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TERM RANK-SUM PRESERVERS OF FUZZY MATRICES

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TERM RANK-SUM PRESERVERS OF FUZZY MATRICES

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⟨**Abstract**⟩

TERM RANK-SUM PRESERVERS OF FUZZY MATRICES

In this thesis, we construct the sets of fuzzy matrix pairs. These sets are naturally occurred at the extreme cases for the (zero) term rank inequalities relative to the sum of fuzzy matrices. These sets were constructed with the fuzzy matrix pairs which are related with the term ranks of the sums and the zero term ranks of the sums of two fuzzy matrices.

That is, we construct the following 5 sets;

$$\mathcal{T}_1(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | t(X+Y) = t(X) + t(Y)\};$$

$$\mathcal{T}_2(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | t(X+Y) = 1\};$$

$$\mathcal{T}_3(\mathcal{F}) = \Big\{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | t(X+Y) = \max\{t(X), t(Y)\}\Big\};$$

$$\mathcal{Z}_1(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | z(X+Y) = \min\{z(X), z(Y)\}\};$$

$$\mathcal{Z}_2(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | z(X+Y) = 0\};$$

For these 5 sets of fuzzy matrix pairs, we consider the linear operators that preserve them. We characterize those linear operators as $T(X) = PXQ$ or $T(X) = PX^tQ$ with appropriate invertible fuzzy matrices P and Q. We also prove that these linear operators preserve above 5 sets.

1 Introduction and Preliminaries

The linear algebra over semiring is a subject of intensive research because of its purely algebraic interest and its numerous applications to matrix algebra and combinatorial theory. During the last century, problems on the characterization of the linear operators that leave certain matrix subsets invariants were actively studied. For survey of these types of problems, we refer to the article of Song([11]) and the papers in [10]. The specified frame of problems is of interest both for matrices with entries from a field and for matrices with entries from an arbitrary semiring such as Boolean algebra, nonnegative integers, and fuzzy sets. It is necessary to note that there are several rank functions over a semiring that are analogues of the classical function of the matrix rank over a field. Detailed research and self-contained information about rank functions over semirings can be found in [1, 11].

There are some results on the inequalities for the rank function of matrices([1, 2, 3, 4]). Beasley and Guterman ([1]) investigated the rank inequalities of matrices over semirings. And they characterized the equality cases for some rank inequalities in [2]. The investigation of linear preserver problems of extreme cases of the rank inequalities of matrices over fields was obtained in [4]. The structure of matrix varieties which arise as extremal cases in the inequalities is far from being understood over fields, as well as semirings. A usual way to generate elements of such a variety is to find a matrix pairs which belongs to it and to act on this set by various linear operators that preserve this variety. Song and his colleagues ([3]) characterized the linear operators that preserve the extreme cases of column rank inequalities over semirings.

There are some results on the linear operators that preserve term rank([7, 8]) and zero-term rank([5]). But in these papers, the authors studied the term rank and zero-term rank function themselves.

In this thesis, we characterize linear operators that preserve the sets of matrix pairs which satisfy extreme cases for the term rank inequalities and zero-term rank inequalities for the sum of matrices over fuzzy semirings.

Definition 1.1. A *semiring* $\mathcal{S}$ consists of a set $\mathcal{S}$ and two binary operations, addition and multiplication, such that:

- $\mathcal{S}$ is an Abelian monoid under addition (identity denoted by 0);
- $\mathcal{S}$ is a semigroup under multiplication (identity, if any, denoted by 1);
- multiplication is distributive over addition on both sides;
- $s0 = 0s = 0$ for all $s \in \mathcal{S}$.

Definition 1.2. A semiring is called *antinegative* if the zero element is the only element with an additive inverse.

Definition 1.3. A semiring is called *chain* if the set $\mathcal{S}$ is totally ordered with universal lower and upper bounds and the operations are defined by $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$.

It is straightforward to see that any chain semiring is commutative and antinegative.

Throughout we assume that $m \leq n$. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J, and O, respectively. The matrix $E_{i,j}$, called a *cell*, denotes the matrix with exactly one nonzero entry, that being a one in the (i, j) entry. Let R_i denote the matrix whose i^{th} row is all ones and is zero elsewhere, and C_j denote the matrix whose j^{th} column is all ones and is zero elsewhere. We let $|A|$ denote the number of nonzero entries in the matrix A.

Definition 1.4. Let $\mathcal{R}$ be the field of reals, let $\mathcal{F}$=$\{\alpha \in \mathcal{R} \mid 0 \leq \alpha \leq 1\}$ denote a subset of reals. Define $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$ for all a,b in $\mathcal{F}$. Then $(\mathcal{F}, +, \cdot)$ is called a *fuzzy* semiring. Let $\mathcal{M}_{m,n}(\mathcal{F})$ denote the set of all $m \times n$ matrices with entries in a *fuzzy* semiring $\mathcal{F}$. We call a matrix in $\mathcal{M}_{m,n}(\mathcal{F})$ as a *fuzzy* matrix.

Definition 1.5. A *line* of a matrix A is a row or a column of the matrix A.

Definition 1.6. A matrix $A \in \mathcal{M}_{m,n}(\mathcal{F})$ has *term rank* k ($t(A) = k$) if the least number of lines needed to include all nonzero elements of A is equal to k. Let us denote by $c(A)$ the least number of columns needed to include all nonzero elements of A and by $r(A)$ the least number of rows needed to include all nonzero elements of A.

Definition 1.7. A matrix $A \in \mathcal{M}_{m,n}(\mathcal{F})$ has *zero-term rank* k ($z(A) = k$) if the least number of lines needed to include all zero elements of A is equal to k.

Example 1.8. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 1 & 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then $t(A) = 3$, $z(A) = 1$, $t(B) = 2$ and $z(B) = 3$.

Definition 1.9. A matrix $A \in \mathcal{M}_{m,n}(\mathcal{F})$ has *factor rank* k ($rank(A) = k$) if there exist matrices $B \in \mathcal{M}_{m,k}(\mathcal{F})$ and $C \in \mathcal{M}_{k,n}(\mathcal{F})$ such that $A = BC$ and k is the smallest positive integer such that such a factorization exists. By definition the only matrix with factor rank equal to 0 is the zero matrix, O.

If $\mathcal{S}$ is a subsemiring of a certain field then there is a usual rank function $\rho(A)$ for any matrix $A \in \mathcal{M}_{m,n}(\mathcal{S})$. It is easy to see that these functions are not equal in general but the inequality $rank(A) \geq \rho(A)$ always holds.

Example 1.10. Consider $\mathcal{Z}_+$, the set of nonnegative integers. The semiring $\mathcal{Z}_+$ is embedded in the real field $\mathcal{R}$. Then the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 3 & 3 & 3 \end{pmatrix}$$

has different values as, where $rank(A)$=3 and $\rho(A)$=2.

Definition 1.11. Let $\mathcal{F}$ be a fuzzy semiring. An operator $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ is called *linear* if $T(X + Y) = T(X) + T(Y)$ and $T(\alpha X) = \alpha T(X)$ for all $X, Y \in \mathcal{M}_{m,n}(\mathcal{F})$, $\alpha \in \mathcal{F}$.

Definition 1.12. We say an operator, T, *preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ implies that $T(X) \in \mathcal{P}$, or, if $(X, Y) \in \mathcal{P}$ implies that $(T(X), T(Y)) \in \mathcal{P}$ when $\mathcal{P}$ is a set of ordered pairs.

Definition 1.13. An operator T *strongly preserves* the set $\mathcal{P}$ if $X \in \mathcal{P}$ if and only if $T(X) \in \mathcal{P}$, or, if $(X, Y) \in \mathcal{P}$ if and only if $(T(X), T(Y)) \in \mathcal{P}$ when $\mathcal{P}$ is a set of ordered pairs.

Definition 1.14. The matrix $X \circ Y$ denotes the *Hadamard* or *Schur product*, i.e., the (i, j) entry of $X \circ Y$ is $x_{i,j} y_{i,j}$.

Definition 1.15. An operator T is called a (P, Q, B)-*operator* if there exist permutation matrices P and Q, and a matrix B with no zero entries, such that $T(X) = P(X \circ B)Q$ for all $X \in \mathcal{M}_{m,n}(\mathcal{F})$, or, if $m = n$, $T(X) = P(X \circ B)^t Q$ for all $X \in \mathcal{M}_{m,n}(\mathcal{F})$. A (P, Q, B)-operator is called a (P, Q)-*operator* if $B = J$, the matrix of all ones.

It was shown in [2, 4, 9] that linear preserves for extremal cases of classical matrix inequalities over fields are types of (P, Q)-operators where P and Q are arbitrary invertible matrices. On the other side, linear preservers for various rank functions over semirings have been the object of much study during the last years, see for example [6, 7, 8, 10], in particular term rank and zero term rank were investigated in the last few years, see for example [5].

Definition 1.16. We say that the matrix A *dominates* the matrix B if and only if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$.

Definition 1.17. If A and B are matrices and $A \geq B$ we let $A \backslash B$ denote the matrix C where

$$c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} \neq 0 \\ a_{i,j} & \text{otherwise} \end{cases}$$

The behaviour of the function ρ with respect to matrix multiplication and addition is given by the following inequalities:

The *rank-sum inequalities*:

$$| \rho(A) - \rho(B) | \leq \rho(A + B) \leq \rho(A) + \rho(B),$$

Sylvester's laws:

$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min\{\rho(A), \rho(B)\},$$

and the *Frobenius inequality*:

$$\rho(AB) + \rho(BC) \leq \rho(ABC) + \rho(B),$$

where A, B, C are conformal matrices with coefficients from a field.

In [2, 3, 4, 9] they considered these sets: $Q_1 - Q_5$.

1. $Q_1 = \{(A, B) | \rho(A + B) = \rho(A) + \rho(B)\}$;

2. $Q_2 = \{(A, B) | \rho(A + B) = |\rho(A) - \rho(B)|\}$;

3. $Q_3 = \{(A, B) | \rho(AB) = \min\{\rho(A), \rho(B)\}\}$;

4. $Q_4 = \{(A, B) | \rho(AB) = \rho(A) + \rho(B) - n\}$;

5. $Q_5 = \{(A, B, C) | \rho(AB) + \rho(BC) = \rho(ABC) + \rho(B)\}$;

They also characterized the linear operators that preserves these sets. For examples, bijective linear operator T preserves Q_5 if and only if $T(X) = \alpha PXP^{-1}$ or $T(X) = \alpha PX^tP^{-1}$.

2 Term Rank Inequality Over Fuzzy Semiring

We obtain various inequalities for term rank of matrix addtion over fuzzy semirings. We also show that these inequalities are exact and best possible.

We denote by $A \bigoplus B$ the block-diagonal matrix of the form

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}.$$

Note that in this sense the operation $\bigoplus$ is not commutative.

Proposition 2.1. *Let $\mathcal{F}$ be an arbitrary fuzzy semiring. For any matrices A, $B \in \mathcal{M}_{m,n}(\mathcal{F})$ we have:*

$$t(A + B) \leq \min\{t(A) + t(B), m, n\}.$$

This bound is exact and the best possible.

Proof. This inequality follows directly from the definition of term rank. The substitution $A_r = I_r \bigoplus O_{n-r}, B_s = O_{n-s} \bigoplus I_s$ for each pair $(r, s), 0 \leq r, s \leq n$ shows that this bound is exact and the best possible in the case $m = n$. It is routine to generalize this example to the case $m \neq n$. □

Example 2.2. A nontrivial additive lower bound for the term rank of a sum does not hold over an arbitrary semiring. It is enough to take $A = B = J_{m,n}$ over a field whose characteristics is equal to 2. Then $t(A + B) = t(0) = 0$. Since $t(A) = t(B) = \min\{m, n\}$, we have $t(A + B) < \max(t(A), t(B))$.

However for antinegative semiring there is a lower bound for the term rank of a sum which is better than the one for fields or arbitrary semirings. Namely, the following is true.

Proposition 2.3. *Let $\mathcal{F}$ be a fuzzy semiring. For any matrices A, $B \in \mathcal{M}_{m,n}(\mathcal{F})$ the following inequality holds:*

$$t(A + B) \geq \max\{t(A), t(B)\}.$$

This bound is exact and the best possible.

Proof. This inequality follows from the antinegativity of $\mathcal{F}$, i.e., $a + b \neq 0$ for any $a, b \in \mathcal{F}$, $a \neq 0$, and the definition of the term rank. To prove that this bound is exact and the best possible we consider the matrices $A_r = I_r \bigoplus O_{n-r}, B_s = O_{n-s} \bigoplus I_s$ for each pair (r, s), $0 \leq r$, $s \leq n$ shows that this bound is exact and the best possible in the case $m = n$. It is routine to generalize this examle to the case $m \neq n$. □

Example 2.4. A nontrivial multiplicative lower bound does not hold over an arbitrary fuzzy semiring. It is enough to take $A = B = J_n$ over a field whose characteristic is a divisor of n. Then $t(AB) = t(nJ_n) = 0$.

Over a fuzzy semiring the Sylvester lower bound holds:

Proposition 2.5. *Let $\mathcal{F}$ be a fuzzy semiring. Then for any $A \in \mathcal{M}_{m,n}(\mathcal{F})$, $B \in \mathcal{M}_{n,k}(\mathcal{F})$ the following inequality holds:*

$$t(AB) \geq \begin{cases} 0 & \text{if } t(A) + t(B) \leq n, \\ t(A) + t(B) - n & \text{if } t(A) + t(B) > n. \end{cases}$$

This bound is exact and best possible.

Proof. Let $A \in \mathcal{M}_{m,n}(\mathcal{F})$, $B \in \mathcal{M}_{n,k}(\mathcal{F})$ be arbitray matrices, $t(A) = t_A$, $t(B) = t_B$. Then A and B have generalized diagonals with t_A and t_B nonzero elements, respectively. Denote them by D_A and D_B, respectively. Then $AB \geq D_A D_B$ since F is antinegative. Since the product of two generalized diagonal matrices, which have t_A and t_B nonzero entries, respectively, has at least $t_A + t_B - n$ nonzero entries, the inequality follows.

In order to show that this bound is exact and the best possible for each pair (r, s), $0 \leq r$, $s \leq n$ let us take $A_r = I_r \bigoplus O_{n-r}$, $B_s = O_{n-s} \bigoplus I_s$ in the case $m = n$. It is routine to generalize this example for the case $m \neq n$. □

Example 2.6. Let $A, B \in \mathcal{M}_{n,n}(\mathcal{F})$. The inequality $t(AB) \leq \min(t(A), t(B))$ does not hold. It is enough to take $A = C_1$, $B = R_1$. Then

$$t(AB) = t(J_n) = n > 1.$$

However the following inequality is true.

Proposition 2.7. *Let $\mathcal{F}$ be a fuzzy semiring. Then for any $A \in \mathcal{M}_{m,n}(\mathcal{F})$, $B \in \mathcal{M}_{n,k}(\mathcal{F})$ the inequality $t(AB) \leq \min(t_r(A), t_c(B))$ holds. This is exact and the best possible bound.*

Proof. This inequality is a direct consequence of the definition of the term rank and antinegativity. The exactness follows from Example 2.6. In order to prove that this bound is the best possible, for each pair (r, s), $0 \leq r \leq m$, $0 \leq s \leq k$, consider the family of matrices $A_r = E_{1,1} + \ldots + E_{r,1}$ and $B_s = E_{1,1} + \ldots + E_{1,s}$. □

Example 2.8. For an arbitrary fuzzy semiring, the triple $(C_1, R_1, 0)$ is a counterexample to the term rank version of the Frobenius inequality, since $t(C_1R_1) + t(R_1 0) = n > t(C_1R_1 0) + t(R_1) = 1$. However if $\mathcal{F}$ is a subsemiring of $\mathcal{R}^+$ the following obvious version is true :

$$\rho(AB) + \rho(BC) \leq t(ABC) + t(B)$$

3 Zero-Term Rank Inequality Over Fuzzy Semiring

We obtain inequalities for the zero-term rank addition over fuzzy semirings. We also show that these inequalities are exact and best possible.

Proposition 3.1. *Let $\mathcal{F}$ be a fuzzy semiring. For A, $B \in \mathcal{M}_{m,n}(\mathcal{F})$ one has that*

$$0 \leq z(A+B) \leq \min\{z(A), z(B)\}.$$

These bounds are exact and the best possible.

Proof. The lower bound follows from the definition of the zero-term rank function.

In order to check that this exact and the best possible for each pair (r,s), $0 \leq r$, $s \leq \min\{m,n\}$ let us consider the family of matrices $A_r = J\backslash(\Sigma_{i=1}^{r} E_{i,i}), B_s = J\backslash(\Sigma_{i=1}^{s} E_{i,i+1})$ if $s < \min\{m,n\}$ and $B_s = J\backslash(\Sigma_{i=1}^{s-1} E_{i,i+1} + E_{s,1})$ if $s = \min\{m,n\}$. Then $z(A_r) = r$, $z(B_s) = s$ by definition and $z(A_r + B_s) = 0$ by antinegativity.

The upper bound follows directly from the definition of zero-term rank and from the antinegativity of $\mathcal{F}$. For the proof of its exactness let us take $A = J$ and $B = O$. In order to check that this bound is the best possible we consider the following family of matrices: for each pair (r,s), $0 \leq r$, $s \leq \min\{m,n\}$ let us consider the matrices $A_r = J\backslash(\Sigma_{i=1}^{r} E_{i,i})$ and $B_s = J\backslash(\Sigma_{i=1}^{s} E_{i,i})$. □

Proposition 3.2. *Let $\mathcal{F}$ be a fuzzy semiring. For $A \in \mathcal{M}_{m,n}(\mathcal{F})$, $B \in \mathcal{M}_{n,k}$ one has that*

$$0 \leq z(AB) \leq \min\{z(A) + z(B), k, m\}.$$

These bounds are exact and the best possible for $n > 2$.

Proof. The lower bound follows from the definition of the zero-term rank function. In order to show that this bound is exact and the best possible let us consider the family of matrices: for each pair (r,s), $0 \leq r \leq \min\{m,n\}$, $0 \leq s \leq \min\{k,n\}$, we take $A_r = J\backslash(\Sigma_{i=1}^{r} E_{i,i})$, $B_s = J\backslash(\Sigma_{i=1}^{s} E_{i,i+1})$ if $s < \min\{k,n\}$ and $B_s = J\backslash(\Sigma_{i=1}^{s-1} E_{i,i+1} + E_{s,1})$ if $s = \min\{k,n\}$. Then $z(A_r) = r$, $z(B_s) = s$ by definition and if $n > 2$ then $A_r B_s$ does not have zero elements by antinegativity. Thus $z(A_r B_s) = 0$.

The upper bound follows directly from the definition of zero-term rank and from the antinegativity of $\mathcal{F}$.

In order to show that this bound is exact and the best possible let us consider the family of matrices: for each pair (r, s), $0 \leq r \leq \min\{m, n\}$, $0 \leq s \leq \min\{k, n\}$, we take $A_r = J \backslash (\Sigma_{i=1}^{r} R_i)$ and $B_s = J \backslash (\Sigma_{i=1}^{s} C_i)$. $\square$

Example 3.3. The triple (C_1, I, R_1) is a counterexample to the zero-term rank version of the Frobenius inequality, since

$$z(C_1) + z(R_1) = 2n - 2 > z(C_1 R_1) + z(I) = n$$

for $n > 2$.

4 Basic results for linear operator over fuzzy semiring

In this section, we obtain some basic results for our main theorems in the section 5 and 6. For a surjective linear operator, we have the followings.

Theorem 4.1. *Let $\mathcal{F}$ be a fuzzy semiring and $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ be a linear operator. Then the following are equivalent:*

1. *T is bijective.*
2. *T is surjective.*
3. *There exists a permutation σ on $\{(i,j) \mid i = 1, 2, \cdots, m; j = 1, 2, \cdots, n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$.*

Proof. That 1) implies 2) and 3) implies 1) is straightforward. We now show that 2) implies 3).

We assume that T is surjective. Then, for any pair (i,j), there exists some X such that $T(X) = E_{i,j}$. Clearly $X \neq O$ by the linearity of T. Thus there is a pair of indexes (r,s) such that $X = x_{r,s}E_{r,s} + X'$ where (r,s) entry of X' is zero and the following two conditions are satisfied: $x_{r,s} \neq 0$ and $T(E_{r,s}) \neq O$. Indeed, if in the contrary for all pairs (r,s) either $x_{r,s} = 0$ or $T(E_{r,s}) = O$ then $T(X) = 0$ which contradicts with the assumption $T(X) = E_{i,j} \neq 0$. Hence

$$T(x_{r,s}E_{r,s}) \leq T(x_{r,s}E_{r,s}) + T(X \setminus (x_{r,s}E_{r,s})) = T(X) = E_{i,j}$$

That is, $x_{r,s}T(E_{r,s}) = T(x_{r,s}E_{r,s}) \leq E_{i,j}$. Thus $T(x_{r,s}E_{r,s}) = \alpha E_{i,j}$ for a certain $\alpha \in \mathcal{F}$. That is there is some permutaion σ on $\{(i,j) \mid i = 1, 2, \cdots, m; j = 1, 2, \cdots, n\}$ such that for some scalars $b_{i,j}$, $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$. we now only need show that the $b_{i,j}$ are all units. Since T is surjective and $T(E_{r,s}) \not\leq E_{\sigma(i,j)}$ for $(r,s) \neq (i,j)$,there is some α such that $T(\alpha E_{i,j}) = E_{\sigma(i,j)}$. But then, since T is linear, $T(\alpha E_{i,j}) = \alpha T(E_{i,j}) = \alpha b_{i,j}E_{\sigma(i,j)} = E_{\sigma(i,j)}$. That is, $\alpha b_{i,j} = 1$, or $b_{i,j}$ is a unit. But 1 is the only unit over fuzzy semiring. □

Lemma 4.2. *Let $\mathcal{F}$ be a fuzzy semiring, $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ be an operator which maps lines to lines and is defined by $T(E_{i,j}) = E_{\sigma(i,j)}$, where σ is a permutation on the set $\{(i,j) \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\}$. Then T is a (P,Q)-operator.*

Proof. Since no combination of u rows and v columns can dominate J where $u+v = m$ unless $v = 0$ (or if $m = n$, if $u = 0$) we have that either the image of each row is a row and the image of each column is a column, or $m = n$ and the image of each row is a column and the image of each column is a row. Thus, there are permutation matrices P and Q such that $T(R_i) \leq PR_iQ$ and $T(C_j) \leq PC_jQ$ or, if $m = n$, $T(R_i) \leq P(R_i)^tQ$ and $T(C_j) \leq P(C_j)^tQ$. Since each cell lies in the intersection of a row and a column and T maps nonzero cells to nonzero (weighted) cells, it follows that $T(E_{i,j}) = PE_{i,j}Q$, or, if $m = n$, $T(E_{i,j}) = PE_{j,i}Q = P(E_{i,j})^tQ$ $\square$

5 The Term Rank Preservers Over Fuzzy Semiring

In this section, we obtain characterizations of the linear operators that preserve the set of matrix pairs which arise as the extremal cases in the inequalities of term rank of matrix sums.

Below, we use the following notations in order to denote sets of matrices that arise as extremal cases in the inequalities of term rank of matrix sums listed in section 2.

$$\mathcal{T}_1(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | t(X+Y) = t(X) + t(Y)\};$$

$$\mathcal{T}_2(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | t(X+Y) = 1\};$$

$$\mathcal{T}_3(\mathcal{F}) = \Big\{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | t(X+Y) = \max\{t(X), t(Y)\}\Big\};$$

5.1 Linear Preservers of $\mathcal{T}_1(\mathcal{F})$

Consider the set of matrix pairs:

$$\mathcal{T}_1(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 \mid t(X+Y) = t(X) + t(Y)\}.$$

We characterize the linear operators that preserve the set $\mathcal{T}_1(\mathcal{F})$.

Theorem 5.1. *Let $\mathcal{F}$ be a fuzzy semiring, $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ be a surjective linear map. Then T preserves the set $\mathcal{T}_1(\mathcal{F})$ if and only if T is a (P,Q)-operator, where P and Q are permutation matrices of appropriate sizes.*

Proof. By Theorem 4.1 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \le i \le m$, $1 \le j \le n$, σ is a permutation on the set of pairs (i, j).

Let us show that T maps lines to lines. Suppose that the images of two cells are in the same line, but the cells are not, say $E_{i,j}, E_{k,l}$ are the cells such that $t(E_{i,j} + E_{k,l}) = 2$ and $t(T(E_{i,j} + E_{k,l})) = 1$. Then $(E_{i,j}, E_{k,l}) \in \mathcal{T}_1$ but $(T(E_{i,j}), T(E_{k,l})) \notin \mathcal{T}_1$, a contradiction. Thus T maps lines to lines. Thus by Lemma 4.2, T is a (P,Q)-operator where P and Q are permutation matrices of appropriate sizes.

Conversely, $(X, Y) \in \mathcal{T}_1$ then $t(T(X) + T(Y)) = t(T(X + Y)) = t(P(X + Y)Q) = t(X+Y) = t(X)+t(Y) = t(PXQ)+t(PYQ) = t(T(X))+t(T(Y))$. Thus $(T(X), T(Y)) \in \mathcal{T}_1$ and T preserves $\mathcal{T}_1$ $\square$

Theorem 5.2. *Let $\mathcal{F}$ be a fuzzy semiring. $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ strongly preserves the set $\mathcal{T}_1(\mathcal{F})$ if and only if T is a (P, Q)-operator, where P and Q are permutation matrices of appropriate sizes.*

Proof. Suppose that T strongly preserves $\mathcal{T}_1$. There is some power of T which is idempotent, say $L = T^d$ and $L^2 = L$. It is easy to see that L strongly preserves $\mathcal{T}_1$.

If $X \in \mathcal{M}_{m,n}(\mathcal{F})$ and $(X, X) \in \mathcal{T}_1$ then necessarily $X = O$. Thus, if $A \neq O$, $L(A) \neq O$ since L strongly preserves $\mathcal{T}_1$.

Suppose that there exists i, $1 \leq i \leq m$, such that $L(R_i)$ is not dominated by R_i. Then there is a pair of indexes (r, s) such that $E_{r,s}$ is not dominated by R_i and $L(R_i) \geq E_{r,s}$. Then $(R_i, E_{r,s}) \in \mathcal{T}_1$, and $L(R_i) = aE_{r,s} + X$ with $x_{r,s} = 0$.

Now,

$$\begin{aligned}
L(R_i + aE_{r,s}) &= L(R_i) + L(aE_{r,s}) \\
&= L^2(R_i) + L(aE_{r,s}) \\
&= L(L(R_i)) + L(aE_{r,s}) \\
&= L(aE_{r,s} + X) + L(aE_{r,s}) \\
&= L(X) + L(aE_{r,s}) + L(aE_{r,s}) \\
&= L(X) + L(aE_{r,s} + aE_{r,s}) \\
&= L(X) + L(aE_{r,s}) \\
&= L(X + aE_{r,s}) \\
&= L(L(R_i)) \\
&= L^2(R_i) \\
&= L(R_i).
\end{aligned}$$

Now, $(R_i, aE_{r,s}) \in \mathcal{T}_1$ but, $L(R_i) + L(aE_{r,s}) = L(R_i + aE_{r,s}) = L(R_i)$ and hence, $(L(R_i), L(aE_{r,s})) \notin \mathcal{T}_1$, a contradiction.

We have established that $L(R_i) \leq R_i$ for all i. Similarly, $L(C_j) \leq C_j$ for all j. By considering that $E_{i,j}$ is dominated by both R_i and C_j we have that $L(E_{i,j}) \leq E_{i,j}$. Since $\mathcal{F}$ is fuzzy semiring, we have that T also maps a cell to a cell, or $|T(E_{i,j})| = 1$ for all i, j, and $T(J)$ has all nonzero entries.

So T induces a permutation, σ, on the set of subscripts $\{1, 2, \cdots, m\} \times \{1, 2, \cdots, n\}$. That is, $T(E_{i,j}) = E_{\sigma(i,j)}$. We have that T is a (P, Q)-operator.

Conversely, all (P, Q) operators preserve the term rank. □

5.2 Linear Preservers of $\mathcal{T}_2(\mathcal{F})$

Consider the set of matrix pairs:

$$\mathcal{T}_2(\mathcal{F}) = \{(X, Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 \mid t(X + Y) = 1\}.$$

We characterize the linear operators that preserve the set $\mathcal{T}_2(\mathcal{F})$.

Theorem 5.3. *Let $\mathcal{F}$ be a fuzzy semiring, $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ be a surjective linear map. Then T preserves the set $\mathcal{T}_2(\mathcal{F})$ if and only if T is a (P, Q)-operator, where P and Q are permutation matrices of appropriate sizes.*

Proof. By Theorem 4.1 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, σ is a permutation on the set of pairs (i, j).

The cells $E_{i,j}$, $E_{r,s}$ are in the same line, if and only if $t(E_{i,j} + E_{r,s}) = 1$ if and only if $(E_{i,j}, E_{r,s}) \in \mathcal{T}_2$ then $(T(E_{i,j}), T(E_{r,s})) \in \mathcal{T}_2$. That is, $t(T(E_{i,j}) + T(E_{r,s})) = 1$. Therefore $T(E_{i,j})$ and $T(E_{r,s})$ are in the same line. Thus lines are mapped to lines, and we have that T is a (P, Q)-operator by Lemma 4.2.

Conversely, let T be a (P, Q)-operator, and $(X, Y) \in \mathcal{T}_2$. Then $1 = t(X + Y) = t(P(X + Y)Q) = t(T(X + Y)) = t(T(X) + T(Y))$. Hence $(T(X), T(Y)) \in \mathcal{T}_2$. That is, T preserves $\mathcal{T}_2$. □

5.3 Linear Preservers of $\mathcal{T}_3(\mathcal{F})$

Consider the set of matrix pairs:

$$\mathcal{T}_3(\mathcal{F}) = \{(X, Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | t(X+Y) = \max(t(X), t(Y))\}.$$

We characterize the linear operators that preserve the set $\mathcal{T}_3(\mathcal{F})$.

Theorem 5.4. *Let $\mathcal{F}$ be a fuzzy semiring, $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ be a surjective linear map. Then T preserves the set $\mathcal{T}_3(\mathcal{F})$ if and only if T is a (P, Q)-operator, where P and Q are permutation matrices of appropriate sizes.*

Proof. By Theorem 4.1 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, where σ is a permutation on the set of pairs (i, j).

Suppose that the images of two cells are not in the same line, but the cells are, say $E_{i,j}, E_{i,l}$ are the cells such that $T(E_{i,j}), T(E_{i,l})$ are not in the same line, i.e., $t(T(E_{i,j} + E_{i,l})) = 2$. Then $(E_{i,j}, E_{i,l}) \in \mathcal{T}_3$ but $(T(E_{i,j}), T(E_{i,l})) \notin \mathcal{T}_3$, a contradiction. Thus T^{-1} maps lines to lines. By Lemma 4.2 it follows that T^{-1} is a (P, Q)-operator where P and Q are permutation matrices of appropriate sizes. Hence, T is also of this type.

Conversely, if $(X, Y) \in \mathcal{T}_3$ then $t(X+Y) = t(X)$, $t(P(X+Y)Q) = t(PXQ)$, $t(T(X+Y)) = t(T(X))$, $t(T(X) + T(Y)) = t(T(X))$. Hence $(T(X), T(Y)) \in \mathcal{T}_3$. That is, T preserves $\mathcal{T}_3$. $\square$

6 The Zero-Term Rank Preservers Over Fuzzy Semiring

In this section, we obtain the characterizations of the linear operators that preserve the set of matrix pairs which arise as the extremal cases in the inequalities of zero-term rank of matrix sums.

Below, we use the following notations in order to denote sets of matrices that arise as extremal cases in the inequalities of zero-term rank of matrix sums listed in section 3.

$$\mathcal{Z}_1(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | z(X+Y) = \min\{z(X), z(Y)\}\};$$

$$\mathcal{Z}_2(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | z(X+Y) = 0\};$$

6.1 Linear Preservers of $\mathcal{Z}_1(\mathcal{F})$

Consider the set of matrix pairs:

$$\mathcal{Z}_1(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | z(X+Y) = \min\{z(X), z(Y)\}\}.$$

We characterize the linear operators that preserve the set $\mathcal{Z}_1(\mathcal{F})$.

Theorem 6.1. *Let $\mathcal{F}$ be a fuzzy semiring, $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ be a surjective linear map. Then T preserves the set $\mathcal{Z}_1(\mathcal{F})$ if and only if T is a (P,Q)-operator, where P and Q is a permutation matrices of appropriate sizes.*

Proof. By Theorem 4.1 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, where σ is a permutation on the set of pairs (i,j).

Let us show that T maps lines to lines. Suppose that the images of two cells are not in the same line, but the cells are, say $E_{i,j}$, $E_{i,k}$ are the cells such that $T(E_{i,j})$, $T(E_{i,k})$ are not in the same line. Then one has that $z((J \backslash E_{i,j} \backslash E_{i,k}) + E_{i,k}) = 1 = z(J \backslash E_{i,j} \backslash E_{i,k})$, i.e. $(J \backslash E_{i,j} \backslash E_{i,k}, E_{i,k}) \in \mathcal{Z}_1$, as far as $z(T(J \backslash E_{i,j} \backslash E_{i,k}) + T(E_{i,k})) = 1 < 2 = \min\{z(T(J \backslash E_{i,j} \backslash E_{i,k})), z(T(E_{i,k}))\}$, i.e. $(T(J \backslash E_{i,j} \backslash E_{i,k}), T(E_{i,k})) \notin \mathcal{Z}_1$, a contradiction. Thus T maps lines to lines. By Lemma 4.2 it follows that T is a (P,Q)-operator where P and Q are permutation matrices of appropriate sizes.

Conversely, if $(X, Y) \in \mathcal{Z}_1$ then $z(X+Y) = z(X)$, $z(P(X+Y)Q) = z(PXQ)$, $z(T(X+Y)) = z(T(X))$. Hence $(T(X), T(Y)) \in \mathcal{Z}_1$. That is, T preserves $\mathcal{Z}_1$. $\square$

6.2 Linear Preservers of $\mathcal{Z}_2(\mathcal{F})$

Consider the set of matrix pairs:

$$\mathcal{Z}_2(\mathcal{F}) = \{(X, Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | z(X+Y) = 0\}.$$

We characterize the linear operators that preserve the set $\mathcal{Z}_2(\mathcal{F})$.

Theorem 6.2. *Let $\mathcal{F}$ be a fuzzy semiring, $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ be a surjective linear map. Then T preserves the set $\mathcal{Z}_2(\mathcal{F})$ if and only if T is a (P, Q)-operator, where P and Q is a permutation matrices of appropriate sizes.*

Proof. By Theorem 4.1 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, where σ is a permutation on the set of pairs $\{(i,j) | 1 \leq i \leq m, 1 \leq j \leq n\}$.

Let us show that T maps lines to lines. For all $i = 1, 2, \dots, n$, let $C_1 + \dots + C_{n-1} = X$, $Y = C_n$. Then $z(X+Y) = z(J) = 0$. Hence $z(T(X) + T(Y)) = 0$ by assumption. Thus each column is mapped to column. Similarly, each row is mapped to row. Thus T maps lines to lines. By Lemma 4.2 it follows that T is a (P, Q)-operator where P and Q are permutation matrices of appropriate sizes.

Conversely, if $z(X+Y) = 0$ that is, sets of zero cells in X and Y are disjoint. Thus the same holds for $T(X)$ and $T(Y)$ since σ is a permutation. Hence in $(T(X)+T(Y))$ there is no zero elements. i.e. $z(T(X)+T(Y)) = 0$. Thus (P, Q)-operator preserves the set $\mathcal{Z}_2(\mathcal{F})$. $\square$

As a concluding remark, we have characterized the linear operators that preserve the extreme sets of the term rank inequalities and zero-term rank inequalities of the matrix sums over fuzzy semirings. For further research, we hope to study the term rank inequalities of matrix product and zero-term inequalities of matrix product over fuzzy semirings. Moreover, we hope to research the linear operators that preserve the extreme sets of the term rank inequalities of the matrix product over fuzzy semiring.

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⟨국문초록⟩

퍼지 행렬의 항별 계수 합의 선형보존자

본 논문에서는 퍼지 행렬의 짝들로 구성되는 집합들을 구성하였다. 이 집합들은 두 퍼지 행렬들의 합의 항별 계수와 영항 계수와 관련된 부등식의 극치인 경우들에서 자연스럽게 나타나는 퍼지 행렬 짝들의 집합들이다. 이 퍼지 행렬 짝들의 집합들은 두 퍼지 행렬의 항별 계수들의 합과 영항 계수들의 합과 관련된 부등식들에서 극치인 경우들로 구성하였다.

곧, 다음과 같은 5가지 집합을 구성하였다;

$$\mathcal{T}_1(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | t(X+Y) = t(X) + t(Y)\};$$

$$\mathcal{T}_2(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | t(X+Y) = 1\};$$

$$\mathcal{T}_3(\mathcal{F}) = \Big\{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | t(X+Y) = \max\{t(X), t(Y)\}\Big\};$$

$$\mathcal{Z}_1(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | z(X+Y) = \min\{z(X), z(Y)\}\};$$

$$\mathcal{Z}_2(\mathcal{F}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{F})^2 | z(X+Y) = 0\};$$

이상의 퍼지 행렬 짝들의 집합을 선형연산자로 보내어 그 집합의 성질들을 보존하는 선형연산자를 연구하여 그 형태를 규명하였다. 곧, 이러한 퍼지 행렬 짝들의 집합을 보존하는 선형연사자의 형태는 $T(X) = PXQ$ 또는 $T(X) = PX^tQ$로 나타남을 보이고, 이들을 증명하였다. 그리고 이 선형연산자가 위의 5가지 집합들을 보존함을 증명하였다.

감사의 글

처음 대학원 합격통보를 받고서 걱정을 많이 했습니다. 학부시절 공부를 열심히 하지 못 한 것이 못내 아쉬워 대학원 진학을 결심했지만 석사과정을 마칠 수 없을 것 같았습니다. 1년간은 혼자 하는 대학원 생활에 많이 힘들기도 했고 마음과는 다르게 공부가 뜻대로 되지 않을때는 포기하고 싶다는 생각뿐 이였습니다. 하지만 이렇게 2년이 지났고 졸업논문이 나오게 되었습니다. 저 혼자만의 힘으로 이 자리에 올 수 있었던 것이 아닙니다. 중간에 힘든 고비도 여러 차례 있었지만 송석준 교수님의 격려에 한 학기, 한 학기 마칠 수 있었습니다. 바쁘신 시간에도 논문지도를 꼼꼼히 해주신 교수님, 감사드립니다. 때로는 꾸지람으로 때로는 칭찬으로 저를 가르쳐 주신 방은숙 교수님, 양영오 교수님, 정승달 교수님, 윤용식 교수님, 유상욱 교수님, 진현성 교수님, 정말 감사드립니다.

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