# On the Uniformizable Space

By

Ahn, Youngsuck



Department of Mathematics
Graduate School of Education
Cheju National University

Supervised By

Associate Prof. Han, Chulsoon

June, 1982

# On the Uniformizable SPace

이를 教育學碩士學位 論文으로 提出함



濟州大學校 敎育大學院 數學敎育專攻

提出者 安 榮 錫

指導敎授 韓 哲 淳

1982年 6月 日

# 安榮錫의 碩士學位 論文을 認准함

濟州大學校教育大學院



主 審 ________________

副 審 ________________

副 審 ________________

1982年 6月 日

## 감 사 의 글

이 논문이 완성되기 까지 연구에 분주한 가운데도 자상한 마음으로 친절하게 지도를 하여 주신 한철순 교수님께 감사드리며 아울러 재화하는 동안 많은 도움을 주신 수학교육과의 여러 교수님께 심심한 사의를 표합니다.

그리고 그동안 저에게 좋은 지도 조언의 말씀과 격려를 하여주신 주위의 많은 분들께 또한 감사드립니다.

1982년 6월 일

안 영 석

# C O N T E N T S

# 국 문 초 록

## Uniform 공 간

제주대학교 교육대학원

수학교육 전공

안 영 석

이 논문은 Uniform 공간이 거리공간을 일반화하고 있음을 증명하고 Proximity 공간으로부터 Uniform 공간을 유도할 수 있음을 증명하였다.

# 1. Introduction

In this paper, We shall study the uniform spaces generalizing the metric spaces.

We shall also show that the topological spaces derivable form the proximity spaces are precisely the uniformizable spaces.

We begin by defining the uniform spaces.

## 2. Preliminary

**Definition 2-1**

Let $S \neq \emptyset$ and $\mu \subset 2^{S \times S}$ satisfying the following axioms:

($U_1$) The diagonal $\Delta = \{(x,x) \mid x \in S\} \subset U$, $\forall U \in \mu$.

($U_2$) If $U \in \mu$ and $U \subset V$, then $V \in \mu$.

($U_3$) If $U$, $V \in \mu$, then $U \cap V \in \mu$.

($U_4$) $\forall U \in \mu$ there exist $V \in \mu$ such that $V \circ V \subset U$.

($U_5$) $U \in \mu$ implies that $U^{-1} \in \mu$.

Then $\mu$ is called a uniformity for $S$ and $(S, \mu)$ is a uniform space.

If $\mu$ satisfies ($U_1$) $\sim$ ($U_4$), then $\mu$ is a quasiuniformity for S.



**Definition 2-2**

If $\mu$ is a uniformity (quasiuniformity) for S, then $B \subset \mu$ is a base for $\mu$ if each member of $\mu$ contains a member of B.

It follows form definition 2-1 that conditions ($U_1$), ($U_4$), and ($U_5$) on a uniformity correspond roughly to conditions ($M_1$) $d(x,y) \geq 0$, $d(x,y) = 0$ iff $x = y$, ($M_3$) $d(x,y) \leqq d(x,z) + d(z,y)$ and ($M_2$) $d(x,y) = d(y,x)$ respectively, on a metric on S.

**Proposition 2-1**

Let$(S,\mu)$ be a uniform space.

For each $x \in S$ and each $U \in \mu$ ,define

$U(x) = \{y \in S \mid (x,y) \in U\}$ by a nbd of $x$.

Then the collection $B_x = \{U(x) \mid U \in \mu\}$ for each $x \in S$ is a nbd system of $x$

Proof,

It follows from definition 2-1 and 2-2 that $B_x$ is a nbd system of $x$.

We use proposition 2-1 to get a base for the topology on S induced by the uniformity $\mu$ on S.



**Proposition 2-2**

Let$(S,\mu)$ be a uniform space and $B = \{B_x \mid x \in S\}$ where each $B_x$ is a nbd system of $x$.

Then B is a base for the topology on S induced by the uniformity $\mu$.

Proof,

Clearly, $S = \bigcup\{B_x : x \in X\}$

Let $U(x)$, $V(y) \in B$ and $z \in U(x) \cap V(y)$.

Then $z \in U(x)$ and $z \in V(y)$

Let $W=U(x)\cap V(y)$, then $W\in\mu$ and $Z\in W$

So, $W\in B_z$ and $W\in B$

Therefore, B is a base for the topolgy on S induced by the uniformity $\mu$.

So, we have the following definition:

**Definition 2-3**

Let $(S,\mu)$ be a uniform space and B a base for the topology on S induced by the uniformity $\mu$.

We call such a topology $\mathcal{J}$ having B as a base induced by the uniformity $\mu$.



We conclude that every uniform space is a topological space induced by the uniformity

**Definition 2-4**

$(S,\delta)$ is a Proximity space iff $S\neq\emptyset$ and $\delta$ is a relation on $2^S$ satisfying the following conditions:

$(P_1)$ $(A,\emptyset)\notin\delta$, $^\forall A\in 2^S$

$(P_2)$ $(\{x\},\{x\})\in\delta$, $^\forall x\in S$

$(P_3)$ $(C, A\cup B)\in\delta$ iff $(C,A)\in\delta$ or $(C,B)\in\delta$ $^\forall A,B,C\in 2^S$.

$(P_4)$ If $(A,B)\notin\delta$, then there exist $C\in 2^S$ such that $(A,C)\notin\delta$ and $(S-C, B)\notin\delta$

$(P_5)$ $(A,B) \in \delta$ iff $(B,A) \in \delta$

The relation $\delta$ is called a proximity for S, and $(A,B) \in \delta$ is read A is near B

**Proposition 2-3**

Every proximity space $(S, \delta)$ is a topological space

Proof,

Let $C(A) = \{x \in S \mid (\{x\},A) \in \delta\}$ for each $A \in 2^S$

We cleam that C satisfies the Kuratowski's closure axioms

(a) $C(\emptyset) = \emptyset$ from the definition of C and$(P_1)$

(b) Let $A \in 2^S$ and $x \in A$

Suppose $x \notin C(A)$, Then$(\{x\},A) \notin \delta$

Since $\{x\} \cup A = A$, then $(\{x\}, \{x\} \cup A) \notin \delta$ iff

$(\{x\},\{x\} \notin \delta$ and $(\{x\}, A) \notin \delta)$

a contradiction to $(P_3)$

So $(\{x\},A) \in \delta \Rightarrow x \in C(A)$

Therefore $A \subset c(A)$

(c) Let $A \in 2^S$

We claim $C(c(A)) = c(A)$

clearly $c(A) \subset c(c(A))$ by (b)

Let $x \in c(c(A))$ and suppose $x \notin c(A)$

Then $(\{x\},A) \notin \delta$

There exist $E \in 2^S$ such that $(\{x\}.E) \notin \delta$ and $(S-E, A) \notin \delta$ by($p_4$)

Now $c(A) \subset E$ and $(\{x\}, E) \notin \delta$, so that $(\{x\}, c(A)) \notin \delta$

We have a contradiction.

Hence $c(c(A)) \subset c(A)$.

Therefore $c(c(A)) = c(A)$.

(d) Let $A, B \in 2^S$, then $x \in c(A \cup B)$ iff $(\{x\}, A \cup B) \in \delta$

iff $(\{x\}, A) \in \delta$ or $(\{x\}, B) \in \delta$

iff $x \in c(A) \cup c(B)$

Therefore $c(A \cup B) = c(A) \cup c(B)$

Let $\mathcal{J} = \{(c(A))^c : A \in 2^S\}$.

Then $\mathcal{J}$ is a topology on S.

## 3. Uniform space induced by metric and proximity

**Proposition 3-1**

Every metric space $(S, \alpha)$ is a uniform space.

Proof.

Let $B_\varepsilon = \{(x, y) \in S \times S \mid d(x, y) < \varepsilon\}$ for each $\varepsilon > 0$, and

$\mu = \{U \subset S \times S : B_\varepsilon \subset U$ for some $\varepsilon > 0\}$

we claim that $\mu$ is a uniformity on S

Then $\triangle \subset U \ \forall U \in \mu$, since $\triangle \subset B_\varepsilon \ \forall \varepsilon > 0$ and $(U_1)$ is satisfied.

If $U \in \mu$ and $U \subset V$, then $B_\varepsilon \subset U \subset V$ for some $\varepsilon > 0$.

Hence $V \in \mu$ and $(U_2)$ is satisfied.

If $U, V \in \mu$, then $B_{\varepsilon_1} \subset U$ and $B_{\varepsilon_2} \subset V$ for some $\varepsilon_1$, $\varepsilon_2 > 0$

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$, then $B_\varepsilon \subset B_{\varepsilon_1} \cap B_{\varepsilon_2}$ and $B_\varepsilon \subset U \cap V$

Hence $U \cap V \in \mu$ and $(U_3)$ is satisfied.

Let $B_\varepsilon \subset U \in \mu$ and $V = B_{\frac{\varepsilon}{2}}$.

Then $V \circ V = \{(x, y) \in S \times S \mid$ There exists $z \in S$ such that $d(x, z) < \frac{\varepsilon}{2}$ and $d(z, y) < \frac{\varepsilon}{2}\}$ $B_\varepsilon \subset U$ So $(U_4)$ is satisfied.

Finally, since d is symmetric.

if $B_\varepsilon \subset U \in \mu$ for some $\varepsilon > 0$, then $B_\varepsilon \subset U^{-1}$ and hence $U^{-1} \in \mu$.

So, $(U_5)$ is satisfieed

Therefore every metric space is a uniform space.

**Definition 3-1**

A nonempty subset $\mu$ of the power set of $S\times S$ is an M-uniformity on S iff the following axioms are satisfied:

(1) $\triangle\subset U$ for each $U\in\mu$

(2) $U\in\mu$ implies $U=U^{-1}$

(3) For every $A\subset S$ and $U,V\in\mu$,

there exist a $W\in\mu$ such that

$W(A)\subset U(A)\cap V(A)$, where $W(A)=\{y:(x,y)\in W$ for some $x\in A\}$

(4) For every pair of subsets A, B of S and every $U\in\mu$.

$V(A)\cap B\neq\emptyset$ for every $V\in\mu$ implies the existence of an $x\in B$ and $W\in\mu$ such that $W(x)\subset U[A]$



(5) $U\in\mu$ and $U\subset V=V^{-1}\subset S\times S$ implies $V\in\mu$.

The pair $(S;\mu)$ is called an M-uniform space.

**Proposition 3-2**

Every M-uniform space$(S,\mu)$ has an associated topology $\mathcal{J}=\mathcal{J}(\mu)$ defined by $G\in\mathcal{J}$

iff for each $x\in G$, there exists a $U\in\mu$ such that $U(x)\subset G$

Proof.

Clearly. $\emptyset\in\mathcal{J}$,

Let $G\alpha\in\mathcal{J}$ $\forall\alpha\in\Lambda$, and choose $x\in\bigcup_{\alpha\in\Lambda}G\alpha$

Then $x\in G\alpha$ for some $\alpha\in\Lambda$

There exists $U \in \mu$ such that $U(x) \subset G_\alpha \subset \bigcup_{\alpha \in \Lambda} G_\alpha$

So $\bigcup_{\alpha \in \Lambda} G_\alpha \in \mathcal{J}$.

Let $G_1$ $G_2$ $\cdots\cdots G_n \in \mathcal{J}$ and $x \in G_1 \cap \cdots\cdots \cap G_n$

Then for each $i = 1, 2 \cdots\cdots n$, $x \in G_i$ and there exists

$U \in \mu$ such that $U(x) \subset G_i$

So, $U(x) \subset \bigcap_{i=1}^{n} G_i$

Hence $\bigcap_{i=1}^{n} G_i \in \mathcal{J}$

Therefore $(S, \mathcal{J})$ is a topological space.

**Proposition 3-3**

Every M-uniformity $\mu$ for S induce a proximity

$\delta = \delta(\mu)$ for S defined by for each $A, B \in 2^S$

$(A, B) \in \delta$

iff $U(A) \cap B \neq \emptyset$ for every $U \in \mu$

iff $(A \times B) \cap U \neq \emptyset$ for every $U \in \mu$

Proof,

Let $A \in 2^S$, then $U(A) \cap \emptyset = \emptyset$ and so $(A, \emptyset) \in \delta$

So, $(P_1)$ is satisfied

Let $x \in S$

$U(x) \cap \{x\} = \{x\} \neq \emptyset$ for $U \in \mu$ and hence $(\{x\}, \{x\}) \in \delta$

So $(P_2)$ is satisfied.

Let $A, B, C \in 2^S$, then

$(C, A \cup B) \in \delta$ iff $U(c) \cap (A \cup B) \neq \emptyset$ for every $U \in \mu$

iff $(U(c)\cap A) \cup (U(c)\cap B) \neq \emptyset$, $^\forall U \in \mu$

iff $U(c)\cap A \neq \emptyset$ or $U(c) \cap B \neq \emptyset$, $^\forall U \in \mu$

iff $(C,A) \in \delta$ or $(C,B) \in \delta$

So $(P_3)$ is satisfied.

$(A,B) \notin \delta$ iff there exists a $U \in \mu$ such that $(A\times B)\cap U = \emptyset$

By$(U_4)$, symmetric there exists $V \in \mu$ such that $V \circ V$ U

Let $E = V^{-1}(B) = V(B)$

then $(A\times E)\cap V = \emptyset$ and$((S-E)\times B)\cap V = \emptyset$

So there exists $E \in 2^S$ such that $(E,A) \notin \delta$

and $(S-E,B) \notin \delta$

Hence $(P_4)$ is satisfied.



It follows from the definition of M-uniformity for s that

$(A,B) \in \delta$ iff $(A\times B)\cap U \neq \emptyset$, $^\forall U \in \mu$

iff $(B\times A)\cap U^{-1} \neq \emptyset$, $^\forall U \in \mu$

iff $(B\times A)\cap U \neq \emptyset$, $^\forall U \in \mu$

iff $(B,A) \in \delta$

So, $(P_5)$ is satisfied

Therefore $\delta$is a proximity for S.

**Definition 3-2**

Let$(S,\mu)$ be a M-uniform space.

Then the proximity $\delta(\mu)$ given by proposition 3-3 and $\mu$ are compatible if $\delta = \delta(\mu)$

**Proposition 3-4**

Let $\delta$ be a binary relation on the power set of S and $\mu$ be a collection of symmetric subsets of $S\times S$ such that $\delta$ and $\mu$ satisfies

$^{\forall}A,B\in 2^S$, $(A,B)\in\delta$

iff $U(A)\cap B\neq\emptyset$ for every $U\in\mu$

iff $(A\times B)\cap U\neq\emptyset$ for every $U\in\mu$

Then $\delta$ and $\mu$ are compatible iff

$\mu$ is a base for an M-uniformity for S.

Proof.

Suppose that $\delta$ and $\mu$ are compatible



Then $\delta=\delta(\mu)$

Let V be an M-uniformity for S containing $\mu$

Then $\mu$ is a base for V

For the converse, let $\mu$ be a base for an M-uniformity for S

It follows from proposition 3-3 that $\delta$ and $\mu$ are compatible

**Proposition 3-5** (Main theorem)

The topological spaces derivable from the proximity spaces are precisely the uniform spaces

Proof.

Let $(S,\mu)$ be a uniform space

If $A, B \in 2^S$; we define $(A,B) \in \delta$

iff $^\forall U \in \mu$ there exist $x \in A, y \in B$ such that $(x, y) \in U$

we claim that $\delta$ is a proximity for S

clearly, $(A, \emptyset) \notin \delta$, $^\forall A \in 2^S$

So $(P_1)$ is satisfied

Since $\triangle \subset U$, $^\forall U \in \mu$, then $(\{x\}, \{x\}) \in \delta$ $^\forall x \in S$

So $(P_2)$ is satisfied

Let $A, B, C \in 2^S$, then $(C, A \cup B) \in \delta$

iff There exists $x \in C, y \in A \cup B$ such that $(x,y) \in U$

iff $(x \in C,\ y \in A)$ or $(x \in C$ or $y \in B)$

such that $(x, y) \in U$ iff $(C,A) \in \delta$ or $(C,B) \in \delta$.



So $(P_3)$ is satisfied.

If $(A,B) \notin \delta$, then there exists $U \in \mu$ such that $^\forall x \in A$, $^\forall y \in B$,

$(x, y) \notin U$

By$(U_4)$, there exists $V \in \mu$ such that $V \circ V \subset U$

Let $E = V^{-1}(B) = \{y : (x, y) \in V^{-1}$ for some $x \in B\}$

Then $(A \times E) \cap V = \emptyset$

since otherwise $(x, y) \in (A \times E) \cap V$

$\Rightarrow$ $x \in A$, $y \in E$, $(y, x) \in V^{-1}$

$\Rightarrow$ $x \in A$, There exists $z \in B$ such that$(y, z) \in V$, $(x, y) \in V$

$\Rightarrow$ $x \in A$, Therefore $(x, z) \in V \circ V \subset U$

We have a contradiction

So $(A\times E)\cap V = \emptyset$

$\Rightarrow (A,E)\notin \delta$

Similarly $((S-E)\times B)\cap V = \emptyset$

$\Rightarrow (S-E,B) \notin \delta$

So $(P_4)$ is satisfied

$(A,B)\in \delta$ iff $^{\forall} U\in\mu$ There exists $x\in A$, $y\in B$ such that $(x,y)\in U$

By$(U_3)$, $(y,x)\in U^{-1}\in\mu$ and so $(B,A)\in\delta$, and conversely.

Hence $(p_5)$ is satisfied

Therefore is a proximity for S induced by the uniformity $\mu$

Suppose that $\delta$ and $\mu$ are compatible

For every pair of subsets A and B of S

define $U(A,B) = X\cdot X-[(A\times B)\cup(B\times A)]$

Let $V = \{U_{A,B}: (A,B) \notin \delta\}$

Then each member of V is clearly symmetric

So $V\in V \Rightarrow V = V^{-1}$

Now, if $(A,B)\notin\delta$, then $U_{A,B}(A)\cap B = \emptyset$

conversely if $U_{C,D}(A)\cap B = \emptyset$ for some pair C.D such that $(C,D)\notin\delta$,

then either $A\subset C$ and $B\subset D$ or $A\subset D$ and $B\subset C$

In either case, $(A,B)\notin \delta$

It follows from proposition 3-3 and 3-4 that

V is a base for a uniformity U and $\delta = \delta(U)$.

# Reference

1. Bejamin T.Sims. Fundamentals of topology Mac Millan co. 1976

2. J. Dugundji, Topology, Allyn and Bacon Inc Boston, 1966

3. C. O. christenson and William L. Voxman "aspects of topology", D Marcal Dekker. Inc, New York and Basel 1977

4. Naimpalky and warrack. "Proximity Spaces". Cambridge at the university PRESS 1970.

5. G. F. Simmons "introduction to topology and Modern analysis. Mcgraw-Hill Book company. Znc. 1963