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Riemannian foliation admitting a transversal conformal Killing field

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CONTENTS

<Abstract>

Riemannian foliation admitting a transversal conformal Killing field

In this paper, we study the transversal conformal Killing field on a Riemannian foliation. In particular, we study the foliations on a compact Riemannian manifold with a transversal conformal Killing field. Namely, let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transversal Einstein foliation $\mathcal{F}$ and a bundle-like metric g_M. If M admits a transversal conformal Killing field which is not Killing, then $\mathcal{F}$ is transversally isometric to the action of a discrete subgroup of $O(q)$ acting on the q-sphere of constant curvature.

1 Introduction

Let (M, g_M) be a compact Riemannian manifold of dimension $n \geq 2$ and g_M a Riemannian metric. It is well-known ([10]) that if the scalar curvature r of M is positive constant, then M admits a conformal transformation, which is not isometric. Furthermore, if a Riemannian manifold of constant scalar curvature r admits an infinitesimal conformal transformation X with $\theta(X)g_M = 2\phi g_M$, where $\theta(X)$ the Lie derivative and ϕ a function, then ϕ satisfies the equation $\Delta\phi = nc\phi$, where $c = r/n(n-1)$. The existence of such a function might give some informations about the topological structure of the Riemannian manifold. In fact, the following theorems are well-known in M.Obata([11]).

Theorem 1.1 *A compact Einstein manifold of constant scalar curvature r admits a non-constant function ϕ such that $\Delta\phi = nc\phi$ if and only if the manifold is isometric with a sphere $S^n(\sqrt{c})$ with radius $\frac{1}{\sqrt{c}}$ in the $(n+1)$-dimensional Euclidean space.*

Theorem 1.2 *Let M be a compact Einstein manifold of dimension $n \geq 2$ with positive constant scalar curvature r. If M admits a conformal Killing field X with a non-Killing field, then M is isometric with a sphere S^n.*

In this paper, we study the properties of a foliated Riemannian manifold M with constant transversal scalar curvature σ^∇ admitting a transversal conformal Killing field. Moreover, we prove corresponding theorem to Theorem 1.2 for foliation. The corresponding theorem to Theorem 1.1 for foliation was given by J. Lee and K. Richardson([8]). This paper is organized by the following. In Chapter 2, we review the known fact on the foliated Riemannian manifold. In Chapter

3, we study the basic Laplacian. In Chapter 4, we investigate the properties of the transversal conformal Killing field. In Chapter 5, we study the Riemannian foliation admitting a transversal conformal Killing field. In fact, we prove the corresponding theorem to theorem 1.2 for foliation.

2 Riemannian foliation

Let M be a smooth manifold of dimension $p+q$.

Definition 2.1 A codimension q *foliation* $\mathcal{F}$ on M is given by an open cover $\mathcal{U} = (U_i)_{i\in I}$ and for each i, a diffeomorphism $\varphi_i : \mathbb{R}^{p+q} \to U_i$ such that, on $U_i \cap U_j \neq \emptyset$, the coordinate change $\varphi_j^{-1} \circ \varphi_i : \varphi_i^{-1}(U_i \cap U_j) \to \varphi_j^{-1}(U_i \cap U_j)$ has the form

$$\varphi_j^{-1} \circ \varphi_i(x,y) = (\varphi_{ij}(x,y), \gamma_{ij}(y)). \tag{2.1}$$

From Definition 2.1, the manifold M is decomposed into connected submanifolds of dimension p. Each of these submanifolds is called a *leaf* of $\mathcal{F}$. Coordinate patches (U_i, φ_i) are said to be *distinguished* for the foliation $\mathcal{F}$. The tangent bundle L of $\mathcal{F}$ is the subbundle of TM, consisting of all vectors tangent to the leaves of $\mathcal{F}$. The normal bundle Q of $\mathcal{F}$ on M is the quotient bundle $Q = TM/L$. Equivalently, Q appears in the exact sequence of vector bundles

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0. \tag{2.2}$$

If $(x_1, \ldots, x_p; y_1, \ldots, y_q)$ are local coordinates in a distinguished chart U, then the bundle $Q|U$ is framed by the vector fields $\pi\frac{\partial}{\partial y_1}, \ldots, \pi\frac{\partial}{\partial y_q}$. For a vector field $Y \in \Gamma TM$, we denote also $\overline{Y} = \pi Y \in \Gamma Q$.

Definition 2.2 A vector field Y on U is *projectable,* if $Y = \sum_i a_i \frac{\partial}{\partial x_i} + \sum_\alpha b_\alpha \frac{\partial}{\partial y_\alpha}$ with $\frac{\partial b_\alpha}{\partial x_i} = 0$ for all $\alpha = 1, \ldots, q$ and $i = 1, \ldots, p$.

Definition 2.2 means that the functions $b_\alpha = b_\alpha(y)$ are independent of x. Then $\overline{Y} = \sum_\alpha b_\alpha \frac{\bar{\partial}}{\partial y_\alpha}$ with b_α independent of x. This property is preserved under the change of distinguished charts. Note that every projectable vector field preserves the leaves in sense of $[Y, Z] \in \Gamma L$ for any $Z \in \Gamma L$.

Let $V(\mathcal{F})$ be the space of all projectable vector fields on M, i.e.,

$$V(\mathcal{F}) = \{Y \in TM | [Y, Z] \in \Gamma L, \quad \forall Z \in \Gamma L\}. \tag{2.3}$$

An element of $V(\mathcal{F})$ is called an *infinitesimal automorphism* of $\mathcal{F}$. Now we put

$$\bar{V}(\mathcal{F}) = \{\bar{Y} = \pi(Y) \in \Gamma Q | Y \in V(\mathcal{F})\}. \tag{2.4}$$

The *transversal geometry* of a foliation is the geometry infinitesimally modeled by Q, while the tangential geometry is infinitesimally modeled by L. A key fact of the transversal geometry is the existence of the *Bott connection* in Q defined by

$$\overset{\circ}{\nabla}_X s = \pi([X, Y_s]), \quad \forall X \in \Gamma L, \tag{2.5}$$

where $Y_s \in TM$ is any vector field projecting to s under $\pi : TM \to Q$. It is a partial connection along L. The right hand side in (2.5) is independent of the choice of Y_s. Namely, the difference of two such choices is a vector field $X' \in \Gamma L$ and $[X, X'] \in \Gamma L$, which implies $\pi([X, X']) = 0$.

Definition 2.3 A Riemannian metric g_Q on the normal bundle Q of a foliation $\mathcal{F}$ is *holonomy invariant* if

$$\theta(X) g_Q = 0, \quad \forall X \in \Gamma L, \tag{2.6}$$

where $\theta(X)$ is the transversal Lie derivative, which is defined by $\theta(X)s = \pi[X, Y_s]$.

Here $\theta(X)g_Q$ is defined by

$$(\theta(X)g_Q)(s,t) = Xg_Q(s,t) - g_Q(\theta(X)s,t) - g_Q(s,\theta(X)t) \quad \forall s,t \in \Gamma Q.$$

Definition 2.4 A *Riemannian foliation* is a foliation $\mathcal{F}$ with a holonomy invariant transversal metric g_Q. A metric g_M is a *bundle-like* if the induced metric g_Q in Q is holonomy invariant.

The study of a Riemannian foliation was initiated by Reinhart in 1959([14]). A simple example of a Riemannian foliation is given by a nonsingular Killing vector field X on (M, g_M), because $\theta(X)g_M = 0$.

Definition 2.5 An *adapted connection* in Q is a connection restricting along L to the partial Bott connection $\overset{\circ}{\nabla}$.

To show that such connections exist, consider a Riemannian metric g_M on M. Then TM splits orthogonally as $TM = L \oplus L^\perp$. This means that there is a bundle map $\sigma : Q \to L^\perp$ splitting the exact sequence (2.2), i.e., satisfying $\pi \circ \sigma = identity$. This metric g_M on TM is then a direct sum

$$g_M = g_L \oplus g_{L^\perp}.$$

With $g_Q = \sigma^* g_{L^\perp}$, the splitting map $\sigma : (Q, g_Q) \to (L^\perp, g_{L^\perp})$ is a metric isomorphism. Let ∇^M be the Levi-Civita connection associated to the Riemannian metric g_M. Then the adapted connection ∇ in Q is given by([5,15])

$$\nabla_X s = \begin{cases} \overset{\circ}{\nabla}_X s = \pi([X, Y_s]) & \forall X \in \Gamma L, \\ \pi(\nabla^M_X Y_s) & \forall X \in \Gamma L^\perp, \end{cases} \tag{2.7}$$

where $s \in \Gamma Q$ and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $Q \cong L^\perp$. For any connection ∇ in Q, there is a torsion T_∇ defined by

$$T_\nabla(Y, Z) = \nabla_Y \pi(Z) - \nabla_Z \pi(Y) - \pi([Y, Z]) \tag{2.8}$$

for any $Y, Z \in \Gamma TM$. Then we have the following proposition ([15]).

Proposition 2.6 *For any metric g_M on M and the adapted connection ∇ in Q defined by (2.7) the torsion is free, i.e., $T_\nabla = 0$.*

Proof. For any vector fields $X \in \Gamma L$, $Y \in \Gamma TM$, we have

$$T_\nabla(X, Y) = \nabla_X \pi(Y) - \pi([X, Y]) = 0.$$

For any vector fields $Z, Z' \in \Gamma L^\perp$, we have

$$T_\nabla(Z, Z') = \pi(\nabla^M_Z Z') - \pi(\nabla^M_{Z'} Z) - \pi([Z, Z']) = \pi(T_{\nabla^M}(Z, Z')) = 0,$$

where T_{∇^M} is the (vanishing) torsion of ∇^M. Finally the bilinearity and skew symmetry of T_∇ imply the desired result. □

The curvature R^∇ of ∇ is defined by

$$R^\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \forall X,\ Y \in TM. \tag{2.9}$$

From the adapted connection ∇ in Q defined by (2.7), its curvature R^∇ coincides with $\mathring{R}$ for $X, Y \in \Gamma L$, hence $R^\nabla(X, Y) = 0$ for $X, Y \in \Gamma L$. And we have the following proposition ([4,5,15]).

Proposition 2.7 *Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and bundle-like metric g_M with respect to*

$\mathcal{F}$. Let ∇ be the connection defined by (2.7) in Q with curvature R^∇. Then for $X \in \Gamma L$ the following holds:

$$i(X)R^\nabla = \theta(X)R^\nabla = 0. \tag{2.10}$$

By Proposition 2.7, we can define the (transversal) Ricci curvature $\rho^\nabla : \Gamma Q \to \Gamma Q$ and the (transversal) scalar curvature σ^∇ of $\mathcal{F}$ by

$$\rho^\nabla(s) = \sum_a R^\nabla(s, E_a)E_a, \quad \sigma^\nabla = \sum_a g_Q(\rho^\nabla(E_a), E_a), \tag{2.11}$$

where $\{E_a\}_{a=1,\cdots,q}$ is a local orthonormal basic frame of Q.

Definition 2.8 The foliation $\mathcal{F}$ is said to be (transversally) *Einsteinian* if the model space N is Einsteinian, that is,

$$\rho^\nabla = \frac{1}{q}\sigma^\nabla \cdot id \tag{2.12}$$

with constant transversal scalar curvature σ^∇.

Definition 2.9 *The mean curvature vector* $\kappa^\sharp$ of $\mathcal{F}$ is defined by

$$\kappa^\sharp = \pi\Big(\sum_{i=1}^p \nabla^M_{E_i} E_i\Big), \tag{2.13}$$

where $\{E_i\}$ is a local orthonormal basis of L. The foliation $\mathcal{F}$ is said to be *minimal* if $\kappa^\sharp = 0$.

For the later use, we recall the divergence theorem on a foliated Riemannian manifold ([19]).

Theorem 2.10 *Let $(M, g_M, \mathcal{F})$ be a closed, oriented, connected Riemannian manifold with a transversally orientable foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then*

$$\int_M div_\nabla(X) = \int_M g_Q(X, \kappa^\sharp) \tag{2.14}$$

for all $X \in \Gamma Q$, where $div_\nabla(X)$ denotes the transversal divergence of X with respect to the connection ∇ defined by (2.7).

Proof. Let $\{E_i\}$ and $\{E_a\}$ be orthonormal basis of L and Q, respectively. Then for any $X \in \Gamma Q$,

$$\begin{aligned} div(X) &= \sum_i g_M(\nabla^M_{E_i} X, E_i) + \sum_a g_M(\nabla^M_{E_a} X, E_a) \\ &= \sum_i -g_M(X, \pi(\nabla^M_{E_i} E_i)) + \sum_a g_M(\pi(\nabla^M_{E_a} X), E_a) \\ &= -g_Q(X, \kappa^\sharp) + \sum_a g_Q(\nabla_{E_a} X, E_a) \\ &= -g_Q(X, \kappa^\sharp) + div_\nabla(X). \end{aligned}$$

By Green's Theorem on an ordinary manifold M, we have

$$0 = \int_M div(X) = \int_M div_\nabla(X) - \int_M g_Q(X, \kappa^\sharp). \quad \square$$

Corollary 2.11 *If $\mathcal{F}$ is minimal, then we have that for any $X \in \Gamma Q$,*

$$\int_M div_\nabla(X) = 0. \tag{2.15}$$

3 The basic Laplacian

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M.

Definition 3.1 Let $\mathcal{F}$ be an arbitrary foliation on a manifold M. A differential form $\omega \in \Omega^r(M)$ is *basic* if

$$i(X)\omega = 0,\ \theta(X)\omega = 0, \quad \forall X \in \Gamma L. \tag{3.1}$$

In a distinguished chart $(x_1, \ldots, x_p; y_1, \ldots, y_q)$ of $\mathcal{F}$, a basic 1-form w is expressed by

$$\omega = \sum_{a_1 < \cdots < a_r} \omega_{a_1 \cdots a_r} dy_{a_1} \wedge \cdots \wedge dy_{a_r},$$

where the functions $\omega_{a_1 \cdots a_r}$ are independent of x, i.e. $\frac{\partial}{\partial x_i}\omega_{a_1 \cdots a_r} = 0$. Let $\Omega_B^r(\mathcal{F})$ be the set of all basic r-forms on M. The foliation $\mathcal{F}$ is said to be *isoparametric* if $\kappa \in \Omega_B^1(\mathcal{F})$, where κ is a g_Q-dual 1-form $\kappa^\sharp$. Then we have the well-known theorem([9,15]).

Theorem 3.2 *Let $\mathcal{F}$ be an isoparametric Riemannian foliation on M. Then the mean curvature form κ is closed, i.e., $d\kappa = 0$.*

We now define the star operator $\bar{*} : \Omega_B^r(\mathcal{F}) \to \Omega_B^{q-r}(\mathcal{F})$ naturally associated to g_Q. The relationships between $\bar{*}$ and $*$ are characterized by

$$\bar{*}\phi = (-1)^{p(q-r)} * (\phi \wedge \chi_{\mathcal{F}}), \tag{3.2}$$

$$*\phi = \bar{*}\phi \wedge \chi_{\mathcal{F}} \tag{3.3}$$

for $\phi \in \Omega_B^r(\mathcal{F})$, where $\chi_{\mathcal{F}}$ is the characteristic form of $\mathcal{F}$ and $*$ is the Hodge star operator([15]). Then the inner product $< \, , \, >_B$ on $\Omega_B^r(\mathcal{F})$ is defined by

$< \phi, \psi >_B = \phi \wedge \bar{*}\psi \wedge \chi_{\mathcal{F}}$ for any $\phi, \psi \in \Omega_B^r$ and the global inner product is given by

$$\ll \phi, \psi \gg_B = \int_M < \phi, \psi >_B . \tag{3.4}$$

With respect to this scalar product, the adjoint $\delta_B : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r-1}(\mathcal{F})$ of d_B is given by

$$\delta_B \phi = (-1)^{q(r+1)+1} \bar{*}(d_B - \kappa\wedge)\bar{*}\phi. \tag{3.5}$$

Then the *basic Laplacian* is given by

$$\Delta_B = d_B \delta_B + \delta_B d_B. \tag{3.6}$$

Lemma 3.3 ([1,2]) *On the Riemannian foliation $\mathcal{F}$, we have*

$$d_B \phi = \sum_a E^a \wedge \nabla_{E_a} \phi, \quad \delta_B \phi = \sum_a -i(E_a)\nabla_{E_a}\phi + i(\kappa^\sharp)\phi, \tag{3.7}$$

when $\{E_a\}$ is a local orthonormal basic frame on Q and $\{E^a\}$ its g_Q-dual 1-form.

Definition 3.4 For any vector field $Y \in V(\mathcal{F})$, we define an operator $A_Y : \Gamma Q \to \Gamma Q$ as

$$A_Y s = \theta(Y)s - \nabla_Y s. \tag{3.8}$$

Remark. Let $Y_s \in \Gamma TM$ with $\pi(Y_s) = s$. Then it is trivial that

$$A_Y s = -\nabla_{Y_s} \pi(Y). \tag{3.9}$$

So A_Y depends only on $s = \pi(Y)$ and is a linear operator. Moreover, A_Y extends in an obvious way to tensors of any type on Q (see [6] for details). Namely, we can define the following.

Definition 3.5 For any basic 1-form $\phi \in \Omega^1_B(\mathcal{F})$, the operator A_Y is given by

$$(A_Y\phi)(X) = -\phi(A_Y X) \quad \forall X \in \Gamma Q. \tag{3.10}$$

Now, we introduce the operator $\nabla^*_{tr}\nabla_{tr} : \Omega^*_B(\mathcal{F}) \to \Omega^*_B(\mathcal{F})$ as

$$\nabla^*_{tr}\nabla_{tr}\phi = -\sum_a \nabla^2_{E_a,E_a}\phi + \nabla_{\kappa^\sharp}\phi, \tag{3.11}$$

where $\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla^M_X Y}$ for any $X, Y \in TM$. Then we have the following.

Proposition 3.6 ([2]) *On the Riemannian foliation $\mathcal{F}$ on a compact manifold M, the operator $\nabla^*_{tr}\nabla_{tr}$ satisfies*

$$\ll \nabla^*_{tr}\nabla_{tr}\phi_1, \phi_2 \gg_B = \ll \nabla\phi_1, \nabla\phi_2 \gg_B \tag{3.12}$$

for all $\phi_1,\ \phi_2 \in \Omega^*_B(\mathcal{F})$, where $< \nabla\phi_1, \nabla\phi_2 >_B = \sum_a < \nabla_{E_a}\phi_1, \nabla_{E_a}\phi_2 >_B$.

By the straight calculation, we have the following theorem.

Theorem 3.7 *On the Riemannian foliation $\mathcal{F}$, we have*

$$\Delta_B\phi = \nabla^*_{tr}\nabla_{tr}\phi + A_{\kappa^\sharp}\phi + F(\phi) \tag{3.13}$$

for $\phi \in \Omega^r_B(\mathcal{F})$, where $F(\phi) = \sum_{a,b} E^a \wedge i(E_b)R^\nabla(E_b, E_a)\phi$. In particular, if ϕ is a basic 1-form, then $F(\phi)^\sharp = \rho^\nabla(\phi^\sharp)$.

Proof. Fix $x \in M$ and let $\{E_a\}$ be an orthonormal basis for Q with $(\nabla E_a)_x = 0$. Then from (3.7) we have

$$\begin{aligned} d_B\delta_B\phi &= \sum_{a,b}(E^a \wedge \nabla_{E_a})(-i(E_b)\nabla_{E_b}\phi + i(\kappa^\sharp)\phi) \\ &= -\sum_{a,b} E^a \wedge \nabla_{E_a}\{i(E_b)\nabla_{E_b}\phi\} + \sum_a E^a \wedge \nabla_{E_a} i(\kappa^\sharp)\phi \\ &= -\sum_{a,b} E^a \wedge i(E_b)\nabla_{E_a}\nabla_{E_b}\phi + d_B i(\kappa^\sharp)\phi \end{aligned}$$

and

$$\begin{aligned}
\delta_B d_B \phi &= -\sum_{a,b} i(E_b)\nabla_{E_b}\{E^a \wedge \nabla_{E_a}\phi\} + i(\kappa^\sharp) d_B\phi \\
&= -\sum_{a,b} (i(E_b)E^a)\nabla_{E_b}\nabla_{E_a}\phi + i(\kappa^\sharp) d_B\phi \\
&\quad + \sum_{a,b} E^a \wedge i(E_b)\nabla_{E_b}\nabla_{E_a}\phi \\
&= -\sum_{a} \nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b} E^a \wedge i(E_b)\nabla_{E_b}\nabla_{E_a}\phi + i(\kappa^\sharp) d_B\phi.
\end{aligned}$$

Summing up the above two equations, we have

$$\begin{aligned}
\Delta_B \phi &= d_B\delta_B\phi + \delta_B d_B\phi \\
&= d_B i(\kappa^\sharp)\phi + i(\kappa^\sharp) d_B\phi - \sum_a \nabla_{E_a}\nabla_{E_a}\phi \\
&\quad + \sum_{a,b} E^a \wedge i(E_b) R^\nabla(E_b, E_a)\phi \\
&= \theta(\kappa^\sharp)\phi - \sum_a \nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b} E^a \wedge i(E_b) R^\nabla(E_b, E_a)\phi \\
&= -\sum_a \nabla_{E_a}\nabla_{E_a}\phi + F(\phi) + A_{\kappa^\sharp}\phi + \nabla_{\kappa^\sharp}\phi \\
&= -\sum_a \nabla^2_{E_a,E_a}\phi + \nabla_{\kappa^\sharp}\phi + F(\phi) + A_{\kappa^\sharp}\phi \\
&= \nabla^*_{tr}\nabla_{tr}\phi + F(\phi) + A_{\kappa^\sharp}\phi.
\end{aligned}$$

The proof is completed. On the other hand, let ϕ be a basic 1-form and $\phi^\sharp$ its g_Q-dual vector field. Then

$$\begin{aligned}
g_Q(F(\phi), E^c) &= \sum_{a,b} g_Q(E^a \wedge i(E_b) R^\nabla(E_b, E_a)\phi, E^c) \\
&= \sum_b i(E_b) R^\nabla(E_b, E_c)\phi = \sum_b g_Q(R^\nabla(E_b, E_c)\phi^\sharp, E_b) \\
&= \sum_b g_Q(R^\nabla(\phi^\sharp, E_b)E_b, E_c) = g_Q(\rho^\nabla(\phi^\sharp), E_c).
\end{aligned}$$

This yields that for any basic 1-form ϕ, $F(\phi)^\sharp = \rho^\nabla(\phi^\sharp)$. $\square$

From (3.10) and Theorem 3.7, we have the following corollary.

Corollary 3.8 *On the Riemannian foliation, we have that for any $X \in \Gamma Q$*

$$\Delta_B X = \nabla_{tr}^* \nabla_{tr} X + \rho^\nabla(X) - A_{\kappa^\sharp}^t X. \tag{3.14}$$

Lemma 3.9 *Let $\mathcal{F}$ be a Riemannian foliation. For any vector fields $Y, Z \in V(\mathcal{F})$ and $s \in \Gamma Q$, we have*

$$(\theta(Y)\nabla)(Z,s) = R^\nabla(Y,Z)s - (\nabla_Z A_Y)s, \tag{3.15}$$

where $(\theta(Y)\nabla)(Z,s) = \theta(Y)\nabla_Z s - \nabla_{\theta(Y)Z} s - \nabla_Z \theta(Y)s$ and $(\nabla_Z A_Y)s = -\nabla_Z \nabla_{Y_s} \pi(Y) + \nabla_{\nabla_Z s}\pi(Y)$.

Proof. By a direct calculation, we have that for any $Y, Z \in V(\mathcal{F})$

$$(\theta(Y)\nabla)(Z,s) - [\nabla_Y, \nabla_Z]s = (\theta(Y) - \nabla_Y)\nabla_Z s - \nabla_Z(\theta(Y) - \nabla_Y)s - \nabla_{[Y,Z]}s. \quad \square$$

4 Transversal conformal Killing field

Let $\mathcal{F}$ be a Riemannian foliation. For any vector field $Y \in V(\mathcal{F})$ and $X, X' \in \Gamma Q$, we have

$$(\theta(Y)g_Q)(X, X') = g_Q(\nabla_X \bar{Y}, X') + g_Q(X, \nabla_{X'} \bar{Y}). \tag{4.1}$$

Definition 4.1 If a vector field $Y \in V(\mathcal{F})$ satisfies $\theta(Y)g_Q = 0$, then $\bar{Y}$ is called a *transversal Killing field* of $\mathcal{F}$.

Definition 4.2 If a vector field $Y \in V(\mathcal{F})$ satisfies $\theta(Y)g_Q = 2fg_Q$, where f is a basic function on M, then $\bar{Y}$ is called a *transversal conformal Killing field* of $\mathcal{F}$.

Note that if Y is a transversal conformal Killing field of $\mathcal{F}$, i.e., $\theta(Y)g_Q = 2fg_Q$, then

$$f = \frac{1}{q} div_\nabla(\bar{Y}) = -\frac{1}{q}\delta_T \bar{Y}, \quad \text{where } \delta_T \phi = -\sum_a i(E_a)\nabla_{E_a}\phi. \tag{4.2}$$

Lemma 4.3 *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. If $\bar{Y} \in \bar{V}(\mathcal{F})$ is the transversal conformal Killing field, i.e., $\theta(Y)g_Q = 2fg_Q$, then we have*

$$g_Q((\theta(Y)\nabla)(E_a, E_b), E_c) = \delta_b^c f_a + \delta_a^c f_b - \delta_a^b f_c, \tag{4.3}$$

$$(\theta(Y)R^\nabla)(E_a, E_b)E_c = (\nabla_a \theta(Y)\nabla)(E_b, E_c) - (\nabla_b \theta(Y)\nabla)(E_a, E_c), \tag{4.4}$$

$$g_Q((\theta(Y)R^\nabla)(E_a, E_b)E_c, E_d) = \delta_b^d \nabla_a f_c - \delta_b^c \nabla_a f_d - \delta_a^d \nabla_b f_c + \delta_a^c \nabla_b f_d, \tag{4.5}$$

$$(\theta(Y)Ric^\nabla)(E_a, E_b) = -(q-2)\nabla_a f_b + \delta_a^b(\Delta_B f - \kappa^\sharp(f)), \tag{4.6}$$

where $\nabla_a = \nabla_{E_a}$, $Ric^\nabla(E_a, E_b) = g_Q(\rho^\nabla(E_a), E_b)$ and $f_a = \nabla_a f$.

Proof. Fix $x \in M$. Let $\{E_a\}$ be a local orthonormal basic frame of Q such that $(\nabla E_a)(x) = 0$. From (4.1), we have

$$\nabla_{E_a}(\theta(Y)g_Q)(E_b, E_c) = g_Q(\nabla_{E_a}\nabla_{E_b}\bar{Y}, E_c) + g_Q(\nabla_{E_a}\nabla_{E_c}\bar{Y}, E_b). \tag{4.7}$$

Now we prove the equation (4.3). From (4.7) and the 1-st Bianchi identity, we have

$$\begin{aligned}
&\nabla_a(\theta(Y)g_Q)(E_b, E_c) + \nabla_b(\theta(Y)g_Q)(E_a, E_c) - \nabla_c(\theta(Y)g_Q)(E_a, E_b) \\
&= g_Q(R^\nabla(E_a, E_c)\bar{Y}, E_b) + g_Q(R^\nabla(E_b, E_c)\bar{Y}, E_a) + g_Q(R^\nabla(E_a, E_b)\bar{Y}, E_c) \\
&\quad + 2g_Q(\nabla_b\nabla_a\bar{Y}, E_c) \\
&= 2\{g_Q(R^\nabla(\bar{Y}, E_a)E_b, E_c) + g_Q(\nabla_a\nabla_b\bar{Y}, E_c)\}.
\end{aligned}$$

On the other hand, a direct calculation with (3.9) gives

$$\begin{aligned}
g_Q((\nabla_a A_Y)E_b, E_c) &= g_Q(\nabla_a A_Y E_b, E_c) - g_Q(A_Y(\nabla_a E_b), E_c) \\
&= -g_Q(\nabla_a\nabla_b\bar{Y}, E_c).
\end{aligned}$$

From the above two equations and (3.15), we have

$$\begin{aligned}
&\frac{1}{2}\{\nabla_a(\theta(Y)g_Q)(E_b, E_c) + \nabla_b(\theta(Y)g_Q)(E_a, E_c) - \nabla_c(\theta(Y)g_Q)(E_a, E_b)\} \\
&\quad = g_Q((\theta(Y)\nabla)(E_a, E_b), E_c).
\end{aligned} \tag{4.8}$$

Since $\bar{Y}$ is a transversal conformal Killing field, i.e., $\theta(Y)g_Q = 2fg_Q$, we have $\nabla_a\{(\theta(Y)g_Q)(E_b, E_c)\} = 2f_a\delta_b^c$. From (4.8), (4.3) is proved.

From (4.3), we have

$$
\begin{aligned}
&(\nabla_a\theta(Y)\nabla)(E_b,E_c)-(\nabla_b\theta(Y)\nabla)(E_a,E_c)\\
&=\nabla_a(\theta(Y)\nabla)(E_b,E_c)-(\theta(Y)\nabla)(\nabla_aE_b,E_c)-(\theta(Y)\nabla)(E_b,\nabla_aE_c)\\
&\quad-\nabla_b(\theta(Y)\nabla)(E_a,E_c)+(\theta(Y)\nabla)(\nabla_bE_a,E_c)+(\theta(Y)\nabla)(E_a,\nabla_bE_c)\\
&=(-\nabla_a\nabla_{\theta(Y)E_b}E_c+\nabla_{\theta(Y)E_b}\nabla_aE_c+\nabla_{[E_a,\theta(Y)E_b]}E_c)\\
&\quad+(-\nabla_{\theta(Y)E_a}\nabla_bE_c+\nabla_b\nabla_{\theta(Y)E_a}E_c+\nabla_{[\theta(Y)E_a,E_b]}E_c)\\
&\quad+(-\nabla_a\nabla_b\theta(Y)E_c+\nabla_b\nabla_a\theta(Y)E_c+\nabla_{\nabla_aE_b}\theta(Y)E_c-\nabla_{\nabla_bE_a}\theta(Y)E_c)\\
&\quad+(\theta(Y)(\nabla_a\nabla_bE_c)-\theta(Y)(\nabla_b\nabla_aE_c)-\theta(Y)(\nabla_{\nabla_aE_b}E_c)+\theta(Y)(\nabla_{\nabla_bE_a}E_c))\\
&=-R^\nabla(E_a,\theta(Y)E_b)E_c-R^\nabla(\theta(Y)E_a,E_b)E_c-R^\nabla(E_a,E_b)\theta(Y)E_c\\
&\quad+\theta(Y)R^\nabla(E_a,E_b)E_c\\
&=(\theta(Y)R^\nabla)(E_a,E_b)E_c,
\end{aligned}
$$

which proves (4.4). The equation (4.5) is trivial from (4.3) and (4.4). Now we prove the equation (4.6). Since

$$\theta(Y)g_Q(R^\nabla(E_c,E_a)E_b,E_c)=\nabla_Yg_Q(R^\nabla(E_c,E_a)E_b,E_c)$$

and

$$
\begin{aligned}
g_Q(\nabla_{R^\nabla(E_c,E_a)E_b}\bar{Y},E_c)&=g_Q(\nabla_d\bar{Y},E_c)g_Q(R^\nabla(E_c,E_a)E_b,E_d)\\
&=g_Q(R^\nabla(\nabla_d\bar{Y},E_a)E_b,E_d)\\
&=-g_Q(R^\nabla(\theta(Y)E_d,E_a)E_b,E_d),
\end{aligned}
$$

The proof is completed from (4.5). □

From equation (4.6), we have the following lemma.

Lemma 4.4 *Under the same assumption as in Lemma 4.3, if $\bar{Y} \in \bar{V}(\mathcal{F})$ is the transversal conformal Killing field, i.e., $\theta(Y)g_Q = 2fg_Q$, then*

$$\theta(Y)\sigma^{\nabla} = 2(q-1)(\Delta_B f - \kappa^{\sharp}(f)) - 2f\sigma^{\nabla}. \tag{4.9}$$

Proof. Equation (4.6) implies that

$$\begin{aligned}\theta(Y)\sigma^{\nabla} =& \sum_a \theta(Y)Ric^{\nabla}(E_a, E_a)\\ =& \sum_a (\theta(Y)Ric^{\nabla})(E_a, E_a) + 2\sum_a Ric^{\nabla}(\theta(Y)E_a, E_a)\\ =& 2(q-1)(\Delta_B f - \kappa^{\sharp}(f)) + 2\sum_a Ric^{\nabla}(\theta(Y)E_a, E_a).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}2f\sigma^{\nabla} =& 2f\sum_a g_Q Ric^{\nabla}(E_a, E_a) = \sum_a (\theta(Y)g_Q)(\rho^{\nabla}(E_a), E_a)\\ =& \sum_a g_Q(\nabla_{\rho^{\nabla}(E_a)}\bar{Y}, E_a) + g_Q(\nabla_{E_a}\bar{Y}, \rho^{\nabla}(E_a)).\end{aligned}$$

Since $g_Q(\nabla_{\rho^{\nabla}(E_a)}\bar{Y}, E_a) = g_Q(\rho^{\nabla}(E_a), E_c)g_Q(\nabla_c\bar{Y}, E_a) = g_Q(\nabla_{E_c}\bar{Y}, \rho^{\nabla}(E_c))$. (4.9) is proved. □

Now we define the tensors G^{∇} and Z^{∇} respectively by

$$G^{\nabla}(X) = \rho^{\nabla}(X) - \frac{\sigma^{\nabla}}{q}X, \tag{4.10}$$

$$Z^{\nabla}(X,Y)Z = R^{\nabla}(X,Y)Z - \frac{\sigma^{\nabla}}{q(q-1)}(g_Q(Y,Z)X - g_Q(X,Z)Y) \tag{4.11}$$

for any fields $X, Y, Z \in \Gamma Q$. We can easily verify the following lemma.

Lemma 4.5 *Under the same assumption as in Lemma 4.3, the following hold.*

$$TrG^{\nabla} = 0, \quad \sum_a Z^{\nabla}(X, E_a)E_a = G^{\nabla}(X) \quad \forall X \in \Gamma Q, \tag{4.12}$$

$$|G^{\nabla}|^2 = |\rho^{\nabla}|^2 - \frac{\sigma^{\nabla}}{q}, \quad |Z^{\nabla}|^2 = |R^{\nabla}|^2 - \frac{2(\sigma^{\nabla})^2}{q(q-1)}. \tag{4.13}$$

Proof. From (4.10) and (4.11), (4.12) is trivial. From (4.11), we have

$$
\begin{aligned}
|G^\nabla|^2 =& \sum_a g_Q(G^\nabla(E_a), G^\nabla(E_a)) \\
=& \sum_a g_Q(\rho^\nabla(E_a) - \frac{\sigma^\nabla}{q}E_a, \rho^\nabla(E_a) - \frac{\sigma^\nabla}{q}E_a) \\
=& |\rho^\nabla|^2 - \frac{(\sigma^\nabla)^2}{q}.
\end{aligned}
$$

and from (4.12), we get

$$
\begin{aligned}
|Z^\nabla|^2 =& \sum_{a,b,c} g_Q(Z^\nabla(E_a,E_b)E_c, Z^\nabla(E_a,E_b)E_c) \\
=& |R^\nabla|^2 - \frac{2\sigma^\nabla}{q(q-1)} \sum_{a,b,c} \{g_Q(R^\nabla(E_a,E_c)E_c,E_a) - g_Q(R^\nabla(E_c,E_b)E_c,E_b)\} \\
&+ \frac{2\sigma^\nabla}{q^2(q-1)^2} \sum_{a,b} (\delta_a^a\delta_b^b - \delta_a^b\delta_a^b) \\
=& |R^\nabla|^2 - \frac{2(\sigma^\nabla)^2}{q(q-1)}. \quad \square
\end{aligned}
$$

Lemma 4.6 *On the Riemannian foliation $\mathcal{F}$, we have*

$$
\delta_T G^\nabla = -\frac{q-2}{2q} d_B \sigma^\nabla. \tag{4.14}
$$

If σ^∇ is a constant scalar curvature, then $\delta_T G^\nabla = 0$.

Proof. Since $Y(\sigma^\nabla) = 2\sum_a g_Q((\nabla_{E_a}\rho^\nabla)(Y), E_a)$ for any $Y \in \Gamma Q$, we have

$$
\begin{aligned}
\delta_T G^\nabla =& -\sum_a (\nabla_{E_a} G^\nabla)(E_a) = -\sum_a \nabla_{E_a} G^\nabla(E_a) \\
=& -\sum_a \nabla_{E_a}\rho^\nabla(E_a) + \frac{1}{q}\sum_a (\nabla_{E_a}\sigma^\nabla)E^a \\
=& -\frac{1}{2} d_B\sigma^\nabla + \frac{1}{q} d_B\sigma^\nabla = -\frac{q-2}{2q} d_B\sigma^\nabla. \quad \square
\end{aligned}
$$

Lemma 4.7 *Under the same assumption as in Lemma 4.3, if $\bar{Y} \in \bar{V}(\mathcal{F})$ is the transversal conformal Killing field, i.e., $\theta(Y)g_Q = 2fg_Q$, then*

$$(\theta(Y)G^{\nabla})(E_a, E_b) = -(q-2)\{\nabla_a f_b + \frac{1}{q}(\Delta_B f - \kappa^{\sharp}(f))\delta_a^b\}, \tag{4.15}$$

$$\begin{aligned} g_Q((\theta(Y)Z^{\nabla})(E_a, E_b)E_c, E_d) =& \delta_b^d \nabla_a f_c - \delta_b^c \nabla_a f_d - \delta_a^d \nabla_b f_c + \delta_a^c \nabla_b f_d \\ & - \frac{2}{q}(\Delta_B f - \kappa^{\sharp}(f))(\delta_a^d \delta_b^c - \delta_b^d \delta_a^c). \end{aligned} \tag{4.16}$$

Proof. First, (4.15) is trivial from (4.6) and (4.9). On the other hand, since

$$\begin{aligned} &(\theta(Y)Z^{\nabla})(E_a, E_b)E_c \\ &= \theta(Y)Z^{\nabla}(E_a, E_b)E_c - Z^{\nabla}(\theta(Y)E_a, E_b)E_c - Z^{\nabla}(E_a, \theta(Y)E_b)E_c \\ &\quad - Z^{\nabla}(E_a, E_b)\theta(Y)E_c \\ &= (\theta(Y)R^{\nabla})(E_a, E_b)E_c - \frac{1}{q(q-1)}(\theta(Y)\sigma^{\nabla})(\delta_b^c E_a - \delta_a^c E_b) \\ &\quad - \frac{2f\sigma^{\nabla}}{q(q-1)}(\delta_b^c E_a - \delta_a^c E_b), \end{aligned}$$

(4.16) is proved from (4.5) and (4.9).

5 Riemannian foliation admitting a transversal conformal Killing field

Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M.

Lemma 5.1 ([7]) *For any basic function f on M, it holds that*

$$\int_M \Delta_B f = 0. \tag{5.1}$$

Proposition 5.2 *If f is a basic function on M such that $\Delta_B f = \lambda f$, then*

$$\Delta_B d_B f = \lambda d_B f. \tag{5.2}$$

Proof. $\Delta_B d_B f = d_B \Delta_B f = d_B \lambda f = \lambda d_B f.$ □

Proposition 5.3 *If M has a constant transversal scalar curvature $\sigma^\nabla (\neq 0)$ and admits a transversal conformal Killing field $\bar{Y}$ with $\theta(Y)g_Q = 2fg_Q$, $f \neq 0$, then*

$$\Delta_B f = \frac{\sigma^\nabla}{q-1} f + \kappa^\sharp(f) \tag{5.3}$$

and consequently

$$\int_M f = -\frac{q-1}{\sigma^\nabla} \int_M \kappa^\sharp(f). \tag{5.4}$$

Proof. Since σ^∇ is a constant, Lemma 4.4 implies that

$$2(q-1)(\Delta_B f - \kappa^\sharp(f)) - 2f\sigma^\nabla = 0,$$

which proves (5.3). On the other hand, (5.4) is followed from

$$0 = \int_M \Delta_B f = \frac{\sigma^\nabla}{q-1} \int_M f + \int_M \kappa^\sharp(f). \quad \square$$

Proposition 5.4 *Under the same assumption as in proposition 5.3, the following holds.*

$$\int_M |\nabla f|^2 = \frac{\sigma^\nabla}{q-1}\int_M f^2 + \frac{1}{2}\int_M \kappa^\sharp(f)f. \tag{5.5}$$

Proof. By a direct calculation, we have

$$\frac{1}{2}\Delta_B f^2 = (\Delta_B f)f - |\nabla f|^2 = \frac{\sigma^\nabla}{q-1}f^2 + \kappa^\sharp(f)f - |\nabla f|^2.$$

By Lemma 5.1, we have

$$0 = \int_M \frac{1}{2}\Delta_B f^2 = \frac{\sigma^\nabla}{q-1}\int_M f^2 + \int_M \kappa^\sharp(f)f - \int_M |\nabla f|^2. \quad \square$$

Theorem 5.5 ([7]) *On the Riemannian foliation $\mathcal{F}$ on M, we have*

$$\begin{aligned}\int_M \{g_Q(\Delta_B X, X) - 2g_Q(\rho^\nabla(X), X) - \frac{1}{2}|\theta(X)g_Q + \frac{2}{q}(\delta_T X)|^2 + \frac{q-2}{q}(\delta_T X)^2 \\ + g_Q(A_{\kappa^\sharp}X, X) - div_\nabla(A_X X) - div_\nabla(div_\nabla(X)X)\} = 0\end{aligned} \tag{5.6}$$

for $X \in \Gamma Q$.

Lemma 5.6 *On the Riemannian foliation $\mathcal{F}$ on M, if $X \in \bar{V}(\mathcal{F})$ satisfies $g_Q(X, \kappa^\sharp) = 0$, then*

$$\int_M \{g_Q(A_{\kappa^\sharp}X, X) + div_\nabla(A_X X)\} = 0. \tag{5.7}$$

Proof. The divergence theorem with (3.9) implies

$$\begin{aligned}&\int_M g_Q(A_{\kappa^\sharp}X, X) + \int_M div_\nabla(A_X X) \\ &= \int_M g_Q(A_{\kappa^\sharp}X, X) + \int_M g_Q(A_X X, \kappa^\sharp) \\ &= -\int_M g_Q(\nabla_X \kappa^\sharp, X) - \int_M g_Q(\nabla_X X, \kappa^\sharp) \\ &= -\int_M X g_Q(X, \kappa^\sharp) = 0. \quad \square\end{aligned}$$

Corollary 5.7 *On the Riemannian foliation* $\mathcal{F}$ *on* M*, if* $X \in \bar{V}(\mathcal{F})$ *satisfies* $g_Q(X, \kappa^\sharp) = 0$*, then*

$$\int_M \{g_Q(\Delta_B X, X) - 2Ric^\nabla(X, X) + \frac{q-2}{q} g_Q(d_B \delta_T X, X) + 2g_Q(A_{\kappa^\sharp} X, X) - \frac{1}{2}|\theta(X) g_Q + \frac{2}{q}(\delta_T X)|^2\} = 0. \tag{5.8}$$

In particular, if $X = d_B f$ *for some basic function* f *with* $\kappa^\sharp(f) = 0$*, then*

$$\int_M \{g_Q(\Delta_B d_B f, d_B f) - 2Ric^\nabla(d_B f, d_B f) + \frac{q-2}{q} g_Q(d_B \Delta_B f, d_B f) + 2g_Q(A_{\kappa^\sharp} d_B f, d_B f) - 2|\nabla\nabla f + \frac{1}{q}(\Delta_B f)|^2\} = 0. \tag{5.9}$$

Proof. For the proof of (5.9), it is sufficient to prove that $\theta(d_B f) g_Q = 2\nabla\nabla f$. From (4.1)

$$(\theta(d_B f) g_Q)(E_a, E_b) = g_Q(\nabla_a d_B f, E_b) + g_Q(\nabla_b d_B f, E_a). \tag{5.10}$$

Since

$$\begin{aligned} g_Q(\nabla_a d_B f, E_b) &= \sum_c g_Q(\nabla_a (\nabla_c f) E_c, E_b) \\ &= \sum_c (\nabla_a \nabla_c f) g_Q(E_c, E_b) = \nabla_a \nabla_b f, \end{aligned}$$

from (5.10), we have $\theta(d_B f) g_Q = 2\nabla\nabla f$. □

Corollary 5.8 *On the Riemannian foliation* $\mathcal{F}$ *on* M*, if a basic function* f *satisfies* $\Delta_B f = \lambda f (\lambda = constant)$ *with* $\kappa^\sharp(f) = 0$*, then*

$$\int_M \{\frac{q-1}{q}\lambda |d_B f|^2 - Ric^\nabla(d_B f, d_B f) + g_Q(A_{\kappa^\sharp} d_B f, d_B f) - |\nabla\nabla f + \frac{\lambda}{q} f g_Q|^2\} = 0.$$

Proof. Let $X = d_B f$. From (5.2) and (5.9), it is trivial. □

Corollary 5.9 *For any transversal conformal Killing field $\bar{Y}$ such that $\theta(Y)g_Q = 2fg_Q$ with $\kappa^\sharp(f) = 0$, we have*

$$\int_M \{Ric^\nabla(d_Bf, d_Bf) - \frac{1}{q}\sigma^\nabla|d_Bf|^2 - g_Q(A_{\kappa^\sharp}d_Bf, d_Bf) + |\nabla\nabla f + \frac{\sigma^\nabla}{q(q-1)}fg_Q|^2\} = 0.$$

Proof. From (5.3) and corollary 5.8, it is trivial. $\square$

Proposition 5.10 *Let $(M, g_M, \mathcal{F})$ be a closed Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 3$ and a bundle-like metric g_M. Assume that M has constant transversal scalar curvature σ^∇ and admits a transversal conformal Killing field $\bar{Y}$ such that $\theta(Y)g = 2fg (f \neq 0)$. Then we have*

$$\int_M G^\nabla(d_Bf, d_Bf) = \int_M [\frac{1}{q-2}(2f^2|G^\nabla|^2 + \frac{1}{2}f\theta(Y)|G^\nabla|^2) + g_Q(G^\nabla(fd_Bf), \kappa^\sharp)] \tag{5.11}$$

Proof. To prove this integral formula, we first compute $\theta(Y)|G^\nabla|^2$. Since

$$\begin{aligned}
&g_Q(G^\nabla(\theta(Y)E_a, E_b), G^\nabla(E_a, E_b)) \\
&= g_Q(\theta(Y)E_a, E_c)g_Q(G^\nabla(E_c, E_b), G^\nabla(E_a, E_b)) \\
&= (-2fg_Q(E_a, E_c) - g_Q(E_a, \theta(Y)E_c))g_Q(G^\nabla(E_c, E_b), G^\nabla(E_a, E_b)) \\
&= -2fg_Q(G^\nabla(E_a, E_b), G^\nabla(E_a, E_b)) - g_Q(G^\nabla(E_c, E_a), G^\nabla(\theta(Y)E_c, E_b)),
\end{aligned}$$

we have $\sum_{a,b} g_Q(G^\nabla(\theta(Y)E_a, E_b), G^\nabla(E_a, E_b)) = -f|G^\nabla|^2$.
Similarly $\sum_{a,b} g_Q(G^\nabla(E_a, \theta(Y)E_b), G^\nabla(E_a, E_b)) = -f|G^\nabla|^2$.
Then we have

$$\begin{aligned}
\theta(Y)|G^\nabla|^2 &= \sum_{a,b} \theta(Y)g_Q(G^\nabla(E_a, E_b), G^\nabla(E_a, E_b)) \\
&= \sum_{a,b} \nabla_Y g_Q(G^\nabla(E_a, E_b), G^\nabla(E_a, E_b))
\end{aligned}$$

$$
\begin{aligned}
=&2\sum_{a,b} g_Q(\nabla_Y G^\nabla(E_a,E_b), G^\nabla(E_a,E_b))\\
=&2\sum_{a,b} g_Q(\theta(Y)G^\nabla(E_a,E_b), G^\nabla(E_a,E_b))\\
=&2\sum_{a,b} g_Q((\theta(Y)G^\nabla)(E_a,E_b), G^\nabla(E_a,E_b))\\
&+2\sum_{a,b} g_Q(G^\nabla(\theta(Y)E_a,E_b), G^\nabla(E_a,E_b))\\
&+2\sum_{a,b} g_Q(G^\nabla(E_a,\theta(Y)E_b), G^\nabla(E_a,E_b))\\
=&-2(q-2)g_Q(\nabla\nabla f, G^\nabla) - 4f|G^\nabla|^2,
\end{aligned}
$$

which implies

$$
g_Q(G^\nabla,\nabla\nabla f) = -\frac{2}{q-2}f|G^\nabla|^2 - \frac{1}{2(q-2)}\theta(Y)|G^\nabla|^2. \tag{5.12}
$$

On the other hand,

$$
\begin{aligned}
-\delta_T\{G^\nabla(fd_Bf)\} =&\sum_a g_Q(\nabla_a(G^\nabla(fd_Bf)), E_a)\\
=&\sum_{a,b} g_Q(\nabla_a(fE_b(f)G^\nabla(E_b)), E_a)\\
=&\sum_{a,b} g_Q(G^\nabla(\nabla_a fE_a), E_b(f)E_b)\\
&+f\sum_{a,b} g_Q(\nabla_a\nabla_b f, G^\nabla(E_b)E_a)\\
=&G^\nabla(d_Bf,d_Bf) + fg_Q(\nabla\nabla f, G^\nabla).
\end{aligned} \tag{5.13}
$$

Thus, from (5.12) and (5.13),

$$
-\delta_T\{G^\nabla(fd_Bf)\} = G^\nabla(d_Bf,d_Bf) - \frac{1}{q-2}(2f^2|G^\nabla|^2 + \frac{1}{2}f\theta(Y)|G^\nabla|^2).
$$

Since $-\int_M \delta_T\{G^\nabla(fd_Bf)\} = \int_M g_Q(G^\nabla(fd_Bf),\kappa^\sharp)$, we have (5.11). □

Proposition 5.11 *Under the same assumptions as in Proposition 5.10, we have*

$$\int_M G^\nabla(d_B f, d_B f) = \int_M [\frac{1}{2} f^2 |Z^\nabla|^2 + \frac{1}{8} f\theta(Y)|Z^\nabla|^2 + g_Q(G^\nabla(f d_B f), \kappa^\sharp)]. \quad (5.14)$$

Proof. To prove this integral formula, we first compute $\theta(Y)|Z^\nabla|^2$. From definition and 2-nd equation of (4.12), we have

$$\begin{aligned}
&\sum_{a,b,c} g_Q((\theta(Y)Z^\nabla)(E_a, E_b)E_c, Z^\nabla(E_a, E_b)E_c) \\
&= \sum_{a,b,c,d} g_Q((\theta(Y)Z^\nabla)(E_a, E_b)E_c, E_d) g_Q(Z^\nabla(E_a, E_b)E_c, E_d) \\
&= -4 \sum_{a,b,c} \nabla_a f_c g_Q(Z^\nabla(E_a, E_b)E_b, E_c) \\
&= -4 \sum_{a,c} \nabla_a f_c g_Q(G^\nabla(E_a), E_c) = -4 g_Q(\nabla\nabla f, G^\nabla)
\end{aligned}$$

and

$$\begin{aligned}
&g_Q(Z^\nabla(\theta(Y)E_a, E_b)E_c, Z^\nabla(E_a, E_b)E_c) \\
&= g_Q(Z^\nabla(E_d, E_b)E_c, Z^\nabla(E_a, E_b)E_c) g_Q(\theta(Y)E_a, E_d) \\
&= \{-2f g_Q(E_a, E_d) - g_Q(E_a, \theta(Y)E_d)\} g_Q(Z^\nabla(E_d, E_b)E_c, Z^\nabla(E_a, E_b)E_c) \\
&= -2f g_Q(Z^\nabla(E_a, E_b)E_c, Z^\nabla(E_a, E_b)E_c) - g_Q(Z^\nabla(\theta(Y)E_d, E_b)E_c, Z^\nabla(E_a, E_b)E_c).
\end{aligned}$$

Therefore $\sum_{a,b,c} g_Q(Z^\nabla(\theta(Y)E_a, E_b)E_c, Z^\nabla(E_a, E_b)E_c) = -f|Z^\nabla|^2$. Then we have

$$\begin{aligned}
\theta(Y)|Z^\nabla|^2 &= \sum_{a,b,c} \theta(Y) g_Q(Z^\nabla(E_a, E_b)E_c, Z^\nabla(E_a, E_b)E_c) \\
&= \sum_{a,b,c} (\theta(Y) g_Q)(Z^\nabla(E_a, E_b)E_c, Z^\nabla(E_a, E_b)E_c) \\
&\quad + 2 \sum_{a,b,c} g_Q(\theta(Y) Z^\nabla(E_a, E_b)E_c, Z^\nabla(E_a, E_b)E_c)
\end{aligned}$$

$$
\begin{aligned}
=&2\sum_{a,b,c} f g_Q(Z^\nabla(E_a,E_b)E_c, Z^\nabla(E_a,E_b)E_c)\\
&+2\sum_{a,b,c} g_Q((\theta(Y)Z^\nabla)(E_a,E_b)E_c, Z^\nabla(E_a,E_b)E_c)\\
&+2\sum_{a,b,c} g_Q(Z^\nabla(\theta(Y)E_a,E_b)E_c, Z^\nabla(E_a,E_b)E_c)\\
&+2\sum_{a,b,c} g_Q(Z^\nabla(E_a,\theta(Y)E_b)E_c, Z^\nabla(E_a,E_b)E_c)\\
&+2\sum_{a,b,c} g_Q(Z^\nabla(E_a,E_b)\theta(Y)E_c, Z^\nabla(E_a,E_b)E_c)\\
=&-8g_Q(\nabla\nabla f, G^\nabla) - 4f|Z^\nabla|^2,
\end{aligned}
$$

which implies

$$
g_Q(G^\nabla, \nabla\nabla f) = -\frac{1}{2}f|Z^\nabla|^2 - \frac{1}{8}\theta(Y)|Z^\nabla|^2. \tag{5.15}
$$

Thus, from (5.13),

$$
-\delta_T\{G^\nabla(fd_Bf)\} = G^\nabla(d_Bf, d_Bf) - \frac{1}{2}f^2|Z^\nabla|^2 - \frac{1}{8}f\theta(Y)|Z^\nabla|^2.
$$

Hence we have (5.14). □

Theorem 5.12 ([8]) (Generalized Lichnerowicz-Obata theorem). *Let $(M,\mathcal{F})$ be a codimension-q Riemannian foliation on a closed, connected Riemannian manifold. Suppose that there exists a positive constant a such that the transversal Ricci curvature satisfies $\rho^\nabla(X) \geq a(q-1)X$ for every $X \in N\mathcal{F}$. Then the smallest nonzero eigenvalue λ_B of the basic Laplacian satisfies*

$$
\lambda_B \geq aq.
$$

The equality holds if and only if:

(1) $(M,\mathcal{F})$ is transversally isometric to the action of a discrete subgroup of

$O(q)$ acting on the q-sphere of constant curvature a. Thus, there are at least two closed leaves (the poles).

(2) If we choose the metric on M so that the mean curvature form is basic, then the mean curvature of the foliation is zero (the foliation is minimal).

Theorem 5.13 *Let $(M, g_M, \mathcal{F})$ be a closed Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M. If $\mathcal{F}$ is transversally Einsteinian, then the followings are equivalent:*

(1) $\mathcal{F}$ is transversally isometric to the action of a discrete subgroup of $O(q)$ acting on the q-sphere of constant curvature c.

(2) $\mathcal{F}$ admits a non-constant basic function f with $\kappa^\sharp(f) = 0$ such that

$$\Delta_B f = cfq.$$

Proof. It is trivial from the generalized Obata theorem. □

Theorem 5.14 *Under the same assumption as theorem 5.13, if M admits a transversal conformal Killing field $\bar{Y} \in \Gamma Q$ such that $\theta(Y)g_Q = 2fg_Q (f \neq 0)$ with $\kappa^\sharp(f) = 0$, then $\mathcal{F}$ is transversally isometric to the action of a discrete subgroup of $O(q)$ acting on the q-sphere of constant curvature c.*

Proof. Let $\bar{Y}$ be a transversal conformal Killing field such that $\theta(Y)g_Q = 2fg_Q$. From (5.3), we have

$$\Delta_B f = \frac{\sigma^\nabla}{(q-1)} f.$$

If we put $c = \frac{\sigma^\nabla}{q(q-1)}$, then this equation satisfies theorem 5.13 (2). The proof is completed. □

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<국문 초록>

횡단적 공형 Killing장을 갖는 엽층적 리만다양체

본 논문에서는 엽층적 리만다양체상에서의 횡단적 공형 Killing장에 대해 다루었다. 특히, 횡단적 공형 Killing장을 갖는 컴팩트 리만다양체상에서 엽층구조들을 다루었다. 즉, 횡단적 Einstein 엽층구조 $\mathcal{F}$와 bundle-like 거리함수 g_M을 갖는 컴팩트 리만다양체 $(M, g_M, \mathcal{F})$가 횡단적 Killing장이 아닌, 횡단적 공형 Killing장을 가질 때 엽층 $\mathcal{F}$는 횡단적으로 q차원의 구와 동형이 된다.

감사의 글

수학이 좋다는 마음 하나만으로 대학원에 들어와서 힘든 적도 많았지만 새로운 내용을 배울 때마다 흥미로웠습니다. 솔직히 논문 쓰는 게 이렇게 힘들 줄은 상상도 못했습니다. 지도교수님께서 안계셨다면 시작도 못했을 것입니다. 이렇게 논문이 완성된 걸 보고 있자니 ‘드디어 끝이구나.’ 라는 생각보다 ‘이제 진짜 시작이구나!’ 라는 생각이 듭니다. 정승달 교수님 정말 감사합니다! 교수님의 가르침에 수학이 더욱 좋아졌습니다. 공부 열심히 하겠습니다.

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