On the Volume of an N-Surface

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감 사 의 글

이 논문이 완성되기 까지 바쁘신 가운데도 자세한 지도를 하여 주신 현진오 교수님께 감사드리며, 그동안 많은 도움을 주신 수학 교육과의 모든 교수님께 심심한 사의를 표합니다. 아울러 그동안 저에게 사랑과 격려를 아끼지 않으신 주위의 많은 분들께 감사드립니다.

1982 년 6 월 일

文 宇 哲

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국 문 초 록

n차 곡면의 체적에 관하여

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문 우 철

본 논문의 목적은 Level Set의 정의를 사용하여 곡면의 호의 길이, 면적, 체적에 관한 몇가지 성질들을 연구하는 것이다.

1. INTRODUCTION

We can introduce the properties of the volume of an n-surface in R^{n+1} by using well-known definition.

In this paper, we begin with defining a volume of an n-surface in R^{n+1} on the definition of a level set. Then we will prove the properties of the arc length, area and volume of an n-surface in R^{n+1} according to its definition.

For our further discussion, several definitions and notations will be given first of all.

2. DEFINITION AND NOTATION

Given a function $f:U \to R$, where U is an open subset in R^{n+1}.

DEFINITION(2.1). Level sets are the sets $f^{-1}(C)$ defined by $f^{-1}(C) = \{(x_1, x_2, \cdots, x_{n+1}) \in U \mid f(x_1, x_2, \cdots, x_n) = C\}$ for each real number C. The number C is called the height of the level set and $f^{-1}(C)$ is called the level set at height C.

DEFINITION(2.2). A vector field $\mathbf{X}$ on $U \subset R^{n+1}$ is a function which assigns to each point of U a vector at that point. Thus $\mathbf{X}(p) = (p, X(p))$ for some function $X:U \to R^{n+1}$.

DEFINITION(2.3). A parametrized curve $\alpha:I \to R^{n+1}$ is said to be an integral curve of the vector field $\mathbf{X}$ on the open set U in R^{n+1} if $\alpha(t) \in U$ and $\dot{\alpha}(t) = X(\alpha(t))$ for all $t \in I$.

DEFINITION(2.4). The lenght $\|\dot{a}\|:I \to R$ defined by $\|\dot{\alpha}\|(t) = \|\dot{\alpha}(t)\|$ along for all $t \in I$ is called the speed of α.

DEFINITION(2.5). A smooth unit normal vector field on an n-surface S in R^{n+1} is called an orientation on S.

PROPOSITION(2.6). The length of a connected oriented plane curve C can be computed from the formular;

$\ell(C) = \ell(\alpha) = \int_a^b \|\dot{\alpha}(t)\| dt$, where a,b are the end points of I.

PROPOSITION(2.7). If $\beta:\tilde{I} \to R^{n+1}$ is a reparametrization of α, then $\ell(\alpha) = \ell(\beta)$.

DEFINITION(2.8). A parametrized n-surface in $R^{n+k}(k \geq 0)$ is a smooth map $\varphi : U \to R^{n+k}$, where U is a connected open set in R^n, which is regular; i.e. which is such that $d\varphi_p$ is a nonsingular(has rank n) for each $p \in U$.

DEFINITION(2.9) Let $\varphi : U \to R^{n+k}$ be any smooth map, U open in R^n. Then $\mathbf{E}_i (i \in \{1, \cdots, n\})$ denote the tangent vector fields along φ defined by $\mathbf{E}_i(p) = (p, 0, \cdots, 1, \cdots, 0)$, where the 1 is in the $(i+1)$-th spot.

PROPOSITION(2.10). The components of $\mathbf{E}_i$ are the entries in the i-th column of the Jacobian matrix for φ at p:

$$\mathbf{E}_i(p) = (\varphi(p), \frac{\partial \varphi}{\partial u_i}(p)) = (\varphi(p), \frac{\partial \varphi_1}{\partial u_i}(p)), \ldots\ldots (\frac{\varphi_{n+k}}{U_i}(P)),$$

where $\varphi(p) = (\varphi_1(p), \cdots, \varphi_{n+k}(p))$ for $p \in U$.

PROPOSITION(2.11). The $\mathbf{E}_i$ are called the coordinate vector field along φ.

PROPOSITION(2.12). Suppose that $\varphi : U \to R^{n+1}$ is a parametrized n-surface in R^{n+1}. Let $\mathbf{N}(p)$ denote the unique vector at $\varphi(p)$ such that $\mathbf{N}(p) \perp$ image $d\varphi_p$ and

$$\det \begin{bmatrix} \mathbf{E}(p_1) \\ \mathbf{E}(p_2) \\ \vdots \\ \mathbf{E}(P_n) \\ \mathbf{N}(p) \end{bmatrix} > 0 \quad \text{for } p \in U.$$

Then $\mathbf{N}$ is a smooth unit normal vector field along φ.

3. VOLUME OF AN N-SURFACE

In the present section, we prove the properties of the volume of an n-surface in R^{n+1} with its definition.

We begin by defining the volume of an n-surface.

DEFINITION(3.1) The volume of a parametrized n-surface $\varphi : U \to R^{n+1}$ is defined by

$$\mathcal{V}(\varphi) = \int_U \det \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \vdots \\ \mathbf{E}_n \\ \mathbf{N} \end{bmatrix} = \int_U \det \begin{bmatrix} \mathbf{E}_1(u_1, \cdots, u_n) \\ \mathbf{E}_2(u_1, \cdots, u_n) \\ \vdots \\ \mathbf{E}_n(u_1, \cdots, u_n) \\ \mathbf{N}(u_1, \cdots, u_n) \end{bmatrix} du_1, \cdots du_n$$

where $\mathbf{E}_1, \cdots, \mathbf{E}_n$ are the coordinate vector field along φ and $\mathbf{N}$ is the orientation vector field along φ.

In the next theorem, using the above definition, we obtain a formular for volume which does not require calculation of the orientation vector field $\mathbf{N}$.

THEOREM(3.2) Let $\varphi : U \to R^{n+1}$ be a parametrized n-surface Then

$$\mathcal{V}(\varphi) = \int_U \det \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \vdots \\ \mathbf{E}_n \\ \mathbf{N} \end{bmatrix} = \int_U \sqrt{\det(\mathbf{E}_i \cdot \mathbf{E}_j)} \quad .$$

PROOF. Using the definition(3.1),

$$\left[\det\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \vdots\\ \mathbf{E}_n\\ \mathbf{N}\end{bmatrix}\right]^2 = \det\left[\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \vdots\\ \mathbf{E}_n\\ \mathbf{N}\end{bmatrix}(\mathbf{E}_1^t,\ \mathbf{E}_2^t,\ \cdots \mathbf{E}_n^t,\ \mathbf{N}^t)\right]$$

$$= \det\begin{bmatrix}\mathbf{E}_1\cdot\mathbf{E}_1 & \cdots & \mathbf{E}_1\cdot\mathbf{E}_n & 0\\ \mathbf{E}_2\cdot\mathbf{E}_1 & \cdots & \mathbf{E}_2\cdot\mathbf{E}_n & 0\\ \vdots & & \vdots & \\ \mathbf{E}_n\cdot\mathbf{E}_1 & \cdots & \mathbf{E}_n\cdot\mathbf{E}_n & 0\\ 0, & \cdots & 0 & 1\end{bmatrix} = \det(\mathbf{E}_j\cdot\mathbf{E}_j)$$

where $\mathbf{E}_i^t$ is the transpose of $\mathbf{E}_i$ and $i, j \in \{1, 2, \cdots, n\}$.

The formular for the lenght of a parametrized curve $\alpha: I \to R^2$ can be rewritten as follows.

THEOREM(3.3) The length of a parametrized curve $\alpha: I \to R^2$ is $\ell(\alpha) = \int_I \|\dot{\alpha}\| = \int_I \det\begin{pmatrix}\mathbf{E}_1(t)\\ \mathbf{N}(t)\end{pmatrix}$, where α is regular and $\mathbf{N}$ is the orientation vector field along α.

PROOF. Since the velocity field $\dot{\alpha}$ is the coordinate vector field E_1 along the parametrized 1-surface α in R^2 and the vector $\mathbf{E}_1(t)/\|\mathbf{E}_1(t)\|$, $\mathbf{N}$ form an orthogonal basis for the vector space $\mathbf{R}^2_{\alpha(t)}$ $\|a(t)\| = \|\mathbf{E}_1(t)\| \det\begin{bmatrix}\mathbf{E}_1(t)/\|\mathbf{E}_1(t)\|\\ \mathbf{N}(t)\end{bmatrix} = \det\begin{bmatrix}\mathbf{E}_1(t)\\ \mathbf{N}(t)\end{bmatrix}$

By definition(3.1), when $n = 1$, the volume of φ is the length of φ. Moreover, when $n = 2$, the volume of φ is the area of φ.

Using the above theorem, we obtain a useful theorem.

THEOREM(3.4) Let C be a connected oriented plane curve and $\tilde{C}$ be the same curve with opposite orientation.

Then $\ell(C) = \ell(\tilde{C})$

PROOF. Consider parametrized curve $\alpha: I \to C$ oriented by $\mathbf{N}$ and $\tilde{\alpha}: \tilde{I} \to \tilde{C}$ oriented by $-\mathbf{N}$.

Then $\int_I \det \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{N} \end{bmatrix} = \int_{\tilde{I}} \det \begin{bmatrix} \mathbf{E}_1 \\ -\mathbf{N} \end{bmatrix}$

Using the definition(3.1) and the theorem(3.2), we have the following theorem.

THEOREM(3.5) Let $\mathbf{E}_i$ be the n-ply orthogonal system along φ. If a function $f: U \to R$ is a smooth function on the open set $U \subset R^n$ and $f: U \to R^{n+1}$ is defined by $\varphi(u_1, \cdots, u_n) = (u_1, \cdots, u_n, f(u_1, \cdots, u_n))$, then the volume of φ may be expressed in the integral of

$$\sqrt{1 + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial u_i} \right)^2} \quad \text{along } \varphi \text{ on } U.$$

PROOF. Since $\mathbf{E}_i(p) = (p, \frac{\partial \varphi}{\partial u_i}(p)) = (p, 0, \cdots, 1, \cdots, 0, \frac{\partial f}{\partial u_i})$ for $p = (u_1, u_2, \cdots, u_n) \in U$,

$$\mathbf{E}_i \cdot \mathbf{E}_j = \begin{cases} 1 + \left(\frac{\partial f}{\partial u_i}\right)^2 & \text{if } i = j \\ \frac{\partial f}{\partial u_i} \cdot \frac{\partial f}{\partial u_j} = 0 & \text{if } i \neq j \end{cases}$$

Hence

$$\det(\mathbf{E}_i \cdot \mathbf{E}_j) = \left[1+(\frac{\partial f}{\partial u_1})^2\right]\left[1+(\frac{\partial f}{\partial u_2})^2\right]\cdots\left[1+(\frac{\partial f}{\partial u_n})^2\right]$$

$$= 1+\sum_{i=1}^{n}(\frac{\partial f}{\partial u_i})^2.$$

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ABSTRACT

ON THE VOLUME OF AN N-SURFACE

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The purpose of the present paper is to study some properties of the arc length, area and volume of surface by using definition of level set.