碩師學位論文

The Basic Harmonic forms on a Non Harmonic Foliation

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CONTENTS

< **Abstract** >

The Basic Harmonic forms on a Non Harmonic Foliation

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We study the basic harmonic forms on non-harmonic foliations and prove that on an isoparametric Riemannian foliation with transverse Killing tension field, (i) if the transversal Ricci curvature is quasi-positive, then $H^1_B(\mathcal{F}) = 0$, (ii) if the transversal curvature operator F is quasi-positive, then $H^r_B(\mathcal{F}) = 0$ for $0 < r < q$.

1 Introduction

Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a closed manifold M. Reinhart([7]) introduced a basic differential form to provide a generalized notion of forms on the quotient space M/F, which is not on a manifold generally.

In particular, the basic de-Rham cohomology $H_B^*(\mathcal{F})$ of the complex of basic differential forms is of great interest and has been studied extensively. In contrast to the special case of Riemannian mainfolds, the operators d and δ defined as usual on the local quotients, are not in general adjoint operators.

The defect is related to the mean curvature of the leaves. In [2], Kamber and Tondeur studied this basic cohomology $H_B^*(\mathcal{F})$ under the additional assumption that the curvature form k of the leaves of $\mathcal{F}$ is a basic 1-form.

In 1991, M.Min-Oo and E.A.Ruh and Ph. Tondeur ([6]) proved the following. Let $\rho^\nabla : Q \to Q$ be the transveral Ricci operator on the normal bundle and $F : \wedge^2 Q \to \wedge^2 Q$ then transveral curvature operator. Then if $\rho^\nabla > 0$, the $H_B^1(\mathcal{F}) = 0$ and if $\mathcal{F}$ is positive definite, then $H_B^2(\mathcal{F}) = \{0\}$ for $0 < r < q$.

In this paper, we study the new operator $\tilde{\Delta} = \Delta_B - A_\tau$, where $A_\tau = \theta(\tau) - \nabla_\tau$.

In particular, we give the Weitzenböck type formula for $\tilde{\Delta}$ and using the

Bochuer technique, we estimate $Ker\tilde{\Delta}$ under some curvature conditions.

2 Riemannian foliation

Let M be a smooth manifold of dimension $p+q$.

Definition 2.1 A codimension q *foliation* $\mathcal{F}$ on M is given by an open cover $\mathcal{U} = (U_i)_{i\in I}$ and for each i, a diffeomorphism $\varphi_i : \mathbb{R}^{p+q} \to U_i$ such that, on $U_i \cap U_j \neq \emptyset$, the coordinate change $\varphi_j^{-1} \circ \varphi_i : \varphi_i^{-1}(U_i \cap U_j) \to \varphi_j^{-1}(U_i \cap U_j)$ has the form

$$\varphi_j^{-1} \circ \varphi_i(x, y) = (\varphi_{ij}(x, y), \gamma_{ij}(y)). \tag{2.1}$$

From Definition 2.1, the manifold M is decomposed into connected submanifolds of dimension p. Each of these submanifolds is called a *leaf* of $\mathcal{F}$. Coordinate patches (U_i, φ_i) are said to be *distinguished* for the foliation $\mathcal{F}$. The tangent bundle L of a foliation is the subbundle of TM, consisting of all vectors tangent to the leaves of $\mathcal{F}$. The normal bundle Q of a codimension q foliation $\mathcal{F}$ on M is the quotient bundle $Q = TM/L$. Equivalently, the normal bundle Q appears in the exact sequence of vector bundles

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0. \tag{2.2}$$

If $(x_1, \ldots, x_p; y_1, \ldots, y_q)$ are local coordinates in a distinguished chart U, the bundle $Q|U$ is framed by the vector fields $\pi\frac{\partial}{\partial y_1}, \ldots, \pi\frac{\partial}{\partial y_q}$. For a vector field $Y \in \Gamma TM$, we denote also $\overline{Y} = \pi Y \in \Gamma Q$.

Definition 2.2 A vector field Y on U is *projectable* if $Y = \sum_i a_i \frac{\partial}{\partial x_i} + \sum_\alpha b_\alpha \frac{\partial}{\partial y_\alpha}$ with $\frac{\partial b_\alpha}{\partial x_i} = 0$ for all $\alpha = 1, \ldots, q$ and $i = 1, \ldots, p$.

This means that the functions $b_\alpha = b_\alpha(y)$ are independent of x. Then $\overline{Y} = \sum_\alpha b_\alpha \frac{\bar{\partial}}{\partial y_\alpha}$, where b_α is independent of x. This property is preserved under the change of distinguished charts, hence makes intrinsic sense.

The *transversal geometry* of a foliation is the geometry infinitesimally modeled by Q, while the tangential geometry is infinitesimally modeled by L. A key fact is the existence of the *Bott connection* in Q defined by

$$\overset{\circ}{\nabla}_X s = \pi([X, Y_s]) \quad \text{for } X \in \Gamma L, \tag{2.3}$$

where $Y_s \in TM$ is any vector field projecting to s under $\pi : TM \to Q$. It is a partial connection along L. The right hand side in (2.3) is independent of the choice of Y_s. Namely, the difference of two such choices is a vector field $X' \in \Gamma L$ and $[X, X'] \in \Gamma L$ so that $\pi[X, X'] = 0$.

Definition 2.3 A Riemannian metric g_Q on the normal bundle Q of a foliation $\mathcal{F}$ is *holonomy invariant* if

$$\theta(X) g_Q = 0 \quad \text{for all } X \in \Gamma L, \tag{2.4}$$

where $\theta(X)$ is the transverse Lie derivative.

Here we have by definition for $s, t \in \Gamma Q$,

$$(\theta(X) g_Q)(s, t) = X g_Q(s, t) - g_Q(\theta(X) s, t) - g_Q(s, \theta(X) t).$$

Definition 2.4 A *Riemannian foliation* is a foliation $\mathcal{F}$ with a holonomy invariant transversal metric g_Q. A metric g_M is a *bundle-like* if the induced metric g_Q on Q is holonomy invariant.

The study of Riemannian foliations was initiated by Reinhart in 1959([7]). A simple example of a Riemannian foliation is given by a nonsingular Killing vector field X on (M, g_M). This means that $\theta(X)g_M = 0$.

Definition 2.5 An *adapted connection* in Q is a connection restricting along L to the partial Bott connection $\overset{\circ}{\nabla}$.

To show that such connections exist, we consider a Riemannian metric g_M on M. Then TM splits orthogonally as $TM = L \oplus L^{\perp}$. This means that there is a bundle map $\sigma : Q \to L^{\perp}$ splitting the exact sequence (2.2), which satisfy $\pi \circ \sigma = identity$. This metric g_M on TM is then a direct sum

$$g_M = g_L \oplus g_{L^{\perp}}.$$

With $g_Q = \sigma^* g_{L^{\perp}}$, the splitting map $\sigma : (Q, g_Q) \to (L^{\perp}, g_{L^{\perp}})$ is a metric isomorphism. Let ∇^M be the Levi-Civita connection associated to the Riemannian metric g_M. Then the adapted connection ∇ in Q is defined by

$$\begin{cases} \nabla_X s = \overset{\circ}{\nabla}_X s = \pi([X, Y_s]) & \text{for } X \in \Gamma L, \\ \nabla_X s = \pi(\nabla^M_X Y_s) & \text{for } X \in \Gamma L^{\perp}, \end{cases} \tag{2.5}$$

where $s \in \Gamma Q$ and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $Q \cong L^\perp$. For any connection ∇ on Q, there is a torsion T_∇ defined by

$$T_\nabla(Y, Z) = \nabla_Y \pi(Z) - \nabla_Z \pi(Y) - \pi[Y, Z] \tag{2.6}$$

for any $Y, Z \in \Gamma TM$. Then we have the following proposition ([9]).

Proposition 2.6 *For any metric g_M on M and the adapted connection ∇ on Q defined by (2.5), we have $T_\nabla = 0$.*

Proof. For $X \in \Gamma L$, $Y \in \Gamma TM$, we have $\pi(X) = 0$ and

$$T_\nabla(X, Y) = \nabla_X \pi(Y) - \pi[X, Y] = 0.$$

For $Z, Z' \in \Gamma L^\perp$, we have

$$T_\nabla(Z, Z') = \pi(\nabla^M_Z Z') - \pi(\nabla^M_{Z'} Z) - \pi[Z, Z'] = \pi(T_{\nabla^M}(Z, Z')) = 0,$$

where T_{∇^M} is the (vanishing) torsion of ∇^M. Finally the bilinearity and skew symmetry of T_∇ imply the desired result. $\square$

The curvature R^∇ of ∇ is defined by

$$R^\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \text{for } X,\ Y \in TM.$$

From an adapted connection ∇ in Q defined by (2.5), its curvature R^∇ coincides with $\overset{\circ}{R}$ for $X, Y \in \Gamma L$, hence $R^\nabla(X, Y) = 0$ for $X, Y \in \Gamma L$. And we have the following proposition ([3,4,9]).

Proposition 2.7 *Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and bundle-like metric g_M with respect to $\mathcal{F}$. Let ∇ be a connection defined by (2.5) in Q with curvature R^∇. Then the following holds:*

$$i(X)R^\nabla = \theta(X)R^\nabla = 0 \tag{2.7}$$

for $X \in \Gamma L$

Proof. (i) Let $Y \in \Gamma TM$ and $s \in \Gamma Q$. Since $\theta(X)s = \pi[X, Y_s]$ for $s \in \Gamma Q, \theta(X)s = \nabla_X s$. Hence

$$\begin{aligned} R^\nabla(X,Y)s &= \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s \\ &= \theta(X)\nabla_Y s - \nabla_Y\theta(X)s - \nabla_{\theta(X)Y}s \\ &= (\theta(X)\nabla)_Y s = 0. \end{aligned}$$

(ii) Let $Y, Z \in \Gamma TM$ and $s \in \Gamma Q$. Then

$$\begin{aligned} (\theta(X)R^\nabla)(Y,Z)s &= \theta(X)R^\nabla(Y,Z)s - R^\nabla(\theta(X)Y,Z)s \\ &\quad -R^\nabla(Y,\theta(X)Z)s - R^\nabla(Y,Z)\theta(X)s \\ &= \theta(X)\{\nabla_Y\nabla_Z s - \nabla_Z\nabla_Y s - \nabla_{[Y,Z]}s\} \\ &\quad -\{\nabla_{\theta(X)Y}\nabla_Z s - \nabla_Z\nabla_{\theta(X)Y}s - \nabla_{[\theta(X)Y,Z]}s\} \\ &\quad -\{\nabla_Y\nabla_{\theta(X)Z}s - \nabla_{\theta(X)Z}\nabla_Y s - \nabla_{[Y,\theta(X)Z]}s\} \\ &\quad -\{\nabla_Y\nabla_Z\theta(X)s - \nabla_Z\nabla_Y\theta(X)s - \nabla_{[Y,Z]}\theta(X)s\} \end{aligned}$$

$$
\begin{aligned}
&= \nabla_Y(\theta(X)\nabla_Z s) - \nabla_Z(\theta(X)\nabla_Y s) - \nabla_{\theta(X)[Y,Z]} s \\
&\quad + \nabla_Z\nabla_{\theta(X)Y} s + \nabla_{[\theta(X)Y,Z]} s - \nabla_Y\nabla_{\theta(X)Z} s \\
&\quad + \nabla_{[Y,\theta(X)Z]} s - \nabla_Y\nabla_Z\theta(X)s + \nabla_Z\nabla_Y\theta(X)s \\
&= -\nabla_{\theta(X)[Y,Z]} s + \nabla_{[\theta(X)Y,Z]} s + \nabla_{[Y,\theta(X)Z]} s \\
&= (-\nabla_{[X,[Y,Z]]} + \nabla_{[[X,Y],Z]} + \nabla_{[Y,[X,Z]]})s = 0. \quad \square
\end{aligned}
$$

By Proposition 2.7, we can define the (transversal) Ricci curvature $\rho^\nabla : \Gamma Q \to \Gamma Q$ and the (transversal) scalar curvature σ^∇ of $\mathcal{F}$ respectively by

$$
\rho^\nabla(s) = \sum_a R^\nabla(s, E_a)E_a, \quad \sigma^\nabla = \sum_a g_Q(\rho^\nabla(E_a), E_a), \tag{2.8}
$$

where $\{E_a\}_{a=1,\cdots,q}$ is an orthonormal basis of Q.

The *second fundamental form* α of $\mathcal{F}$ is given by

$$
\alpha(X, Y) = \pi(\nabla^M_X Y) \quad \text{for } X, Y \in \Gamma L. \tag{2.9}
$$

Proposition 2.8 *α is Q-valued, bilinear and symmetric.*

Proof. By definition, it is trivial that α is Q-valued and bilinear. Next, by torsion freeness of ∇^M, we have that for any $X, Y \in \Gamma L$,

$$
\alpha(X, Y) = \pi(\nabla^M_X Y) = \pi(\nabla^M_Y X) - \pi([X, Y]).
$$

Since $[X, Y] \in \Gamma L$ for any $X, Y \in \Gamma L$, we have

$$
\alpha(X, Y) = \pi(\nabla^M_Y X) = \alpha(Y, X). \quad \square
$$

Definition 2.9 The *mean curvature vector field* of $\mathcal{F}$ is then defined by

$$\tau = \sum_i \alpha(E_i, E_i) = \sum_i \pi(\nabla^M_{E_i} E_i), \tag{2.10}$$

where $\{E_i\}_{i=1,\cdots,p}$ is an orthonormal basis of L. The dual form κ, the *mean curvature form* for L, is then given by

$$\kappa(X) = g_Q(\tau, X) \quad \textit{for } X \in \Gamma Q. \tag{2.11}$$

The foliation $\mathcal{F}$ is said to be *minimal* (or *harmonic*) if $\kappa = 0$.

For the later use, we recall the divergence theorem on a foliated Riemannian manifold ([9]).

Theorem 2.10 *Let $(M, g_M, \mathcal{F})$ be a closed, oriented, connected Riemannian manifold with a transversally orientable foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then*

$$\int_M div_\nabla(X) = \int_M g_Q(X, \tau) \tag{2.12}$$

for all $X \in \Gamma Q$, where $div_\nabla(X)$ denotes the transverse divergence of X with respect to the connection ∇ defined by (2.5).

Proof. Let $\{E_i\}$ and $\{E_a\}$ be orthonormal basis of L and Q, respectively. Then for any $X \in \Gamma Q$,

$$\begin{aligned} div(X) &= \sum_i g_M(\nabla^M_{E_i} X, E_i) + \sum_a g_M(\nabla^M_{E_a} X, E_a) \\ &= \sum_i -g_M(X, \pi(\nabla^M_{E_i} E_i)) + \sum_a g_M(\pi(\nabla^M_{E_a} X), E_a) \\ &= -g_Q(X,\tau) + \sum_a g_Q(\nabla_{E_a} X, E_a) \\ &= -g_Q(X,\tau) + div_\nabla(X). \end{aligned}$$

By Green's Theorem on an ordinary manifold M, we have

$$0 = \int_M div(X) d_M = \int_M div_\nabla(X) d_M - \int_M g_Q(X,\tau).$$

This completes the proof of this Theorem. □

Corollary 2.11 *If $\mathcal{F}$ is minimal, then we have that for any $X \in \Gamma Q$,*

$$\int_M div_\nabla(X) = 0. \tag{2.13}$$

3 The Basic Cohomology

Definition 3.1 Let $\mathcal{F}$ be an arbitrary foliation on a manifold M. A differential form $\omega \in \Omega^r(M)$ is *basic* if

$$i(X)\omega = 0 \ \theta(X)\omega = 0, \ \text{for } X \in \Gamma L. \tag{3.1}$$

In a distinguished chart $(x_1, \ldots, x_p; y_1, \ldots, y_q)$ of $\mathcal{F}$, a basic form w is expressed by

$$\omega = \sum_{a_1 < \cdots < a_r} \omega_{a_1 \cdots a_r} dy_{a_1} \wedge \cdots \wedge dy_{a_r},$$

where the functions $\omega_{a_1 \cdots a_r}$ are independent of x, i.e. $\frac{\partial}{\partial x_i} \omega_{a_1 \cdots a_r} = 0$.

Let $\Omega^r_B(\mathcal{F})$ be the set of all basic r-forms on M. The exterior differential on the de Rham complex $\Omega^*(M)$ restricts by the cartan formula $\theta(X) = di(X) + i(X)d$ to a differential $d_B : \Omega^r_B(\mathcal{F}) \to \Omega^{r+1}_B(\mathcal{F})$. then the differential d_B defines then *basic* De Rham complex :

$$0 \xrightarrow{d_B} \Omega^0_B(\mathcal{F}) \xrightarrow{d_B} \ldots \xrightarrow{d_B} \Omega^r_B(\mathcal{F}) \xrightarrow{d_B} \Omega^{r+1}_B(\mathcal{F}) \xrightarrow{d_B} \ldots \xrightarrow{d_B} 0. \tag{3.2}$$

Definition 3.2 The basic cohomology $H^*_B(\mathcal{F}) = H_B(\Omega^*_B(\mathcal{F}), d_B)$ is define by $H_B(\Omega^*_B(\mathcal{F}), d_B) = Ker(d_B)/Im(d_B)$.

The basic cohomology $H^*_B(\mathcal{F}) = H_B(\Omega^*_B(\mathcal{F}), d_B)$ plays the role of the De Rham cohomology of the leaf space $M/\mathcal{F}$ of the foliation.

We also need the star operator $\bar{*} : \Omega_B^r(\mathcal{F}) \to \Omega_B^{q-r}(\mathcal{F})$ naturally associated to g_Q. The relations between $\bar{*}$ and $*$ are characterized by

$$\bar{*}\phi = (-1)^{p(q-r)} * (\phi \wedge \chi_{\mathcal{F}}),$$

$$*\phi = \bar{*}\phi \wedge \chi_{\mathcal{F}}$$

for $\phi \in \Omega_B^r(\mathcal{F})$, where $\chi_{\mathcal{F}}$ is the characteristic form of $\mathcal{F}$ and $*$ is the Hodge star operator. So we can define a Riemannian metric $< , >_B$ on $\Omega_B^r(\mathcal{F})$ by $< \phi, \psi >_B = \phi \wedge \bar{*}\psi \wedge \chi_{\mathcal{F}}$ for any $\phi, \psi \in \Omega_B^r(\mathcal{F})$ and the global inner product is given by

$$\ll \phi, \psi \gg_B = \int_M < \phi, \psi >_B .$$

With respect to this scalar product, the adjoint $\delta_B : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r-1}(\mathcal{F})$ of d_B is given by

$$\delta_B \phi = (-1)^{q(r+1)+1} \bar{*}(d_B - \kappa \wedge)\bar{*}\phi.$$

Let $\{E_a\}_{a=1,\cdots,q}$ be an orthonormal basis with $(\nabla E_a)_x = 0$ for Q and $\{\theta_a\}$ its g_Q-dual. Then we have the following proposition

Proposition 3.3 ([1]) *On the Riemannian foliation $\mathcal{F}$, we have*

$$d_B \phi = \sum_a \theta_a \wedge \nabla_{E_a} \phi, \quad \delta_B \phi = -\sum_a i(E_a) \nabla_{E_a} \phi + i(\tau)\phi.$$

Definition 3.4 The basic Laplacian acting on $\Omega_B^*(\mathcal{F})$ is defined by $\Delta_B = d_B \delta_B + \delta_B d_B$.

Trivially, the basic Laplacian Δ_B involve the mean curvature k. Let

$$\mathcal{H}_B^r = Ker\Delta_B \tag{3.3}$$

be the set of the *basic harmonic forms* of degree r. It is well known ([2]) that for $\kappa \in \Omega_B^1(\mathcal{F})$,

$$\Omega_B^r(\mathcal{F}) = Imd_B \oplus Im\delta_B \oplus \mathcal{H}_B^r$$

with finite dimensional $\mathcal{H}_B^r$.

Theorem 3.5 *Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a closed oriented manifold (M, g_M). Assume g_M to be bundle-like metric with $\kappa \in \Omega_B^1(\mathcal{F})$. Then*

$$H_B^r(\mathcal{F}) \cong \mathcal{H}_B^r(\mathcal{F}).$$

4 The basic harmonic forms

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with $\kappa \in \Omega_B^1(\mathcal{F})$. For any $Y \in \Gamma TM$, We define an operator $A_Y : \Omega_B^r(\mathcal{F}) \to \Omega_B^r(\mathcal{F})$ by

$$A_Y \phi = \theta(Y)\phi - \nabla_Y \phi \tag{4.1}$$

for $\phi \in \Omega_B^r(\mathcal{F})$, where $\theta(Y)$ is a Lie derivative. We also introduce the operator $\nabla_{tr}^* \nabla_{tr} : \Omega_B^*(\mathcal{F}) \to \Omega_B^*(\mathcal{F})$ as

$$\nabla_{tr}^* \nabla_{tr} = -\sum_a \nabla_{E_a,E_a}^2 + \nabla_Y, \tag{4.2}$$

where $\nabla_{X,Y}^2 = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$ for any $X, Y \in TM$. Then we have

Proposition 4.1 *The operator $\nabla_{tr}^* \nabla_{tr}$ satisfies*

$$\ll \nabla_{tr}^* \nabla_{tr} \phi_1, \phi_2 \gg_B = \ll \nabla \phi_1, \nabla \phi_2 \gg_B \tag{4.3}$$

for all $\phi_1,\ \phi_2 \in \Omega_B^*(\mathcal{F})$ provided that one of ϕ_1 and ϕ_2 has compact support, where $< \nabla\phi_1, \nabla\phi_2 >_B = \sum_a < \nabla_{E_a}\phi_1, \nabla_{E_a}\phi_2 >$.

Proof. Fix $x \in M$. We choose an orthonormal frame $\{E_a\}$ satisfying $(\nabla E_a)_x = 0$. For any $\phi_1,\ \phi_2 \in \Omega_B^*(\mathcal{F})$,

$$\begin{aligned} < \nabla_{tr}^* \nabla_{tr} \phi_1, \phi_2 >_B &= < -\sum_a \nabla_{(E_a,E_a)}^2 \phi_1 + \nabla_\tau \phi_1, \phi_2 >_B \\ &= -\sum_a < \nabla_{E_a} \nabla_{E_a} \phi_1, \phi_2 >_B + < \nabla_\tau \phi_1, \phi_2 >_B \end{aligned}$$

$$
\begin{aligned}
&= -\sum_a \{E_a < \nabla_{E_a}\phi_1, \phi_2 >_B - < \nabla_{E_a}\phi_1, \nabla_{E_a}\phi_2 >_B\} \\
&\quad + < \nabla_\tau \phi_1, \phi_2 >_B \\
&= -div_\nabla(v) + \sum_a < \nabla_{E_a}\phi_1, \nabla_{E_a}\phi_2 >_B \\
&\quad + < \nabla_\tau \phi_1, \phi_2 >_B \\
&= -div_\nabla(v) + < \nabla\phi_1, \nabla\phi_2 >_B + < \nabla_\tau\phi_1, \phi_2 >_B
\end{aligned}
$$

where $v \in \Gamma(Q)$ is defined by the condition that $g_Q(v, w) =< \nabla_w \phi_1, \phi_2 >_B$ for all $w \in \Gamma(Q)$. The last line is proved as follows: At $x \in M$,

$$
div_\nabla(v) = \sum_a g_Q(\nabla_{E_a} v, E_a) = \sum_a E_a < \nabla_{E_a}\phi_1, \phi_2 >_B .
$$

By the divergence theorem in Theorem 2.9 on a foliated Riemannian manifold,

$$
\int_M div_\nabla(v) = \ll \tau, v \gg_B = \ll \nabla_\tau \phi_1, \phi_2 \gg_B .
$$

Hence we have

$$
\begin{aligned}
\ll \nabla_{tr}^* \nabla_{tr} \phi_1, \phi_2 \gg_B &= \int_M div_\nabla(v) + \ll \nabla\phi_1, \nabla\phi_2 \gg_B + \ll \nabla_\tau\phi_1, \phi_2 \gg_B \\
&= - \ll \nabla_\tau \phi_1, \phi_2 \gg_B + \ll \nabla\phi_1, \nabla\phi_2 \gg_B \\
&\quad + \ll \nabla_\tau\phi_1, \phi_2 \gg_B \\
&= \ll \nabla\phi_1, \nabla\phi_2 \gg_B .
\end{aligned}
$$

Therefore, the proof is completed. $\square$

Put $\tilde{\Delta} = \Delta_B - A_\tau$. Then $\tilde{\Delta}$ is a transversally elliptic but it is not self-adjoint. We call $\tilde{\Delta}$ as the generalized basic Laplacian. By the straight calculation, we have the following theorem.

Theorem 4.2 *On the Riemannian foliation $\mathcal{F}$, we have*

$$\tilde{\Delta}\phi = \nabla_{tr}^* \nabla_{tr}\phi + F(\phi) \tag{4.4}$$

for $\phi \in \Omega_B^r(\mathcal{F})$, where $F(\phi) = \sum_{a,b} \theta_a \wedge i(E_b) R^\nabla(E_b, E_a)\phi$.

Proof. Let ϕ be a basic r-form. Let $\{E_a\}$ be an orthonormal basis for Q with $\nabla E_a = 0$ and $\{\theta_a\}$ its g_i dual basis. Then we have

$$\begin{aligned} d_B\delta_B\phi &= \sum_a \theta_a \wedge \nabla_{E_a} - \sum_b \{i(E_b)\nabla_{E_b}\phi + i(\tau)\phi\} \\ &= -\sum_{a,b} \theta_a \wedge \nabla_{E_a}\{i(E_b)\nabla_{E_b}\phi\} + \sum_a \theta_a \wedge \nabla_{E_a} i(\tau)\phi \\ &= -\sum_{a,b} \theta_a \wedge i(E_b)\nabla_{E_a}\nabla_{E_b}\phi + d_B i(\tau)\phi \end{aligned}$$

and

$$\begin{aligned} \delta_B d_B\phi &= -\sum_{a,b} i(E_b)\nabla_{E_b}\{\theta_a \wedge \nabla_{E_a}\phi\} + i(\tau)d_B\phi \\ &= -\sum_{a,b}\{i(E_b)\theta_a\}\nabla_{E_b}\nabla_{E_a}\phi + i(\tau)d_B\phi + \sum_{a,b} \theta_a \wedge i(E_b)\nabla_{E_b}\nabla_{E_a}\phi \\ &= -\nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b} \theta_a \wedge i(E_b)\nabla_{E_b}\nabla_{E_a}\phi + i(\tau)d_B\phi. \end{aligned}$$

Summing up the above two equations, we have

$$\begin{aligned}\Delta_B\phi &= d_B\delta_B\phi + \delta_B d_B\phi \\ &= d_B i(\tau)\phi + i(\tau)d_B\phi - \nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b}\theta_a\wedge i(E_b)R^{\nabla}(E_b,E_a)\phi \\ &= \theta(\tau)\phi - \nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b}\theta_a\wedge i(E_b)R^{\nabla}(E_b,E_a)\phi. \\ &= -\nabla_{E_a}\nabla_{E_a}\phi + F(\phi) + A_{\nabla}(\tau)\phi + \nabla_\tau\phi \\ &= -\nabla^2_{(E_a,E_a)}\phi + \nabla_\tau\phi + F(\phi) + A_{\nabla}(\tau)\phi \\ &= \nabla^*_{tr}\nabla_{tr}\phi + F(\phi) + A_{\nabla}(\tau)\phi\end{aligned}$$

Hence we have

$$\tilde{\Delta}_B\phi = \Delta_B\phi - A_\tau\phi = \nabla^*_{tr}\nabla_{tr}\phi + F(\phi).$$

Therefore the proof is completed. □

From the Proposition 4.1 and Theorem 4.2, we have the following theorem.

Theorem 4.3 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with $\kappa \in \Omega^1_B(\mathcal{F})$. If F is non-negative, $\tilde{\Delta}$-harmonic forms are parallel. If F is quasi-positive, then $Ker\tilde{\Delta} = \{0\}$.*

On the other hand, it is known ([7]) that if $\pi(Y)$ is a transverse Killing field, i.e., $\theta(Y)g_Q = 0$ if and only if

$$< A_Y\phi, \psi >_B + < \phi, A_Y\psi >_B = 0 \quad \text{for } \phi,\ \psi \in \Omega^r_B(\mathcal{F}). \tag{4.5}$$

From this equation, if τ is a transverse Killing field, then for any $\phi \in \Omega_B^r(\mathcal{F})$

$$< A_\tau \phi, \phi >_B = 0. \tag{4.6}$$

Hence we have

$$< \tilde{\Delta}\phi, \phi >_B = < \Delta_B \phi, \phi >_B \quad \text{for any } \phi \in \Omega_B^r(\mathcal{F}).$$

By Theorem 2.3, if $\phi \in Ker\Delta_B$, then we have

$$0 = |\nabla_{tr}\phi|^2 + < F(\phi), \phi >_B .$$

Hence we have the following theorem.

Theorem 4.4 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with $\kappa \in \Omega_B^1(\mathcal{F})$. Assume that the tension field τ is a transverse Killing field. If F is quasi-positive, then every basic harmonic r-forms is zero. i.e., $\mathcal{H}_B^r(\mathcal{F}) = 0$.*

Remark. If $\mathcal{F}$ is minimal, $\Delta_B = \tilde{\Delta}$.

Let ϕ be a basic 1-form and ϕ^* its g_Q-dual. Then we have

$$\begin{aligned}
< F(\phi), \phi > &= < \sum_{a,b} \theta_a \wedge i(E_b) R^\nabla(E_b, E_a)\phi, \phi > \\
&= \sum_{a,b} i(E_b) R^\nabla(E_b, E_a)\phi < \theta_a, \phi > \\
&= \sum_{a,b} < R^\nabla(E_b, E_a)\phi^*, E_b >< E_a, \phi^* >
\end{aligned}$$

$$= \sum_a < R^{\nabla}(\phi^*, E_a)E_a, \phi^* >$$

$$=< \rho^{\nabla}(\phi^*), \phi^* >,$$

where ρ^{∇} is the transversal Ricci curvature. From this equation, we have the following corollary.

Corollary 4.5 *Under the same assumptions as in Theorem 4.4. If the transversal Ricci curvature is non-negative, then every basic harmonic 1-form is parallel. If the transversal Ricci curvature is quasi positive, then every basic harmonic 1-form is zero, i.e.,* $\mathcal{H}^1_B(\mathcal{F}) = 0$.

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<국 문 초 록>

엽층구조를 가지는 조화적이 아닌 구조에서의 basic harmonic 형식

본 연구는 엽층들이 조화적(또는 극소적)이 아닌 경우 basic harmonic 형식들의 성질을 조사하였다.
더구나 basic cohomology군의 특성을 횡단적 Ricci곡률의 조건하에 일반다양체의 특성과 얼마나 차이가 있는지 조사 연구하였다.
실제로 횡단적 Ricci 곡률이 0보다 크거나 같으면 basic cohomology군 $H_B^1(F)=0$임을 보였다.

감사의 글

제 이름 석자가 새겨진 책 한권이 완성이 되었습니다.

미흡하고 부족하기만 한 저를 제자로 맞이하여 아낌없는 조언과 지도를 해주신 정승달 교수님께 감사의 마음을 전합니다.

또한 아는 것과 모르는 것의 차이를 알게 해주신 정보수학과 교수님들(양영오 교수님, 송성준 교수님, 방은숙 교수님, 윤용식 교수님, 정승달 교수님)께 깊은 감사를 드립니다.

2년 6개월 동안 아는 것을 어떻게 가르쳐야 하는지를 지도해주신 수학교육학과 교수님들(현진오 교수님, 양성호 교수님, 김도현 교수님, 고봉수 교수님, 고윤희 교수님, 박진원 교수님)께도 깊은 감사를 드립니다.

그리고 논문을 심사해주신 유상욱 교수님과 현진오 교수님, 논문 작성을 도와주신 강은희 선배, 강경태 선생님, 항상 옆에서 격려해 주신 고연순 선생님, 문영봉 선생님, 함께 공부한 대학원 동기이자 선배인 정유철과 홍경호 선생님, 강정석 선생님, 이상헌 선생님께도 감사드립니다.

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공부는 혼자만의 싸움이 아니라는 걸 느꼈습니다. 누군가의 응원과 도움이 없다면 발전 할 수 없다는 것을 알게 되었습니다. 무엇을 하든 주위에 도움을 주는 많은 사람들의 격려와 사랑을 잊지 않고 최선을 다하는 사람이 되겠습니다.

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