Scalar Curvatures of Left Invariant Metrics on a Lie Group

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Hwang Soon - Ik

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좌불변거리가 주어진 리군에서의 스칼라곡률

黃 淳 翼

濟州大學校 教育大學院 數學教育專攻

(指導教授 玄 進 五)

좌불변거리가 주어진 리군에서 스칼라곡률의 부호가 항상 음이 아닐 조건과 항상 양이 될 조건에 대하여 연구하여 정리화하였다.

Scalar Curvatures of Left Invariant Metrics on a Lie Group

1. Introduction

When studying relationships between curvature of a complete Riemannian manifold and other topological or geometric properties, it is useful to have many examples. This paper will give aboundant good curvature properties of a Lie group equipped with a Riemannian metric invariant under left translations. And they give us many good examples of a Riemannian manifold.

In this paper, we will pay attention to the scalar curvature of left invariant metrics on a Lie group.

2. Preliminaries

Some basic concepts of Differential Geometry will be stated here. The object of section 2 is to give a rapid outline of some basic concepts of Riemannian Geometry which will be needed later.

Let M be a Riemannian manifold with a particul-ar Riemannian metrc g also denoted by $\langle\ \ \rangle$. And let TM and TM_p denote its tangent bundle and tangent space at p. If $f:M\to N$ is a smooth map, f_*

will denote its differential of tangent bundles.

Definition. An affine connection at a point $p \in M$ is a function which assigns to each tangent vector $X_p \in TM_p$ and to each vector field Y a new tangent vector

$$X_p \vdash Y \in TM_p$$

called the covariant derivative of Y in the direction X_p. This is required to be bilinear as a function of X_p and Y. Furthermore, if $f: M \to R$ is a real-valued function, and if fY denotes the vector field

$$(fY)_p = f(p)Y_p$$

then $\vdash$ is required to satisfy the identity

$$X_p \vdash (fY) = (X_p f)Y_p + f(p)X_p \vdash Y.$$

A global affine connection on M is a function which assigns to each p M an affine connection at p, satisfying the following smoothness condition: if X and Y are smooth vector fields on M then the vector field $X \vdash Y$, defined by the identity

$$(X \vdash Y)_p = X_p \vdash_p Y ,$$

must also be smooth.

Definition. Let c be a parametrized curve from the real numbers to M. A vector field V along the curve c is a function which assigns to each $t \in R$ a tangent vector

$$V_t \in TM_{c(t)} .$$

This is required to be smooth in the following sense: for any smooth function f on M the corres-pondence

$$t \to V_t f$$

should define a smooth function on R.

Definition. Let M be a smooth Riemannian manifold with an affine connection. Any vector field V along c determines a new vector field $\frac{DV}{dt}$ along c called the covariant derivative of V. The operation $V \to \frac{DV}{dt}$ is characterized by the following three axioms:

a) $\frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$.

b) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$ for any smooth function f on R.

c) If V is induced by a vector field Y on M, that is, if $V_t = Y_{c(t)}$ for each t, then $\frac{DV}{dt}$ is equal to

$$\nabla_{\frac{dc}{dt}} Y$$ (= the covariant derivative of Y in the direction of the velocity of c)

Lemma 2-1. There is one and only one operation $\frac{DV}{dt}$ which satisfies three conditions in the definition.

For the proof, see (8).

Definition. A vector field V along c is said to be a <u>parallel vector field</u> if the covariant derivative $\frac{DV}{dt}$ is identically zero.

Lemma 2-2. Given a curve c and a tangent vector v_o at the point c(0), there is one and only one parallel vector field V along c which extends v_o.

For the proof, see (8).

Definition. A connection on M is <u>compatible</u> with the Riemannian metric if parallel translation preserves inner products.

A connection $\vdash$ is called <u>symmetric</u> if it satisfies the identity

$$(X \vdash Y) - (Y \vdash X) = [X , Y]$$

where [X , Y] denotes the poisson bracket $[X , Y]f = X(Yf) - Y(Xf)$ of two vector fieleds.

Now we will state "Fundamental Theorem of Riemannian Geometry": a Riemannian manifold possesses one and only one symmetric connection which is compatible with its metric.

Note that such a connection is called the Riemannian connection of a metric, and denoted by ∇.

Definition. A <u>Lie group</u> is a group which is also a manifold with a C^∞ structure such that

$$(x,y) \to xy$$
$$x \to x^{-1}$$

are C^∞ functions.

For any Lie group G, if a G we define the left and right translations, $L_a : G \to G$ and $R_a : G \to G$ by

$$L_a(b) = ab$$
$$R_a(b) = ba .$$

A vector field X on G is called <u>left invariant</u> if

$L_{a*}X_b = X_{ab}$ for all $a,b \in G$, where $(L_{a*}X_b)g = X_b(g \circ L_a)$, $g \in C^\infty(ab)$.

Definition. A <u>Lie algebra</u> is a finite dimensional vector space V, with a bilinear operation satisfying

$$[X , X] = 0 ,$$

$$[[X,Y] ,Z] + [[Y,Z] ,X] + [[Z,X] ,Y] = 0$$

for all X,Y,Z V .

The vector space G_e is called the <u>Lie algebra</u> and denoted by *G* if it has an operation [,] defined by [v,w] = [X,Y](e) where X,Y are the left invariant vector fields with X(e) = v, Y(e) = w and [X,Y] is the poisson bracket operation.

3. Scalar curvature

Let G be an n-dimensional Lie group, and let *G* be the associated Lie algebra, consisting of all smooth vector fields on G which are invariant under left translations. Choosing some basis $e_1,\cdots,e_n$ for the vector space *G* , it is easy to check that there is one and only one Riemannian metric on G so that these vector fields $e_1,\cdots,e_n$ are everywhere orthonormal. More generally, given any n n positive definite symmetric matrix (β_{ij}) of real numbers, there is one and only one Riemannian metric so that the inner product $<e_i, e_j>$ is everywhere equal to the constant function β_{ij}. Thus each n-dimensional Lie group possesses (1/2)n(n+1)-dimensional family of distinct left invariant metrics.We will see that different metrics on the same Lie group may exhibit drastically different curvature properties.

Definition. Given vector fields x,y,z of a smooth Riemannian manifold M define a new vector field $R_{xy}(z)$ by the identity

$$R_{xy}(z) = - x \vdash (y \vdash z) + y \vdash (x \vdash z) + [x, y] \vdash z.$$

Such R is called a <u>Riemann curvature tensor.</u>

The curvature of a Riemannian manifold at a point can be described most easily by the bi-quadratic curvature function

$$\kappa(x,y) = \langle R_{xy}(x), y\rangle .$$

Here x and y ranges over all tangent vectors at the given point.

If u and v are orthonormal, then the real number $K = \kappa(u,v)$ is called <u>the sectional curvature</u> of the tangential 2-plane spanned by u and v.

Choosing any orthonormal basis $e_1, \cdots, e_n$ for the tangent vectors at a point of a Riemannian manifold, the real number

$$\rho = 2\sum_{i<j} \kappa(e_i, e_j)$$

is called <u>the scalar curvature</u> at the point.

In order to study a Lie group with left invariant metric, it is best to choose an orthonormal basis $e_1, \cdots, e_n$ for the left invariant vector fields. The Lie algebra structure can then be described by an $n \times n \times n$ array of <u>structure constants</u> α_{ijk} where

$$[e_i, e_j] = \sum_k \alpha_{ijk} e_k$$

or equivalently

$$\alpha_{ijk} = \langle [e_i, e_j], e_k\rangle .$$

This array is skew-symmetric in the first two indices. The curvature function κ can then be expressed as a complicated quadratic function of the ijk .

Lemma 3-1. With the structure constants as above, the sectional curvature $\kappa(e_i, e_j)$ is given by the formula

$$\kappa(e_i,e_j) = \Sigma\ (\ \frac{1}{2}\alpha_{ijk}(-\alpha_{ijk}+\alpha_{jki}-\alpha_{kij}) - \frac{1}{4}(\alpha_{ijk}-\alpha_{jki}+\alpha_{kij})(\alpha_{ijk} + \alpha_{jki}-\alpha_{kij}) - \alpha_{kii}\,\alpha_{kjj}),$$

to be summed over k.

Proof. Let ∇ be the Riemannian connection with a Riemannian metric. Recall that ∇ is always uniquely defined, that $\nabla_x y$ is bilinear as a function of x and y, that it satisfies the "symmetry" condition

$$\nabla_x y - \nabla_y x = [x,y] \qquad (3.1)$$

and that the identity

$$<\nabla_x y, z> + <y, \nabla_x z> = 0 \qquad (3.2)$$

is satisfied whenever y and z are vector fields such that the Riemannian inner product $<y,z>$ is a constant function.

If x,y,z are all left invariant vector fields, then combining (3.1) and (3.2) with the various identities obtained by permuting the variables, we can solve to obtain the following formula:

$$<\nabla_x y, z> = \frac{1}{2}(<[x,y],z> - <[y,z],x> + <[z,x],y>).$$

In particular, it follows that

$$<\nabla_{e_i} e_j, e_k> = \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) .$$

Since $\nabla_{e_i} e_j = \sum_k \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij})$, we have

$$<\nabla_{e_i} \nabla_{e_j} e_i, e_j>$$

$$= -\sum_k \frac{1}{4}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij})(\alpha_{ijk} + \alpha_{jki} - \alpha_{kij}),$$

$$< \nabla_{e_j} \nabla_{e_i} e_i, e_j >$$

$$= -\sum_k \alpha_{kii} \alpha_{kjj} \quad , \text{ and}$$

$$<\nabla_{[e_i, e_j]} e_i, e_j>$$

$$= \sum_k \frac{1}{2} \alpha_{ijk}(-\alpha_{ijk} + \alpha_{jki} + \alpha_{kij}) .$$

Consequently, we have the required formula.

We have defined the scalar curvature at a point. But it is really well-defined? We will look whether it is well-defined or not.

Lemma 3-2. Let G be an n-dimensional Lie group and p is in G. For any two orthonormal bases $\{e_1, \cdots, e_n\}$ and $\{E_1 \cdots, E_n\}$, $\sum_{i<j} \kappa(e_i, e_j) = \sum_{i<j} \kappa(E_i, E_j)$, and therefore the scalar curvature at p in G is well-defined.

Proof. Let T be a linear map such that $T(e_i) = E_i$ for $i=1,2,\cdots,n$. Then T becomes a unitary operator. Since G_p is an n-dimensional vector space over R, G_p can be expressed as a direct sum

$$G_p = G_1 \oplus \cdots \oplus G_r$$

of T-invariant subspaces, which are mutually orthogonal and dim $G_i = 1$ or 2, for each $i=1,2\cdots,r$.(See 6.)

Case 1. If e_i is in G_j and dim $G_j = 1$, then E_i is also in G_j and $E_i = ae_i$. Since E_i is orthonormal, $a^2=1$ and therefore

$$<R_{xe_i}(x), e_i> = <R_{xE_i}(x), E_i> .$$

Case 2. If e_k, e_l are in G_j and dim $G_j=2$, then E_k, E_l are also in G_j, $E_k = ae_k + be_l$, and $E_l = -be_k$

$+ ae_1$, where a,b are reals such that $a^2 + b^2 = 1$.
Recall that R is a trilinear map satisfying:

(1) $R_{xy}(z) + R_{yx}(z) = 0$

(2) $R_{xy}(z) + R_{yz}(x) + R_{zx}(y) = 0$

(3) $\langle R_{xy}(z), w\rangle + \langle R_{xy}(w), z\rangle = 0$

(4) $\langle R_{xy}(z), w\rangle = \langle R_{zw}(x), y\rangle$.

Hence $\langle R_{xE_k}(x), E_k\rangle + \langle R_{xE_1}(x), E_1\rangle$

$= \langle R_{xe_k}(x), e_k\rangle + \langle R_{xe_1}(x), e_1\rangle$.

Combining these two cases, we have:

for every tangent x,

$$\langle R_{xe_i}(x), e_i\rangle = \langle R_{xE_i}(x), E_i\rangle .$$

Note that $2\sum_{i<j}\kappa(e_i, e_j) = \sum\sum \langle R_{e_j e_i}(e_j), e_i\rangle$

and $2\sum_{i<j}(E_i, E_j) = \sum\sum \langle R_{E_j E_i}(E_j), E_i\rangle$.

Now using methods by which case 1 and case 2 was proved, we obtain

$$\sum\sum \langle R_{E_j E_i}(E_j), E_i\rangle = \sum\sum \langle R_{e_j E_i}(e_j), E_i\rangle$$

$$= \sum\sum \langle R_{e_j e_i}(e_j), e_i\rangle$$

and therefore $2\sum_{i<j}\kappa(E_i, E_j) = 2\sum_{i<j}\kappa(e_i, e_j)$

This completes the assertion.

Recall that the adjoint L^* of a linear transformation L between metric vector spaces is defined by the formula

$$\langle Lx, y\rangle = \langle x, L^* y\rangle .$$

The transformation L is skew-adjoint if $L^* = -L$

and is self-adjoint if $L^* = L$. For any element x in a Lie algebra G the linear transformation

$$y \mapsto [x, y]$$

from G to itself is called ad(x).

Theorem 3-1. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of G, which is a Lie algebra of a Lie group with dimension n. If the linear transformations $ad(e_i)$ are skew-adjoint, then $\rho \geqq 0$ at any point in G.

Proof. Since $ad(e_i)$ is skew-adjoint,

$$\begin{aligned} \langle [e_i, e_j], e_k \rangle &= \langle ad(e_i)e_j, e_k \rangle \\ &= \langle e_j, -ad(e_i)e_k \rangle \\ &= -\langle [e_i, e_k], e_j \rangle \end{aligned}$$

, that is the statement that $ad(e_i)$ is skew-adjoint means that the array α_{ijk} is skew in the last two indices j and k.

Lemma 3-1 reduces to

$$\kappa(e_i, e_j) = \sum_k (\alpha_{ijk})^2/4 .$$

Thus $\kappa(e_i, e_j) \geqq 0$ whenever $i \neq j$. Therefore $\rho \geqq 0$, as asserted.

Some Lie groups may possess a metric which is invariant not only under left translation but also under right translation. The basic facts about such bi-invariant metrics can be summarized as follows.

Theorem 3-2. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $\mathcal{G}$, which is a Lie algebra of an n-dimensional Lie group G. A left invariant metric on G is also right invariant if all e_i's belong to the center of the Lie algebra $\mathcal{G}$.

Proof. We can easily see that ad(x) = 0 for every x in $\mathcal{G}$, since $ad(e_i) = 0$ whenever e_i is in the center of $\mathcal{G}$.

If g is sufficiently close to the identity in G, then g = exp(x) for some uniquely defined x in $\mathcal{G}$ close to zero. We have already known Ad(g) is a linear isometry. Recall that Ad(g) means $(L_g R_g^{-1})_*$, L_g means left translation by g and R_g right translation by g.

Since a connected Lie group is generated by any neighborhood of the identity, and since products of linear isometries are also linear isometric, we may conclude that Ad(g) is a linear isometry for any g in G.

Let μ be a left invariant metric on G. Since Ad(g) is a linear isometry, evidently

$$(L_g R_{g^{-1}})^* \mu = \mu$$

and therefore

$$\begin{aligned} R_g^* \mu &= R_g^*(L_g R_{g^{-1}})^* \mu \\ &= (R_{g^{-1}} R_g)^* L_g^* \mu \\ &= L_g^* \mu = \mu \end{aligned}$$

. This completes our assertion.

Theorem 3-3. A connected Lie group G admits a left invariant metric with $\rho > 0$ at every point in G if G is compact with finite fundamental group.

Proof. If G is compact, then we can choose a bi-invariant metric so that each ad(x) is skew-adjoint.(See 4.) If G also has finite fundamental group, so that the universal covering group $\tilde{G}$ is compact, note that $\mathcal{G}$ must be equal to its commutator ideal $[\mathcal{G}\ \mathcal{G}]$. For otherwise there would be a non-trivial Lie alge-bra homomorphism from $\mathcal{G}$ to the commutative Lie algebra R. This would induce a non-trivial homomorphism from G to the additive Lie group R, contradicting the hypothesis that G is compact.

If ad(u) is skew-adjoint, then

$$\kappa(u,v) \geqq 0$$

for all v, where equality holds if and only if u is orthogonal to the image $[v, \mathcal{G}]$. (See 4.) Since $[e_i, \mathcal{G}] = \mathcal{G}$ for each e_i, $\kappa(e_i, e_j) > 0$ if $i \neq j$. Hence $\rho = 2\sum_{i<j} \kappa(e_i, e_j) > 0$ and our assertion is proved.

Theorem 3-4. If the Lie algebra of G is non-commutative, then G possesses a left invariant metric of strictly negative scalar curvature.

Proof. First suppose that there exist linealy independent vectors x,y,z in the Lie algebra with $[x,y] = z$. Choose a fixed basis $\{b_1, \cdots, b_n\}$ with $b_1 = x$, $b_2 = y$, $b_3 = z$. For any real number $\lambda > 0$, consider an auxiliary basis $\{e_1, \cdots, e_n\}$ defined by $e_1 = \lambda b_1$, $e_2 = \lambda b_2$, and $e_i = \lambda^2 b_i$ for $i \geq 3$. Define a left invariant metric by requiring that $\{e_1, \cdots, e_n\}$ should be orthonormal. Let $\mathcal{G}_\lambda$ denote the Lie algebra $\mathcal{G}$ provided with this particular metric and this particular orthonormal basis.

Setting $[e_i, e_j] = \sum_k \alpha_{ijk} e_k$, the structure constants α_{ijk} are cleary functions of λ. Now con-sider the limit as $\lambda \to 0$. Inspection shows that each α_{ijk} tends to a well defined limit. Thus we obtain a limit Lie algebra G_0 with prescribed metric and prescribed orthonormal basis. Further-more the bracket product in G_0 is given by

$$[e_1, e_2] = -[e_2, e_1] = e_3,$$

with $[e_i, e_j] = 0$ otherwise. Note that G_0 is nilpotent but not commutative. Applying Lemma 3-1, we obtain $\rho[G_0] < 0$. It follows by continuity that $\rho(G_\lambda) < 0$ whenever λ is sufficiently close to zero.

On the other hand, suppose x, y and $[x, y]$ are always linearly dependent. Then there exists a well-defined linear mapping l from G to the real numbers such that $[x, y] = l(x)y - l(y)x$. Choosing any positive definite metric, the sectional curvatures are constant:

$$K = -\| l \|^2 .$$

Thus, in the noncommutative case $l \neq 0$, every possible metric has constant negative scalar curvature. Our theorem is proved.

If the Lie algebra of G is commutative, we can easily obtain that $\rho = 0$ at any point for any left invariant metric on G.

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis. Then $[e_i, e_j] = 0$ for each pair i,j if the Lie algebra of G is commutative. Hence $\alpha_{ijk} = 0$ and therefore $\kappa(e_i, e_j) = 0$ if $i \neq j$. This implies $\rho = 0$.

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