\{X_i|\ i \in I\}$ of convergence spaces. To any point $x \in X$ we assign all those filters $\mathcal{F}$ on X for which $f_i(\mathcal{F})$ converges to $f_i(x)$ for every $i \in I$. The convergence structure on X defined in this way is called the *initial convergence structure induced by the family* $\{f_i|\ i \in I\}$. It is of course the coarsest of all the convergence structures on X which allow every f_i to be continuous [1].

Based on the notion of the initial convergence structure we define subspaces and products in the obvious way. A subset A of a convergence space X is turned into a *subspace* of X if it is endowed with the initial convergence structure induced by the inclusion map. The *product* ΠX_i of a family $\{X_i|\ i \in I\}$ of convergence spaces is the product of the underlying sets of the family endowed with the initial convergence structure induced by the family of all the canonical projections.

For a family $\{X_i|\ i \in I\}$ of convergence spaces and a family of maps $\{f_i : X_i \to X|\ i \in I\}$ into a set X, we define the *final convergence structure induced by the family* $\{f_i|\ i \in I\}$ as follows : A filter $\mathcal{F}$ on X converges to $x \in X$ if and only if $\mathcal{F} \supseteq \dot{x}$ or $\mathcal{F} \supseteq \cap_{k=1}^{n} f_{i_k}(\mathcal{G}_k)$, where the filters $\mathcal{G}_k$ converge to a preimage under f_{i_k} of x for $i_k \in I$ and $k = 1, 2, \cdots, n$ [1].

Next we introduce some separation axioms for the convergence spaces.

Definition 2.3. [8] A convergence space X is said to be T_0 if $x \neq y$, $\dot{x}$ does not converge to y or $\dot{y}$ does not converge to x.

Definition 2.4. [8] A convergence space X is said to be T_1 if $x \neq y$, $\dot{x}$ does not converge to y and $\dot{y}$ does not converge to x.

Definition 2.5. [8] A convergence space X is said to be *Hausdorff* if a filter $\mathcal{F}$ on X converges to x and y then $x = y$.

Now we intorduce the notion of a convergence spaces over a base.

Let X and B be convergence spaces and $p : X \to B$ be a continuous map. In this case, X is called a *convergence space over a base* B and p is called a *projection*. For a convergence space X over B with a projection p, the *fibre* of $b \in B$, denoted by X_b, is the subset $p^{-1}(b)$ of X.

For convergence spaces X and Y over B with projections p and q, respectively, a continuous map $f : X \to Y$ with $q \circ f = p$ is called a *continuous map over* B.

For convergence spaces X and Y over B, with projections p and q, respectively, let $X \times_B Y = \{(x, y) |\ p(x) = q(y)\}$ be endowed with the initial convergence structure induced by the $\{\pi_1 : X \times_B Y \to X, \pi_2 : X \times_B Y \to Y\}$. Then $X \times_B Y$, considered as a convergence space over B with the projection $p \circ \pi_1$, is the product of X and Y in the sense of convergence spaces over a base. This product is called the *fibre product* of X and Y.

Similarly for a family $\{X_i |\ i \in I\}$ of convergence spaces over B, we obtain the product $\Pi_B X_i$.

Now we will extend a property of convergence spaces to convergence spaces over B, in a natural way, satisfying some specific conditions.

Definition 2.6. [2] A property P_B of convergence spaces over B is said to be *well-behaved* if it satisfies the following three conditions :

(Condition 1) If X and Y are homeomorphic convergence spaces over B and if X has property P_B then so does Y.

(Condition 2) A convergence space X has property P if and only if the convergence space X over the point $*$ has property P_*.

(Condition 3) If a convergence space X over B has property P_B then the convergence space ξ^*X over B' has property $P_{B'}$ for each convergence space B' and continuous map $\xi : B' \to B$, where $\xi^*X = B' \times_B X$ is the convergence space over B' with the projection π_1.

Now, we introduce some separation axioms for the convergence spaces over B.

Definition 2.7. [8] A convergence space X over B is said to be *fibrewise* T_0 if $x \neq y$, where x and y belong to the same fibre, $\dot{x}$ does not converge to y or $\dot{y}$ does not converge to x.

Definition 2.8. [8] A convergence space X over B is said to be *fibrewise* T_1 if $x \neq y$, where x and y belong to the same fibre, $\dot{x}$ does not converge to y and $\dot{y}$ does not converge to x.

Definition 2.9. [8] A convergence space X over B is said to be *fibrewise Hausdorff* if a filter $\mathcal{F}$ on X converges to x and y, where x and y belong to the same fibre, then $x = y$.

Proposition 2.10. [8] *The properties fibrewise* T_0, *fibrewise* T_1, *and fibrewise Hausdorff are well-behaved.*

3. Regular convergence spaces

In this section, we introduce the notion of regular convergence spaces and investigate some properties of regular convergence spaces.

Definition 3.1. [1] Let (X, c) be a convergence space. X is said to be *regular* if for any filter $\mathcal{F}$ converging to x, $\{\overline{F} \mid F \in \mathcal{F}\}$ converges to x, where $\overline{F} = \{x \in X \mid$ there exists an ultrafilter $\mathcal{U}$ containing F converging to $x\}$.

Let $(X, \mathcal{T})$ be a topological space. By assigning to each point $x \in X$ the set $c_{\mathcal{T}}(x)$ of all filters on X which converge to x with respect to the given topology, we obtain a convergence structure on X. This convergence space is denoted by $(X, c_{\mathcal{T}})$ [1]. Then we can have the following result.

Proposition 3.2. *Let $(X, \mathcal{T})$ be a topological space. $(X, \mathcal{T})$ is regular if and only if $(X, c_{\mathcal{T}})$ is regular.*

Proof. Suppose $\mathcal{F}$ converges to x in $(X, c_{\mathcal{T}})$. Since $(X, \mathcal{T})$ is regular, for each $U \in \mathcal{N}_x$, there exists a $V \in \mathcal{N}_x$ such that $\overline{V} \subseteq U$. Since $\mathcal{N}_x$ is contained in $\mathcal{F}$ and $V \in \mathcal{N}_x$, $\overline{V} \in \{\overline{F} \mid F \in \mathcal{F}\}$. Thus $U \in \{\overline{F} \mid F \in \mathcal{F}\}$ and hence $\mathcal{N}_x$ is contained in $\{\overline{F} \mid F \in \mathcal{F}\}$. Therefore $\{\overline{F} \mid F \in \mathcal{F}\}$ converges to x.

Conversely, let $x \in X$ and U be an open set in $(X, \mathcal{T})$ containing x. Since $\mathcal{N}_x$ converges to x in $(X, c_{\mathcal{T}})$ and $(X, c_{\mathcal{T}})$ is regular, $\{\overline{V} \mid V \in \mathcal{N}_x\}$ converges to x. Thus $\mathcal{N}_x$ is contained in $\{\overline{V} \mid V \in \mathcal{N}_x\}$. This implies that there exists a $V \in \mathcal{N}_x$ such that $x \in \overline{V} \subseteq U$. Hence $(X, \mathcal{T})$ is regular.

Proposition 3.3. *Let X be a regular T_1 convergence space. Then X is Hausdorff.*

Proof. Suppose $\mathcal{F}$ converges to x and y. Then since X is regular, $\{\overline{F}|\ F \in \mathcal{F}\}$ converges to x and y. So $x \in \overline{F}$ and $y \in \overline{F}$ for all $F \in \mathcal{F}$, and hence $\{\overline{F}|\ F \in \mathcal{F}\}$ is contained in $\dot{x}$ and $\dot{y}$. This means that $\dot{x}$ converges to x and y, simultaneously. Since X is a T_1 space, $x = y$. Hence X is Hausdorff.

The following example explains that there exists a Hausdorff convergence space, which is not a regular convergence space.

Example Let $A = \left\{\frac{1}{n}|\ n \in \mathbb{N}\right\}$ in $\mathbb{R}$, let $\mathcal{B} = \{B \in \mathcal{P}(\mathbb{R})|\ B$ is an open interval that does not contain 0 or $B = (-x, x) - A\}$ for $x > 0$. Then we note that $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}$, and the space $(\mathbb{R}, \mathcal{T})$ is a Hausdorff space. Thus $(\mathbb{R}, c_{\mathcal{T}})$ is a Hausdorff convergence space. Consider the subset $A = \left\{\frac{1}{n}|\ n \in \mathbb{N}\right\}$ of $\mathbb{R}$, then A is closed in $(\mathbb{R}, \mathcal{T})$. We note that for any open sets U and V containing A and $\{0\}$, respectively, $U \cap V \neq \emptyset$. Hence $(\mathbb{R}, \mathcal{T})$ is not regular. Therefore $(\mathbb{R}, c_{\mathcal{T}})$ is not regular, by proposition 3.2 .

Proposition 3.4. *Let $f : X \to Y$ be an embedding and Y be a regular convergence space. Then X is regular.*

Proof. Let $\mathcal{F}$ converge to x in X. Then $f(\mathcal{F})$ converges to $f(x)$ in Y, since f is continuous. Since Y is regular, $\{\overline{f(F)}|\ F \in \mathcal{F}\}$ converges to $f(x)$ in Y.

Then $\{f^{-1}(\overline{f(F)})|\ F \in \mathcal{F}\}$ converges to x in X, since f is an embedding. Note that

$$\overline{F} \subseteq f^{-1}(f(\overline{F})) \subseteq f^{-1}(\overline{f(F)}) \ .$$

Thus

$$\{f^{-1}(\overline{f(F)})|\ F \in \mathcal{F}\} \subseteq \{\overline{F}|\ F \in \mathcal{F}\} \ .$$

Hence $\{\overline{F}|\ F \in \mathcal{F}\}$ converges to x in X. Therefore X is regular.

By the above proposition, we have the following corollary.

Corollary 3.5. *Every subspace of a regular convergence space is regular.*

Proposition 3.6. *Let $\{X_i|i \in I\}$ be a class of convergence spaces. Then the product space $\prod X_i$ is regular if and only if X_i is regular for all $i \in I$.*

Proof. Let $\{X_i|i \in I\}$ be a class of regular convergence spaces. Let $\mathcal{F}$ converge to $x = (x_i)$ in $\prod X_i$. Then $\pi_i(\mathcal{F})$ converges to x_i in X_i for all $i \in I$. Since X_i is regular, $\{\overline{\pi_i(F)}|\ F \in \mathcal{F}\}$ converges to x_i in X_i for all $i \in I$. Since $F \subset \prod \pi_i(F)$, $\overline{F} \subseteq \overline{\prod \pi_i(F)} = \prod \overline{\pi_i(F)}$. Therefore

$$\left\{\prod \overline{\pi_i(F)}|\ F \in \mathcal{F}\right\} \subseteq \{\overline{F}|\ F \in \mathcal{F}\} \ .$$

Thus $\{\overline{F}|\ F \in \mathcal{F}\}$ converges to x in $\prod X_i$. Hence $\prod X_i$ is regular.

Conversely, we note that X_i is homeomorphic to a subspace of $\prod X_i$ for all $i \in I$. Hence X_i is regular for all $i \in I$, by corollary 3.5.

Proposition 3.7. *Let $f : X \to Y$ be an initial surjection. Then if X is regular, Y is regular.*

Proof. Let $\mathcal{G}$ converge to y in Y. Since f is surjective, $f^{-1}(\mathcal{G})$ is a filter in X and $\mathcal{G} = f(f^{-1}(\mathcal{G}))$. Thus $f(f^{-1}(\mathcal{G}))$ converges to y in Y, and hence $f^{-1}(\mathcal{G})$ converges to x for some $x \in f^{-1}(y)$ in X, since f is an initial. Since X is regular, $\{\overline{f^{-1}(G)} |\ G \in \mathcal{G}\}$ converges to x in X. Moreover, since f is continuous, $\{f(\overline{f^{-1}(G)}) |\ G \in \mathcal{G}\}$ converges to y in Y. Note that $\overline{G} \subseteq f(\overline{f^{-1}(G)})$. In fact, let $p \in \overline{G}$, then there exists an ultrafilter $\mathcal{U}$ containing G converging to p. Then $f^{-1}(\mathcal{U})$ contains $f^{-1}(G)$ and converges to $q \in f^{-1}(p)$ in X. Thus $q \in \overline{f^{-1}(G)}$. Hence $p = f(q) \in f(\overline{f^{-1}(G)})$. So we have

$$\{f(\overline{f^{-1}(G)}) |\ G \in \mathcal{G}\} \subseteq \{\overline{G} |\ G \in \mathcal{G}\}$$

Hence $\{\overline{G} |\ G \in \mathcal{G}\}$ converges to y in Y. Therefore Y is regular.

Proposition 3.8. *Let $f : X \to Y$ be a final injection. Then if Y is regular, X is regular.*

Proof. Let $\mathcal{F}$ converge to x in X. Then $f(\mathcal{F})$ converges to $f(x)$ in Y. Since Y is regular, $\{\overline{f(F)} |\ F \in \mathcal{F}\}$ converges to $f(x)$ in Y. Since f is a final injection, there exists a filter $\mathcal{G}$ on X converging to x such that $f(\mathcal{G}) \subseteq \{\overline{f(F)} |\ F \in \mathcal{F}\}$. So, for each $G \in \mathcal{G}$ there exists an $F \in \mathcal{F}$ such that $\overline{f(F)} \subseteq f(G)$. Then

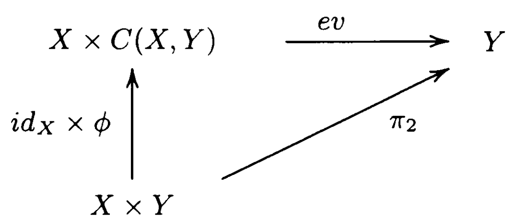
$$\overline{F} = f^{-1}(f(\overline{F})) \subseteq f^{-1}(\overline{f(F)}) \subseteq f^{-1}(f(G)) = G\ .$$

Hence $\mathcal{G} \subseteq \{\overline{F} | \ F \in \mathcal{F}\}$, and so $\{\overline{F} | \ F \in \mathcal{F}\}$ converges to x in X. Therefore X is regular.

Let X and Y be convergence spaces and $C(X,Y)$ be the set of all continuous functions from X to Y. Define a filter $\mathcal{F}$ converges to f in $C(X,Y)$ if and only if for any filter $\mathcal{A}$ which converges to x in X, $\mathcal{F}(\mathcal{A})$ converges to $f(x)$ in Y, where $\mathcal{F}(\mathcal{A})$ is the filter generated by $\{F(A) | \ F \in \mathcal{F}, A \in \mathcal{A}\}$. Then it is well known that $C(X,Y)$ with this structure is a convergence space and this structure is called the continuous convergence structure on $C(X,Y)$ [5]. Moreover, it is also known that for any convergence space Z and a function $f : Z \to C(X,Y)$, f is continuous if and only if $ev \circ (1_X \times f) : X \times Z \to Y$ is continuous, where $ev : X \times C(X,Y) \to Y$ is an evaluation map which is defined by $ev(x,f) = f(x)$, by the cartesian closedness of the category of convergence spaces [5].

Proposition 3.9. *Let $K = \{f \in C(X,Y) | \ f : \text{constant map}\}$, then K is homeomorphic to Y.*

Proof. Define $\phi : Y \to C(X,Y)$ by $\phi(y) = c_y$, where c_y is the constant map from X to Y with the value y. Clearly, ϕ is well-defined and injective. Note that $\phi(Y) = K$. Let $\psi : Y \to K$ be the corestriction of ϕ. Consider the following diagram



Note that $ev \circ (id_X \times \phi) = \pi_2$. Since π_2 is continuous, $ev \circ (id_X \times \phi)$ is continuous. Thus ϕ is continuous, and hence ψ is continuous. Furthermore, we note that for some $x \in X$,

$$\psi^{-1} : K \xrightarrow{\ j\ } \{x\} \times K \xrightarrow{ev|_{\{x\} \times K}} Y$$

where $j(f) = (x, f)$ for $f \in K$. Since j and ev are continuous, ψ^{-1} is continuous. Thus ψ is a homeomorphism from Y to K. Hence K is homeomorphic to Y.

By the above proposition, we have the following necessary and sufficient condition for the regularity of $C(X,Y)$.

Theorem 3.10. *Let X and Y be convergence spaces. Then Y is regular if and only if $C(X,Y)$ is regular.*

Proof. Let $\mathcal{F}$ converge to f in $C(X,Y)$. Then for any filter $\mathcal{A}$ on X which converges to x, the filter $\mathcal{F}(\mathcal{A})$ on Y converges to $f(x)$. Since Y is regular, $\{\overline{F(A)} |\ F \in \mathcal{F}, A \in \mathcal{A}\}$ converges to $f(x)$.

We want to show that $\{\overline{F(A)} |\ F \in \mathcal{F}, A \in \mathcal{A}\}$ is contained in $\{\overline{F}(A) |\ F \in \mathcal{F}, A \in \mathcal{A}\}$. It is enough to show that for $F \in \mathcal{F}$ and $A \in \mathcal{A}$, $\overline{F}(A) \subseteq \overline{F(A)}$.

Let $f(x) \in \overline{F}(A)$. Since $f \in \overline{F}$, there exists an ultrafilter $\mathcal{G}$ on $C(X,Y)$ which contains F and converges to f. Then $\mathcal{G}(\dot{x})$ converges to $f(x)$. Thus let $\mathcal{H}$ be an ultrafilter on Y containing $\mathcal{G}(\dot{x})$, then $F(A) \in \mathcal{G}(\dot{x}) \subset \mathcal{H}$ and $\mathcal{H}$ converges to $f(x)$. Thus $f(x) \in \overline{F(A)}$. So $\{\overline{F}(A) |\ F \in \mathcal{F}\}$ converges to $f(x)$ and hence $\{\overline{F} |\ F \in \mathcal{F}\}$ converges to f. Therefore $C(X,Y)$ is regular.

Conversely, note that Y is homeomorphic to a subspace of $C(X,Y)$ by the above proposition. Hence the result follows by corollary 3.5.

4. Fibrewise regular convergence spaces

In this section, we define the notion of the fibrewise regular convergence spaces, which can be regarded as a generalization of the notion of regular convergence spaces.

Definition 4.1. A convergence space X over B is said to be *fibrewise regular* (or *regular over* B) if for any filter $\mathcal{F}$ on X converging to $x \in X_b$ with $X_b \cap \overline{F} \neq \emptyset$ for all $F \in \mathcal{F}$, $\{X_b \cap \overline{F} |\ F \in \mathcal{F}\}$ converges to x.

Theorem 4.2. *The property "fibrewise regular" is well-behaved.*

Proof. (Condition 1) Let X and Y be convergence spaces over B and $f : X \to Y$ be a homeomorphism. Let Y be fibrewise regular. Suppose $\mathcal{F}$ converges to $x \in X_b$ such that $X_b \cap \overline{F} \neq \emptyset$ for all $F \in \mathcal{F}$. Then $f(\mathcal{F})$ converges to $f(x) \in Y_b$. Since

$$\emptyset \neq f(X_b \cap \overline{F}) = f(X_b) \cap f(\overline{F}) = Y_b \cap \overline{f(F)} \tag{1}$$

and since Y is fibrewise regular, $\{Y_b \cap \overline{f(F)} |\ F \in \mathcal{F}\}$ converges to $f(x)$. By (1),

$$f^{-1}(\{Y_b \cap \overline{f(F)} |\ F \in \mathcal{F}\}) \subseteq \{X_b \cap \overline{F} |\ F \in \mathcal{F}\}\ .$$

Thus $\{X_b \cap \overline{F} |\ F \in \mathcal{F}\}$ converges to x. Hence X is fibrewise regular.

(Condition 2) It is obvious.

(Condition 3) Let X be fibrewise regular and $\xi : B' \to B$ be a continuous map. Let $\mathcal{F}$ converge to $(b', x) \in (\xi^* X)_{b'} = \{b'\} \times X_{\xi(b')}$ such that $(\xi^* X)_{b'} \cap$

$\overline{F} \neq \emptyset$ for all $F \in \mathcal{F}$. Since π_2 is continuous, $\pi_2(\mathcal{F})$ converges to x in X. Since

$$\emptyset \neq \pi_2((\xi^*X)_{b'} \cap \overline{F}) \subseteq \pi_2((\xi^*X)_{b'}) \cap \pi_2(\overline{F}) \subseteq X_{\xi(b')} \cap \overline{\pi_2(F)} \qquad (2)$$

and since X is fibrewise regular, $\{X_{\xi(b')} \cap \overline{\pi_2(F)} |\ F \in \mathcal{F}\}$ converges to x. By (2), $\{X_{\xi(b')} \cap \overline{\pi_2(F)} |\ F \in \mathcal{F}\} \subseteq \{\pi_2((\xi^*X)_{b'} \cap \overline{F}) |\ F \in \mathcal{F}\}$, and hence $\{\pi_2((\xi^*X)_{b'} \cap \overline{F}) |\ F \in \mathcal{F}\}$ converges to x. But, $\{\pi_2((\xi^*X)_{b'} \cap \overline{F}) |\ F \in \mathcal{F}\} = \pi_2\left(\{(\xi^*X)_{b'} \cap \overline{F} |\ F \in \mathcal{F}\}\right)$. Hence $\pi_2\left(\{(\xi^*X)_{b'} \cap \overline{F} |\ F \in \mathcal{F}\}\right)$ converges to x. And since $\pi_1((\xi^*X)_{b'} \cap \overline{F}) = b'$, $\pi_1(\{(\xi^*X)_{b'} \cap \overline{F} |\ F \in \mathcal{F}\})$ is the filter generated by $\{b'\}$ and thus $\pi_1(\{(\xi^*X)_{b'} \cap \overline{F} |\ F \in \mathcal{F}\})$ converges to b'. Thus $\{(\xi^*X)_{b'} \cap \overline{F} |\ F \in \mathcal{F}\}$ converges to (b', x). Hence ξ^*X is fibrewise regular.

Remark [3] Let $(X, \mathcal{T})$ be a topological space over B. Then $(X, \mathcal{T})$ is said to be *fibrewise regular* (or *regular over* B) if for any open set U containing $x \in X_b$, there exists an open set V such that $x \in X_b \cap \overline{V} \subset U$.

Proposition 4.3. *Let* $(X, \mathcal{T})$ *be a topological space. Then* $(X, \mathcal{T})$ *is fibrewise regular if and only if* $(X, c_{\mathcal{T}})$ *is fibrewise regular.*

Proof. Let $\mathcal{F}$ converge to $x \in X_b$ and $X_b \cap \overline{F} \neq \emptyset$ for all $F \in \mathcal{F}$. Then $\mathcal{N}_x \subset \mathcal{F}$. So, for $U \in \mathcal{N}_x$, $X_b \cap \overline{U} \neq \emptyset$. Since $(X, \mathcal{T})$ is fibrewise regular, for any $V \in \mathcal{N}_x$, there exists a $U \in \mathcal{N}_x$ such that $X_b \cap \overline{U} \subseteq V$. Hence $V \in \{X_b \cap \overline{F} |\ F \in \mathcal{F}\}$. Thus $\mathcal{N}_x \subseteq \{X_b \cap \overline{F} |\ F \in \mathcal{F}\}$. Hence $\{X_b \cap \overline{F} |\ F \in \mathcal{F}\}$ converges to x. Therefore $(X, c_{\mathcal{T}})$ is fibrewise regular.

Conversely, note that $\mathcal{N}_x$ converges to $x \in X_b$ and $X_b \cap \overline{V} \neq \emptyset$ for all $V \in \mathcal{N}_x$. Since $(X, c_\mathcal{T})$ is fibrewise regular, $\{X_b \cap \overline{V} |\ V \in \mathcal{N}_x\}$ converges to x. Thus $\mathcal{N}_x \subseteq \{X_b \cap \overline{V} |\ V \in \mathcal{N}_x\}$. This means that for each $U \in \mathcal{N}_x$, there exists a $V \in \mathcal{N}_x$ such that $x \in X_b \cap \overline{V} \subseteq U$. Hence $(X, \mathcal{T})$ is fibrewise regular.

Proposition 4.4. *Let X be a fibrewise T_1 convergence space. Then if X is fibrewise regular, X is fibrewise Hausdorff.*

Proof. Let $\mathcal{F}$ converge to x and y with $x, y \in X_b$. Then for each $F \in \mathcal{F}$, $x \in X_b \cap \overline{F}$ and $y \in X_b \cap \overline{F}$. Hence $\{X_b \cap \overline{F} |\ F \in \mathcal{F}\}$ is contained in $\dot{x}$ and $\dot{y}$. Since X is fibrewise regular, $\{X_b \cap \overline{F} |\ F \in \mathcal{F}\}$ converges to x and y. Thus $\dot{x}$ converges to x and y, simultaneously. Since X is fibrewise T_1, $x = y$. Hence X is fibrewise Hausdorff.

Proposition 4.5. *Let X and Y be convergence spaces over B and $f : X \to Y$ be an embedding. Then if Y is fibrewise regular, so is X.*

Proof. Let $\mathcal{F}$ converge to x in X and $X_b \cap \overline{F} \neq \emptyset$ for all $F \in \mathcal{F}$. Then $f(\mathcal{F})$ converges to $f(x)$ in Y, since f is continuous. Since Y is fibrewise regular and

$$\emptyset \neq f(\overline{F} \cap X_b) \subseteq \overline{f(F)} \cap Y_b \ ,$$

$\{\overline{f(F)} \cap Y_b |\ F \in \mathcal{F}\}$ converges to $f(x)$ in Y. Thus $\{f^{-1}(\overline{f(F)} \cap Y_b) |\ F \in \mathcal{F}\}$ converges to x in X, since f is an embedding. Note that

$$\overline{F} \cap X_b \subseteq f^{-1}(f(\overline{F})) \cap f^{-1}(Y_b) \subseteq f^{-1}(\overline{f(F)} \cap Y_b) \ .$$

Thus

$$\{f^{-1}(\overline{f(F)} \cap X_b) |\ F \in \mathcal{F}\} \subseteq \{\overline{F} \cap X_b |\ F \in \mathcal{F}\} .$$

Hence $\{\overline{F} \cap X_b |\ F \in \mathcal{F}\}$ converges to x in X, and so X is fibrewise regular.

The following corollary is obtained from the above proposition immediately.

Corollary 4.6. *Let X be a fibrewise regular convergence space. Then a subspace of X is also fibrewise regular.*

Proposition 4.7. *Let $\{X_i | i \in I\}$ be a family of fibrewise regular convergence spaces. Then $\prod_B X_i$ is fibrewise regular.*

Proof. Let $\mathcal{F}$ converge to $(x_i) \in (\prod_B X_i)_b$ and $(\prod_B X_i)_b \cap \overline{F} \neq \emptyset$ for all $F \in \mathcal{F}$. Then for each $i \in I$,

$$\emptyset \neq \pi_i \left((\textstyle\prod_B X_i)_b \cap \overline{F}\right) \subseteq \pi_i \left((\textstyle\prod_B X_i)_b\right) \cap \pi_i(\overline{F}) \subseteq (X_i)_b \cap \overline{\pi_i(F)} .$$

For each $i \in I$, since $\pi_i(\mathcal{F})$ converges to x_i and since X_i is fibrewise regular, $\{(X_i)_b \cap \overline{\pi_i(F)} |\ F \in \mathcal{F}\}$ converges to x_i. Since $F \subseteq \prod \pi_i(F)$, $\overline{F} \subseteq \overline{\prod \pi_i(F)} = \prod \overline{\pi_i(F)}$ and thus $(\prod_B X_i)_b \cap \overline{F} \subseteq (\prod_B X_i)_b \cap \prod \overline{\pi_i(F)} = \prod \left((X_i)_b \cap \overline{\pi_i(F)}\right)$. Therefore

$$\left\{\prod \left((X_i)_b \cap \overline{\pi_i(F)}\right) |\ F \in \mathcal{F}\right\} \subseteq \left\{ (\textstyle\prod_B X_i)_b \cap \overline{F} |\ F \in \mathcal{F}\right\} .$$

Thus $\{(\prod_B X_i)_b \cap \overline{F} |\ F \in \mathcal{F}\}$ converges to (x_i). Hence $\prod_B X_i$ is fibrewise regular.

Proposition 4.8. *Let X and Y be convergence spaces over B and $f : X \to Y$ be an initial surjection. Then if X is fibrewise regular, Y is fibrewise regular.*

Proof. Let $\mathcal{G}$ converge to y in Y and $\overline{G} \cap Y_b \neq \emptyset$ for all $G \in \mathcal{G}$. Note that $f^{-1}(\mathcal{G})$ converges to x for some $x \in f^{-1}(y)$ in X, since f is initial and surjective. We know that $f^{-1}(\overline{G}) \subseteq \overline{f^{-1}(G)}$. Thus

$$\emptyset \neq f^{-1}(\overline{G} \cap Y_b) = f^{-1}(\overline{G}) \cap f^{-1}(Y_b) \subseteq \overline{f^{-1}(G)} \cap X_b \,.$$

Since X is fibrewise regular, $\{\overline{f^{-1}(G)} \cap X_b |\ G \in \mathcal{G}\}$ converges to x in X. Moreover, since f is continuous, $\{f(\overline{f^{-1}(G)} \cap X_b) |\ G \in \mathcal{G}\}$ converges to y in Y. So we have

$$\overline{G} \cap Y_b = f(f^{-1}(\overline{G} \cap Y_b)) = f(f^{-1}(\overline{G}) \cap f^{-1}(Y_b)) \subseteq f(\overline{f^{-1}(G)} \cap X_b)$$

and hence

$$\{f(\overline{f^{-1}(G)} \cap X_b) |\ G \in \mathcal{G}\} \subseteq \{\overline{G} \cap Y_b |\ G \in \mathcal{G}\} \,.$$

Therefore $\{\overline{G} \cap Y_b |\ G \in \mathcal{G}\}$ converges to $y \in Y$. In all, Y is fibrewise regular.

Proposition 4.9. *Let X and Y be convergence spaces over B and $f : X \to Y$ be a final injection. Then if Y is fibrewise regular, X is fibrewise regular.*

Proof. Let $\mathcal{F}$ converge to x in X and $F \cap X_b \neq \emptyset$ for all $F \in \mathcal{F}$. Then $f(\mathcal{F})$ converges to $f(x)$ in Y. Note that $\overline{f(F)} \cap Y_b \neq \emptyset$ for all $F \in \mathcal{F}$. Since Y is fibrewise regular, $\{\overline{f(F)} \cap Y_b |\ F \in \mathcal{F}\}$ converges to $f(x)$ in Y. Since f

is a final injection, there exists a filter $\mathcal{G}$ on X converging to x such that $f(\mathcal{G}) \subseteq \{\overline{f(F)} \cap Y_b |\ F \in \mathcal{F}\}$. So, for each $G \in \mathcal{G}$ there exists $F \in \mathcal{F}$ such that $\overline{f(F)} \cap Y_b \subseteq f(G)$. Thus

$$\overline{F} \cap X_b = f^{-1}(f(\overline{F} \cap X_b)) \subseteq f^{-1}(\overline{f(F)} \cap Y_b) \subseteq f^{-1}(f(G)) = G .$$

Hence $\mathcal{G} \subset \{\overline{F} \cap X_b |\ F \in \mathcal{F}\}$. Since $\mathcal{G}$ converges to x, $\{\overline{F} \cap X_b |\ F \in \mathcal{F}\}$ converges to x in X. Therefore X is fibrewise regular.

Let X and Y be convergence spaces over B and $C_B(X,Y) = \cup_{b \in B} C(X_b, Y_b)$ as a set, where $C(X_b, Y_b)$ is the set of all continuous functions from X_b to Y_b. Define a filter $\mathcal{F}$ converges to f in $C_B(X,Y)$, where $f \in C(X_b, Y_b)$ if and only if

(1) for any filter $\mathcal{A}$ in X which converges to $x \in X_b$, $(\mathcal{F} \cap \dot{f})(\mathcal{A} \cap \dot{x})$ converges to $f(x)$ in Y and

(2) $p(\mathcal{F})$ converges to $p(f)$ in B, where $p : C_B(X,Y) \to B$ is defined by $p(g) = b$ if $g \in C(X_b, Y_b)$.

Then it is well known that $C_B(X,Y)$ with this structure is a convergence space and this structure is called the *fibrewise continuous convergence structure* on $C_B(X,Y)$ [6].

Proposition 4.10. *Let X and Y be convergence spaces over B. If Y is fibrewise regular, then $C_B(X,Y)$ is fibrewise regular.*

Proof. Suppose $\mathcal{F}$ converges to $f \in C(X_b, Y_b)$ and $C(X_b, Y_b) \cap \overline{F} \neq \emptyset$ for all $F \in \mathcal{F}$. Then we have to show that $\mathcal{G} = \{C(X_b, Y_b) \cap \overline{F} |\ F \in \mathcal{F}\}$ converges to f in $C_B(X,Y)$. Let $\mathcal{A}$ converge to $x \in X_b$, then it is enough to show that $(\mathcal{G} \cap \dot{f})(\mathcal{A} \cap \dot{x})$ converges to $f(x)$ in Y. Since $\mathcal{F}$ converges to f in $C_B(X,Y)$, $(\mathcal{F} \cap \dot{f})(\mathcal{A} \cap \dot{x})$ converges to $f(x) \in Y_b$. Hence $f(x) \in \overline{(F \cup \{f\})(A \cup \{x\})}$ for all $F \in \mathcal{F}$ and $A \in \mathcal{A}$. So $Y_b \cap \overline{(F \cup \{f\})(A \cup \{x\})} \neq \emptyset$ for all $F \in \mathcal{F}$ and $A \in \mathcal{A}$. Therefore $\{Y_b \cap \overline{(F \cup \{f\})(A \cup \{x\})} |\ F \in \mathcal{F}, A \in \mathcal{A}\}$ converges to $f(x)$, since Y is fibrewise regular. We note that $(\overline{F} \cup \{f\}) \subseteq \overline{F \cup \{f\}}$ and $(A \cup \{x\}) \subseteq \overline{A \cup \{x\}}$. Thus $\{Y_b \cap \overline{(F \cup \{f\})(A \cup \{x\})} |\ F \in \mathcal{F}, A \in \mathcal{A}\}$ is contained in $\{Y_b \cap (\overline{F} \cup \{f\})(A \cup \{x\}) |\ F \in \mathcal{F}, A \in \mathcal{A}\}$, and hence $\{Y_b \cap (\overline{F} \cup \{f\})(A \cup \{x\}) |\ F \in \mathcal{F}, A \in \mathcal{A}\}$ converges to $f(x)$. We want to show that $\{Y_b \cap (\overline{F} \cup \{f\})(A \cup \{x\}) |\ F \in \mathcal{F}, A \in \mathcal{A}\}$ is contained in $\{(C(X_b, Y_b) \cap \overline{F}) \cup \{f\} |\ F \in \mathcal{F}\}(\mathcal{A} \cap \dot{x})$. It is equivalent to show that $((C(X_b, Y_b) \cap \overline{F}) \cup \{f\})(A \cup \{x\}) \subseteq Y_b \cap (\overline{F} \cup \{f\})(A \cup \{x\})$ for all $F \in \mathcal{F}$ and $A \in \mathcal{A}$. In fact, if $g \in C(X_b, Y_b) \cap \overline{F}$, then $g(A \cup \{x\}) \subseteq Y_b$ and $g(A \cup \{x\}) \in (\overline{F} \cup \{f\})(A \cup \{x\})$. So $\{Y_b \cap (\overline{F} \cup \{f\})(A \cup \{x\}) |\ F \in \mathcal{F}, A \in \mathcal{A}\} \subseteq \{(C(X_b, Y_b) \cap \overline{F}) \cup \{f\} |\ F \in \mathcal{F}\}(\mathcal{A} \cap \dot{x})$, and hence $\{(C(X_b, Y_b) \cap \overline{F}) \cup \{f\} |\ F \in \mathcal{F}\}(\mathcal{A} \cap \dot{x})$ converges to $f(x)$. But,

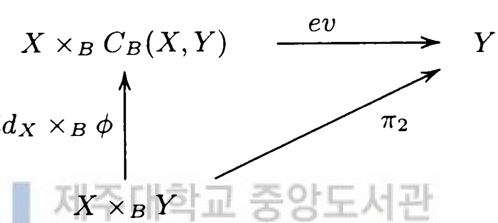
$$\{(C(X_b, Y_b) \cap \overline{F}) \cup \{f\} |\ F \in \mathcal{F}\} = \{C(X_b, Y_b) \cap \overline{F} |\ F \in \mathcal{F}\} \cap \dot{f}.$$

In all, $C_B(X,Y)$ is fibrewise regular.

It is also well known that for a convergence space B, the category of convergence spaces over B is cartesian closed [6]. So, for any convergence space Z over B and a function $f : Z \to C_B(X,Y)$, f is continuous if and only if $ev \circ (1_X \times_B f) : X \times_B Z \to Y$ is continuous, where $ev : X \times_B C_B(X,Y) \to Y$ is an evaluation map which is defined by $ev(x,f) = f(x)$.

Proposition 4.11. *Let X and Y be convergence spaces over B and suppose the projection $p : X \to B$ is surjective. Let $K = \{f \in C_B(X,Y) | \ f :$ constant map$\}$. Then K is homeomorphic to Y.*

Proof. Define $\phi : Y \to C_B(X,Y)$ by, for $y \in Y_b$, $\phi(y) = c_y$, where c_y is the constant map from X_b to Y_b with value $y \in Y_b$. Clearly, ϕ is well-defined and injective. Note that $\phi(Y) = K$. Let $\psi : Y \to K$ be the corestriction of ϕ. Consider the following diagram



Note that $\pi_2 = ev \circ (id_X \times_B \phi)$, since $\pi_2(x,y) = y = c_y(x) = ev(x, c_y)$. Since π_2 is continuous, $ev \circ (id_X \times_B \phi)$ is continuous. Hence ϕ is continuous, by the cartesian closedness of the category of convergence spaces over B. Therefore, $\psi : Y \to K$ is continuous. Pick $x_b \in X_b$ for all $b \in B$ and let $A = \{x_b | \ b \in B\}$. Then we know that

$$\psi^{-1} : K \xrightarrow{\ j\ } A \times_B K \xrightarrow{ev|_{A \times_B K}} Y,$$

where $j(f) = (x_b, f)$ for $f \in C(X_b, Y_b)$. Since j and ev are continuous, ψ^{-1} is continuous. In all, K is homeomorphic to Y.

By the above proposition, we have the following proposition which is the partial converse of the proposition 4.10.

Proposition 4.12. *Let X and Y be convergence spaces over B and suppose the projection $p : X \to B$ is surjective. Then if $C_B(X, Y)$ is fibrewise regular, Y is fibrewise regular.*

Proof. By the above proposition and corollary 4.6, the proof follows immediately.

References

[1] E. Binz, *Continuous Convergence on* $C(X)$, Lecture notes in Mathematics #469, Springer-Verlag, Berlin, 1975.

[2] I. M. James, *General Topology over a Base*, Aspects of Topology, London Math. Soc Lecture Notes #93, 1984.

[3] I. M. James, *General Topology and Homotopy Theory*, Springer-Verlag, New York, 1984.

[4] I. M. James, *Fibrewise Topology*, Cambridge University Press, London, 1989.

[5] C. Y. Kim, S. S. Hong, Y. H. Hong and P. U. Park, *Algebras in Catesian Closed Topological Categories*, Yonsei Univ. Lecture Notes, Seoul, 1978.

[6] K. C. Min and S. J. Lee, *Fibrewise convergence and exponential laws*, Tsukuba L. Math. Vol.16, No.1 (1992), 53-62.

[7] K. C. Min, S. J. Lee and J. W. Park, *Fibrewise convergence*, Comm. K.M.S., 8, No.2 (1992), 335-344.

[8] S. J. Lee and E. P. Lee, *Fibrewise Hausdorff convergence spaces*, Jour. C.M.S. Vol.5 (1992), 167-172.

[9] S. Willard, *General Topology*, Addison-Wesley, Reading(Miss.), 1970.

< 국문초록 >

정규 수렴 공간의 성질 연구

본 논문에서는 정규수렴공간의 개념을 소개하고 이 공간들의 여러 가지 성질에 관하여 조사하였다. 또한, 최근에 위상수학을 주어진 기저 공간을 갖는 위상공간으로 생각하는 파이버 관점으로의 일반화에 대한 연구가 많이 진행되어지고 있는데 이러한 관점에서 주어진 수렴공간을 기저 공간으로 갖는 파이버 정규수렴공간을 정의하고 이 정의가 매우 잘 정의된 정규수렴공간의 파이버 관점에서의 일반화라는 사실을 보였다. 그리고 정규수렴공간들의 성질에 관한 파이버 관점으로의 일반화도 얻어내었다. 마지막으로 주어진 기저 공간을 갖는 수렴공간들에 관한 함수공간을 소개하고 이 함수공간의 정규성에 관하여 조사하였다.

감사의 글

어느 사이 2년의 석사과정 기간이 훌쩍 지나버린 것 같습니다. 막상 석사학위논문을 쓰고 나니 좀 더 공부를 하였더라면 하는 생각이 듭니다. 아무튼 석사학위논문을 내놓게 되어 대단히 기쁘게 생각합니다. 석사학위논문을 쓰는 동안 세세히 지도를 해주신 박진원 교수님, 대학원을 망설이지 않고 선택을 할 수 있도록 조언을 아끼지 않으셨던 고윤희 교수님, 학부에서부터 석사과정까지 항상 관심과 격려 그리고 채찍질, 사랑을 아끼시지 않으신 수학과, 수학교육과 교수님들께 감사의 마음을 전합니다. 그리고 서투른 조교 생활을 하는데 여러모로 도움을 주신 사범대학 조교선생님들께도 글로써 감사의 마음을 대신합니다. 생각해보면 짧아져버린 대학원 2년간 같이 부둥키며 생활한 대학원 동료들, 선배님들 그리고 가족 여러분, 특히 형이 바쁠때마다 마다하지 않고 말을 잘 따라준 동생 정규에게 고맙다는 말을 전합니다. 앞으로 남은 대학원 기간에 더욱 의미있게 보내고자 노력하겠으며, 제가 뜻 하는 소망을 이룰 수 있도록 끝없이 노력·정진하는 자세로 열심히 하겠습니다.

1998년 12월