碩士學位論文

Transversal Infinitesimal Automorphisms On The Kähler Foliation

濟州大學校 大學院

數學科

洪 昌 基

1997年 6月

Transversal Infinitesimal Automorphisms On The Kähler Foliation

Hong, Chang-Gi

(Supervised by professor Jin-Oh, Hyun)

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL
CHEJU NATIONAL UNIVERSITY

1997. 6.

Transversal Infinitesimal Automorphisms On The Kähler Foliation

指 導 敎 授　玄 進 五

洪 昌 基

이 論文을 理學 碩士學位 論文으로 提出함

1997年　6月

洪昌基의 理學 碩士學位 論文을 認准함

審 查 委 員 長 ______________

委　　　　員 ______________

委　　　　員 ______________

濟 州 大 學 校　大 學 院

1997年　6月

< Abstract >

Transversal Infinitesimal Automorphisms On The Kähler Foliation

In this thesis we extend the results which were studied on the harmonic Kähler foliation by S.Nishikawa and Ph.Tondeur([5]) to non-harmonic Kähler foliation, and then prove the following theorem.

Theorem. *Let $\mathcal{F}$ be a non-harmonic Kähler foliation on a closed manifold M with transversal Ricci operator $\rho_\nabla \leq 0$. If the transversally holomorphic infinitesimal automorphism is parallel in direction to the mean curvature vector, then $\nabla\pi(Y) = 0$. Moreover, if $\rho_\nabla < 0$ at some point $x \in M$ and $\nabla_\tau \pi(Y) = 0$, then $\pi(Y) = 0$. i.e, $x \in \Gamma L$.*

In addition, we also study the relation between transversal Killing vector and Killing vector.

CONTENTS

1. Introduction

The foliation theory has its origin in the global analysis of solutions of ordinary differential equations. The general notion of a foliation was defined by Ehresmann and Reeb([1]). In 1959, Reinhart([8]) introduced a particular type of foliation, namely, Riemannian foliation, which admits a particular metric "bundle-like metric", that is, a metric for which the leaves of the foliation remain locally at constant distance from each other. By virtue of this bundle-like metric, we can study the quotient manifold by leaves. We call this area as "transverse geometry". Many authors have studied the transverse geometry of Riemannian foliations. In particular, in 1988, S.Nishikawa and Ph. Tondeur([5]) introduced the Kähler foliation which admit complex structure on normal bundle of the leaves and proved the following theorem.

Theorem A. *Let $\mathcal{F}$ be a harmonic Kähler foliation on a closed orientable manifold M with transversal Ricci operator $\rho_\nabla \le 0$. Then every transversally holomorphic infinitesimal automorphism $Y \in V(\mathcal{F})$ satisfies $\nabla\pi(Y) = 0$. If $\rho_\nabla < 0$ at some point $x \in M$, then every $Y \in V(\mathcal{F})$ with tranrversally holomorphic s satisfies $Y \in \Gamma L$.*

In general, many results were obtained on harmonic foliations which all leaves are minimal submanifolds. In this thesis, we study some geometric transverse fields and extend Theorem A to non-harmonic Kähler foliations. Throughout this paper, we have the following indices:

$$1 \le i, j, \cdots \le p; \quad 1 \le a, b, \cdots \le n,$$

$$1 \le \alpha, \beta, \cdots \le q(= 2n), \quad 1 \le A, B, \cdots \le p + q.$$

2. Preliminaries

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$([8]). Let ∇^M be the Levi-Civita connection with respect to g_M. Let TM be the tangent bundle of M and L the integrable subbundle of TM given by $\mathcal{F}$. The normal bundle Q of $\mathcal{F}$ is given by $Q = TM/L$. The metric g_M defines a splitting σ of the exact sequence

$$O \longrightarrow L \longrightarrow TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \longrightarrow O \tag{2.1}$$

with $\sigma(Q) = L^{\perp}$(the orthonormal complement bundle of L in TM)([2]). Then g_M induces a metric g_Q on Q:

$$g_Q(s,t) = g_M(\sigma(s), \sigma(t)) \quad \text{for any} \quad s, t \in \Gamma Q.$$

A connection ∇ in Q is defined by

$$\begin{aligned} \nabla_X s &= \pi([X, Y]) \quad \text{for} \quad X \in \Gamma L, \quad \pi(Y) = s, \\ \nabla_X s &= \pi(\nabla^M_X Y_s) \quad \text{for} \quad X \in \Gamma Q, \quad Y_s = \sigma(s), \end{aligned} \tag{2.2}$$

where $s \in \Gamma Q$([2]). Then we have

Proposition 2.1([2]). *The connection ∇ in Q is torsion-free and metrical with respect to g_Q.*

The curvature R_∇ of ∇ is defined by

$$R_\nabla(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \tag{2.3}$$

for any $X, Y \in \Gamma TM$ and $s \in \Gamma Q$. Since $i(X)R_\nabla = 0$ for any $X \in \Gamma L$([2]), we can define the transversal Ricci operator $\rho_\nabla : \Gamma Q \to \Gamma Q$ of $\mathcal{F}$ by

$$\rho_\nabla(s) = \sum_{\alpha=1}^{q} R_\nabla(s, E_a)E_a \tag{2.4}$$

where $\{E_\alpha\}$ is an orthonormal basis of Q. Let $V(\mathcal{F})$ be the space of all vector fields Y on M satisfying

$$[Y, Z] \in \Gamma(L) \tag{2.5}$$

for any $Z \in \Gamma L$. An element of $V(\mathcal{F})$ is called an *infinitesimal automorphism* of $\mathcal{F}$([3]). We set

$$\Gamma Q^L = \{s \in \Gamma(Q) | s = \pi(Y), Y \in V(\mathcal{F})\}$$

Then $s \in \Gamma Q^L$ satisfies $\nabla_X s = 0$ for any $X \in \Gamma L$([3]). From (2.1), we have associated exact sequence of Lie algebras

$$O \longrightarrow \Gamma L \longrightarrow V(\mathcal{F}) \xrightarrow{\pi} \Gamma Q^L \longrightarrow O$$

From now on, the foliation is assumed to be transversally Kähler. By a K ähler foliation $\mathcal{F}$ we mean a foliation satisfying the following conditions: (i) $\mathcal{F}$ is Riemannian,with a bundle-like metric g_M on M inducing the holonomy invariant metric g_Q on $Q = L^\perp$, (ii) there is a holonomy invariant almost complex structure $J : Q \to Q$ where $dimQ = q = 2n$(real dimension), with respect to which g_Q is Hermition, i.e. $g_Q(Js, Jt) = g_Q(s,t)$ for $s, t \in \Gamma Q$, and (iii) if ∇ denotes the unique metric and torsion-free connection in Q, then ∇ is almost complex, i.e. $\nabla J = 0$. Then we have the following identities:

$$R_\nabla(s, t)J = JR_\nabla(s, t) \tag{2.6-1}$$

$$R_\nabla(Js, Jt) = R_\nabla(s, t) \tag{2.6-2}$$

$$R_\nabla(s, t)u + R_\nabla(t, u)s + R_\nabla(u, s)t = 0 \tag{2.6-3}$$

for s, t and u are elements of ΓQ. In fact, (2.6-1) follows from $\nabla J = 0$, (2.6-2) follow from $g_Q(Js, Jt) = g_Q(s, t)$.

Finally (2.6-3) is a consequence of the Jacobi identity. Let $\{e_A\}$ be an oriented orthonormal basis of T_xM with e_i in L_x and e_α in $L_x^\perp$ ($\mathcal{F}$ is of codimension $q = 2n$ on M^{p+q}). The transversal Kähler property of $\mathcal{F}$ allows then to extend e_a, Je_a to local vector fields $E_a, JE_a \in \Gamma L^\perp$ such that

$$(\nabla_{E_a} E_b)_x = (\nabla_{E_a} JE_b)_x = (\nabla_{JE_a} E_b)_x = (\nabla_{JE_a} JE_b)_x = 0 \tag{2.7}$$

As a consequence of torsion freeness([2])

$$[E_a, E_b]_x, \quad [E_a, JE_b]_x, \quad [JE_a, JE_b]_x \in L_x \tag{2.8}$$

The E_a, JE_a can be chosen as (local) infinitesimal automorphisms of $\mathcal{F}$, so that

$$\nabla_X E_a = \pi[X, E_a] = 0 \tag{2.9}$$

for $X \in \Gamma L$. We can complete E_a, JE_a by the Gram-Schmidt process to a moving local frame by adding $E_i \in \Gamma L$ with $(E_i)_x = e_i$. In terms of such a moving frame the transversal Ricci operator $\rho_\nabla : Q \to Q$ is given by

$$\rho_\nabla = \sum JR_\nabla(E_a, JE_a) \tag{2.10}$$

. In fact, let $s \in \Gamma Q$. Then by (2.6-1) and (2.6-3)

$$\begin{aligned} R_\nabla(E_a, JE_a)s &= -R_\nabla(JE_a, E_a)s = R_\nabla(E_a, s)JE_a + R_\nabla(s, JE_a)E_a \\ &= JR_\nabla(E_a, s)E_a - JR_\nabla(s, JE_a)JE_a \\ &= -J(R_\nabla(s, E_a)E_a + R_\nabla(s, JE_a)JE_a) \end{aligned}$$

or

$$JR_\nabla(E_a, JE_a)s = R_\nabla(s, E_a)E_a + R_\nabla(s, JE_a)JE_a$$

. It follows that

$$\rho_\nabla s = \sum (R_\nabla(s, E_a)E_a + R_\nabla(s, JE_a)JE_a)$$

is given by (2.10).

3. Infinitesimal automorphisms

The transverse Lie derivative $\theta(Y)$ with respect to $Y \in V(\mathcal{F})$ is defined by

$$\theta(Y)s = \pi([Y, Y_s]) \tag{3.1}$$

for any $s \in \Gamma Q$ with $\pi(Y_s) = s$.

Definition 3.1. *If $Y \in V(\mathcal{F})$ satisfies $\theta(Y)g_Q = 0$, then $s = \pi(Y)$ is Called a transverse Killing field of $\mathcal{F}$.*

If g_M is a bundle - like metric on M and $Y \in \Gamma TM$ a Killing vector field for (M, g_M), then $\pi(Y)$ is transversal Killing field for g_Q. But the converse is not necessarily true: $Y \in V(\mathcal{F})$ may satisfy $\theta(Y)g_Q = 0$ without satisfying $\theta(Y)g_M = 0$. Under what condition, is it true? In orther to this, let us introduce the following tensors([6]):

$$A_X Y = \pi D_{\pi X} \pi^{\perp} Y + \pi^{\perp} D_{\pi X} \pi Y$$

$$T_X Y = \pi D_{\pi^{\perp} X} \pi^{\perp} Y + \pi^{\perp} D_{\pi^{\perp} X} \pi Y$$

for any vector field $X, Y \in TM$. Since the Riemannian foliation can be considered as Riemannian submersion locally, it is possible. And the following properties hold;

$$A_X = 0 \quad \text{and} \quad A_U V = -A_V U$$

$$T_U = 0 \quad \text{and} \quad T_X Y = T_Y X$$

for any $X, Y \in \Gamma L$ and $U, V \in \Gamma Q$. The Riemannian foliation is said to be totally geodesic if all the leaves are totally geodesic submanifolds, that is,

$T = 0$. Moreover, the normal bundle $L^\perp \equiv Q$ is integrable if and only if $A = 0$(in this case the integral submanifolds of $L^\perp$ are totally geodesic). For any $Y \in V(\mathcal{F})$, we have

$$(\theta(Y)g_M)(Z,W)$$
$$= Yg_M(Z,W) - g_M(\theta(Y)Z,W) - g_M(Z,\theta(Y)W)$$

for any $Z, W \in \Gamma TM$. On the other hand, since $TM = L + L^\perp$, and $g_M = g_L + g_Q$, we have

$$Yg_M(Z,W) = Yg_L(\pi^\perp(Z), \pi^\perp(W)) + Yg_Q(\pi(Z), \pi(W)).$$

Also, using the property $\theta(Y)s = \pi[Y, Y_s], \pi(Y_s) = s$, we have

$$g_M(\theta(Y)Z,W) = g_M(\theta(Y)\pi(Z), \pi(W)) + g_M(\theta(Y)\pi^\perp(Z), \pi^\perp(W)).$$

Similarly

$$g_M(Z,\theta(Y)W) = g_M(\pi(Z), \theta(Y)\pi(W)) + g_M(\theta(Y)\pi^\perp(W), \pi^\perp(Z)).$$

Summing up the above equations, we have

$$(3.2)\quad (\theta(Y)g_M)(Z,W) = (\theta(Y)g_L)(\pi^\perp(Z), \pi^\perp(W)) + (\theta(Y)g_Q)(\pi(Z), \pi(W)).$$

On the other hand, since we get

$$
\begin{aligned}
&(\theta(Y)g_L)(\pi^{\perp}(Z), \pi^{\perp}(W)) \\
&\qquad = Y g_M(\pi^{\perp}(Z), \pi^{\perp}(W)) - g_M(\theta(Y)\pi^{\perp}(Z), \pi^{\perp}(W)) \\
&\qquad\quad - g_M(\pi^{\perp}(Z), \theta(Y)\pi^{\perp}(W)) \\
&\qquad = Y g_M(\pi^{\perp}(Z), \pi^{\perp}(W)) - g_M([Y, \pi^{\perp}(Z)], \pi^{\perp}(W)) \\
&\qquad\quad - g_M(\pi^{\perp}(Z), [Y, \pi^{\perp}(W)]) \\
&\qquad = Y g_M(\pi^{\perp}(Z), \pi^{\perp}(W)) - g_M((\nabla^M_Y \pi^{\perp}(Z) - \nabla^M_{\pi^{\perp}(Z)} Y), \pi^{\perp}(W)) \\
&\qquad\quad - g_M(\pi^{\perp}(Z), (\nabla^M_Y \pi^{\perp}(W) - \nabla^M_{\pi^{\perp}(W)} Y)) \\
&\qquad = Y g_M(\pi^{\perp}(Z), \pi^{\perp}(W)) - g_M(\nabla^M_Y \pi^{\perp}(Z), \pi^{\perp}(W)) \\
&\qquad\quad - g_M(\pi^{\perp}(Z), \nabla^M_Y \pi^{\perp}(W)) + g_M(\nabla^M_{\pi^{\perp}(Z)} Y, \pi^{\perp}(W)) \\
&\qquad\quad + g_M(\pi^{\perp}(Z), \nabla^M_{\pi^{\perp}(W)} Y) \\
&\qquad = (\nabla^M_Y g_M)(\pi^{\perp}(Z), \pi^{\perp}(W)) + g_M(\nabla^M_{\pi^{\perp}(Z)} Y, \pi^{\perp}(W)) \\
&\qquad\quad + g_M(\pi^{\perp}(Z), \nabla^M_{\pi^{\perp}(W)} Y) \\
&\qquad = g_M(\nabla^M_{\pi^{\perp}(Z)} Y, \pi^{\perp}(W)) + g_M(\pi^{\perp}(Z), \nabla^M_{\pi^{\perp}(W)} Y).
\end{aligned}
$$

we have

$$
\begin{aligned}
(3.3)\quad (\theta(Y)g_M)(Z, W) &= (\theta(Y)g_Q)(\pi(Z), \pi(W)) \\
&\quad + g_M(\nabla^M_{\pi^{\perp}(Z)} Y, \pi^{\perp}(W)) + g_M(\pi^{\perp}(Z), \nabla^M_{\pi^{\perp}(Z)} Y)
\end{aligned}
$$

From this equality, we have

Proposition 3.2. *Let $\pi(Y)$ be the transversal Killing field of M, Then Y is a Killing field of M if and only if*

$$
g_M(\nabla^M_Z Y, W) + g_M(Z, \nabla^M_W Y) = 0 \tag{3.4}
$$

for any $Z, W \in \Gamma L$.

From this Proposition 3.2, we have

Corollary 3.3. *Let πY be the transversal Killing field of M. If the infinitesimal automorphism Y is parallel along the leaves, then Y is a Killing field of M.*

On the other hand, we have

(3.5)

$$g_M(\nabla^M_{\pi^\perp(Z)}Y, \pi^\perp(W)) = g_M(T_Z Y, \pi^\perp(W)) + g_M(\nabla^M_{\pi^\perp(Z)}\pi^\perp(Y), \pi^\perp(W))$$

From this equation, we have

Coroally 3.4. *Let $\mathcal{F}$ be the totally geodesic foliation. Let $\pi(Y)$ be the transversal Killing field of M. If the tangential part of the infinitesimal automorphism Y is parallel along the leaves, then Y is Killing field of M.*

Definition 3.5. *If $Y \in V(\mathcal{F})$ satisfies $\nabla_X s = 0$, then $s = \pi(Y)$ is called* **a** *projectable.*

Definition 3.6. *If $Y \in V(\mathcal{F})$ satisfies $\theta(Y)J = 0$, then $s = \pi(Y)$ is called* **a** *transversally holomorphic.*

By definition, for $Z \in \Gamma L^\perp$

$$(\theta(Y)J)(Z) = \theta(Y)(JZ) - J(\theta(Y)Z) \tag{3.6}$$

But this expression equals $\pi[Y, JZ] - J\pi[Y, Z]$, which yields the formula

$$(\theta(Y)J)(Z) = -\nabla_{JZ}s + J\nabla_Z s \tag{3.7}$$

for $Y \in V(\mathcal{F})$ and $s = \pi(Y)$. Hence $s = \pi(Y)$ is a transversally holomorphic if and only if

$$\nabla_{JZ} s = J \nabla_Z s \quad \text{for all} Z \in \Gamma L^{\perp}. \tag{3.8}$$

4. Vanishing Theorem

Let $\Omega^r(M.Q) \cong \Gamma(Q) \otimes \Omega^r(M)$ be the space of Q- valued r-forms over M, where $\Omega^r(M)$ is a space of differential r-forms on M. For any $s \in \Gamma(Q)$ and $\eta \in \Omega^r(M)$, the element $s \otimes \eta \in \Omega^r(M, Q)$ is usually abbreviated to $s\eta$. We can consider the connection ∇ given in (2.2) as an R-Liner map $\nabla : \Omega^0(M, Q) \to \Omega^1(M, Q)$ such that

$$\nabla(fs) = f\nabla s + sdf \tag{4.1}$$

for any $f \in \Omega^0(M), s \in \Gamma(Q)$ and such that

$$d < s_1, s_2 >= \nabla s_1 \wedge s_2 + s_1 \wedge \nabla s_2 \tag{4.2}$$

for any $s_1, s_2 \in \Omega^0(M, Q)$, where we define

$$s_1\eta_1 \wedge s_2\eta_2 =< s_1, s_2 > \eta_1 \wedge \eta_2$$

for any $s_1\eta_1 \in \Omega^r(M, Q)$ and $s_2\eta_2 \in \Omega^s(M, Q)$. By the usual algebraic formalism, $\nabla : \Omega^0(M, Q) \to \Omega^1(M, Q)$ can be extended to an anti-derivation

$$d_\nabla : \Omega^r(M, Q) \to \Omega^{r+1}(M, Q)$$

by the following rule: if $s\eta \in \Omega^r(M, Q)$, then

$$d_\nabla(s\eta) = \nabla s \wedge \eta + s(d\eta) \tag{4.3}$$

for $s \in \Gamma(Q), \eta \in \Omega^r(M, Q)$. for a Riemannian metric g_M on M, we extend the star operator $* : \Omega^r(M) \to \Omega^{n-r}(M) (n = dim\text{M})$ to

$$* : \Omega^r(M, Q) \to \Omega^{n-r}(M, Q)$$

as follows : if $s \in \Gamma(Q)$ and $\eta \in \Omega^r(M)$, then

(4.4) $$*(s\eta) = s(*\eta)$$

Moreover the operator $d^*_\nabla : \Omega^r(M,Q) \to \Omega^{r-1}(m,Q)$ given by

(4.5) $$d^*_\nabla \phi = (-1)^{n(r+1)+1} * d_\nabla * \phi, \phi \in \Omega^r(M,Q)$$

is adjoint of d_∇ with respect to an inner product $< \cdot, \cdot >$ defined by

(4.6) $$< s_1\eta_1, s_2\eta_2 >= g_Q(s_1, s_2)(\eta_1, \eta_2).$$

The Laplacian Δ for $\Omega^*(M,Q)$ is given by

(4.7) $$\Delta = d_\nabla d^*_\nabla + d^*_\nabla d_\nabla$$

Let $e_1, \cdot, e_n$ be orthonotmal basis of T_xM and $E_1, \cdot, E_n$ a local framing of TM in a neighborhood of x, coinciding with $e_1, \cdot, e_n$ at x and satisfying $\nabla^M_{e_A} E_B = (\nabla^M_{E_A} E_B)_x = 0(A, B = 1, \cdot, n)$, where$\nabla^M$ denotes the Riemannian connection of (M, g_M). Let w^A be a coframe field of e_A. Then on $\Omega^*(M,Q)$ We have

(4.8) $$d_\nabla = \sum w^A \wedge \tilde{\nabla}_{e_A}, \quad d^*_\nabla = -\sum i(e_a)\tilde{\nabla}_{e_A},$$

where $\tilde{\nabla}$ is a connection on $\Omega^*(M,Q)$ defined as

$$\nabla_X(s\eta) = (\nabla_X s)\eta + s(\nabla^M_X \eta)$$

and

$$i(X)(s\eta) = s[i(X)\eta]$$

for $s \in \Gamma L, \eta \in A^*(M)$. The Q-valued bilinear form α on M is defined by

$$(4.9) \qquad \alpha(X,Y) = -(\nabla_X \pi)(Y)$$

for all $X, Y \in \Gamma TM$ ([9]). Since $\alpha(X,Y) = \pi(\nabla^M_X Y)$ for all $X, Y \in \Gamma L$, we call α the *second fundamental form* of $\mathcal{F}$([9]). The tension field τ of $\mathcal{F}$ is defined by

$$(4.10,) \qquad \tau = \sum_{i=1}^{p} \alpha(E_i, E_i)$$

where $\{E_i\}_{i=1,\cdots,p}$ is an orthonormal basis of L. We remark that $\tau = d^*_\nabla \pi \in \Gamma Q$([9]). The foliation $\mathcal{F}$ is *minimal (or harmonic)* if $\tau = 0$([9]).

Theroem 4.1([2]). *(Transversal divergence theroem) Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a closed oriented Riemannian manifold (M, g_M). Let $Y \in V(\mathcal{F})$, then*

$$\int_M div_B Y \cdot \mu = \int_M g_Q(\tau, Y) \cdot \nu =< \tau, Y >$$

(the global scalar product of sections τ and Y of Q).

We calculate the Laplacian of $S = \pi(Y)$:

$$(\triangle s)_x = (d^*_\nabla d_\nabla s)_x = -\sum_{A=1}^{m} (\nabla_{e_A}(d_\nabla s))(e_A) = -\sum_{A=1}^{m} (\nabla_{e_A} \nabla_{E_A} s - \nabla_{\nabla^M_{e_A} E_A} s).$$

Since $\nabla_{e_a} E_a = \pi(\nabla^M_{e_a} E_a) = 0$, we have $\nabla^M_{e_a} E_a \in \Gamma L_x$. This implies that $\nabla_{\nabla^M_{e_A} E_A} s = \nabla_{\tau_x} s$, where $\tau_x = \sum_i^p \pi(\nabla^M_{e_i} E_i)$ is the mean curvature vector field of $\mathcal{F}$. Hence we have

$$(4.11) \qquad \triangle s = -\sum_{a=p+1}^{p+n} \nabla_{e_a} \nabla_{E_a} s - \sum_{a=p+1}^{p+n} \nabla_{Je_a} \nabla_{JE_a} s + \nabla_\tau s.$$

Lemma 4.2. *If $s = \pi(Y)$ is a transversally holomorphic, then*

$$\triangle s = \rho_\nabla s + \nabla_\tau s$$

Proof. Since $s = \pi(Y)$ is projectable, $\nabla_{[Je_a, e_a]} s = 0$. By (2.3), we have

$$R_\nabla(Je_a, e_a)s = J\nabla_{e_a}\nabla_{E_a} s + J\nabla_{Je_a}\nabla_{JE_a} s$$

Therefore we get

$$-\sum_a \nabla_{e_a}\nabla_{E_a} s - \sum_a \nabla_{Je_a}\nabla_{JE_a} s = \sum_a JR_\nabla(Je_a, e_a)s = \rho_\nabla s.$$

From (4.11), we obtain the result. □

Next we evaluate for $x \in M$

$$\begin{aligned}
&< \theta(Y)J, \theta(Y)J >_x \\
&= \sum_{a=p+1}^{p+n} g_Q((\theta(Y)J)e_a, (\theta(Y)J)e_a) \\
&\quad + \sum_{a=p+1}^{p+n} g_Q((\theta(Y)J)(Je_a), (\theta(Y)J)(Je_a)) \\
&= \sum_a g_Q(\nabla_{Je_a} s - J\nabla_{e_a} s, \nabla_{Je_a} s - J\nabla_{e_a} s) \\
&\quad + \sum_a g_Q(\nabla_{e_a} s + J\nabla_{Je_a} s, \nabla_{e_a} s + J\nabla_{Je_a} s)
\end{aligned}$$

The second sum equals

$$\sum_a g_Q(J\nabla_{e_a} s - \nabla_{Je_a} s, J\nabla_{e_a} s - \nabla_{Je_a} s).$$

and thus equals the first sum. It follows that

$$
\begin{aligned}
&< \theta(Y)J, \theta(Y)J >_x \\
&\quad = 2\sum_a g_Q(\nabla_{Je_a}s - J\nabla_{e_a}s, \nabla_{Je_a}s) \\
&\qquad + 2\sum_a g_Q(J\nabla_{Je_a}s + \nabla_{e_a}s, \nabla_{e_a}s) \\
&\quad = 2\sum_a Je_a g_Q(\nabla_{Je_a}s - J\nabla_{e_a}s, s) \\
&\qquad - 2\sum_a g_Q(\nabla_{Je_a}\nabla_{Je_a}s - \nabla_{Je_a}J(\nabla_{e_a}s), s) \\
&\qquad + 2\sum_a e_a g_Q(J\nabla_{Je_a}s + \nabla_{e_a}s, s) \\
&\qquad - 2\sum_a g_Q(\nabla_{e_a}J(\nabla_{Je_a}s) + \nabla_{e_a}\nabla_{e_a}s, s) \\
&\quad = 2\sum Je_a g_Q(\nabla_{Je_a}s - J\nabla_{e_a}s, s) - 2\sum_a g_Q(\nabla_{Je_a}\nabla_{Je_a}s, s) \\
&\qquad + 2\sum_a g_Q(J\nabla_{Je_a}\nabla_{e_a}s, s) + 2\sum_a e_a g_Q(J\nabla_{Je_a}s + \nabla_{e_a}s, s) \\
&\qquad - 2\sum_a g_Q(J\nabla_{e_a}\nabla_{Je_a}s, s) - 2\sum_a g_Q(\nabla_{e_a}\nabla_{e_a}s, s)
\end{aligned}
$$

Since $R_\nabla(Je_a, e_a)s = \nabla_{Je_a}\nabla_{E_a}s - \nabla_{e_a}\nabla_{JE_a}s - \nabla_{\{[JE_a, E_a]\}}s$ where the last term vanishes by (2.8) and $\triangle s = -\sum_a \nabla_{e_a}\nabla_{E_a}s - \sum_a \nabla_{Je_a}\nabla_{JE_a}s + \nabla_\tau s$. Hence we have

$$
\begin{aligned}
&< \theta(Y)J, \theta(Y)J >_x \\
&\qquad = 2\sum_a g_Q(JR_\nabla(JE_a, E_a)s, s)_x + 2g_Q(\triangle s - \nabla_\tau s, s)_x + 2(div_B Z)_x
\end{aligned}
$$

where $Z \in \Gamma Q$ is defined by

$$
g_Q(Z, X) = g_Q(\nabla_X s + J\nabla_{JX}s, s) \quad \text{for} \quad X \in \Gamma Q
$$

and the transversal divergence $div_B Z$ of Z is defined as the unique scalar satisfying $\theta(Z)\nu = div_B Z \cdot \nu$, ν being the transversal volume form defined by g_Q([4]). From (2.10), It follows that

(4.12)
$$< \theta(Y)J, \theta(Y)J >_x = 2(div_B Z)_x + 2g_Q(\triangle s, s)_x - 2g_Q(\rho_\nabla(s), s)_x$$
$$- 2g_Q(\nabla_\tau s, s)_x$$

By Theorem 4.1, we have

$$\int_M div_B Z = \ll Z, \tau \gg$$
$$= \ll \nabla_\tau s + J\nabla_{J\tau} s, s \gg .$$

Integrating (4.12), we have

Proposition 4.3. *For $Y \in V(\mathcal{F})$, we get*

$$\frac{1}{2} \ll \theta(Y)J, \theta(Y)J \gg = \ll \triangle s, s \gg - \ll \rho_\nabla s, s \gg + \ll J\nabla_{J\tau} s, s \gg$$

where $s = \pi(Y)$.

Coroally 4.4. *If $Y \in V(\mathcal{F})$ is a transversally holomorphic, then we have*

$$\ll \triangle s, s \gg = \ll \rho_\nabla s, s \gg + \ll \nabla_\tau s, s \gg$$

where $s = \pi(Y)$.

Let $s \in \Gamma Q$ be any section. We obtain the classical identity([3],[5])

(4.13)
$$-\frac{1}{2}\triangle g_Q(s, s) = g_Q(\nabla s, \nabla s) - g_Q(\triangle s, s).$$

The Laplacian on the LHS is the ordinary Laplacian d^*d of the function on M. The first term on the RHS is the induced norm square on $\nabla s \in \Omega^1(M, Q)$. In fact, let $x \in M$ and $e_1, \ldots, e_n \in T_xM$ an orthonormal frame. Let $E_1, \ldots, E_n$ be an extension of $e_1, \ldots, e_n$ to an orthonormal frame of TM in a neighborhood of x, and satisfying $\nabla^M_{e_i} E_j = 0$, i.e. the value at x of $\nabla^M_X E_j$ equals 0 for any vertor field X such that $X_x = e_i$. With these notations, we have

$$
\begin{aligned}
-\frac{1}{2}\triangle g_Q(s,s)_x &= -\frac{1}{2}(d^* dg_Q(s,s))_x = \sum_{i=1}^{n}(\nabla^M_{e_i} dg_Q(s,s))(e_i)/2 \\
&= \sum_i [\nabla^M_{e_i}(dg_Q(s,s)(e_i)) - dg_Q(s,s)(\nabla^M_{e_i} E_i)]/2 \\
&= \sum_i \nabla^M_{e_i}(E_i g_Q(s,s))/2 \\
&= \sum_i \nabla^M_{e_i}(g_Q(\nabla_{E_i}s,s) + g_Q(s,\nabla_{E_i}s))/2 \\
&= \sum_i \nabla^M_{e_i} g_Q(\nabla_{E_i}s,s) \\
&= \sum_i g_Q(\nabla_{e_i}\nabla_{E_i}s,s) + \sum_i g_Q(\nabla_{E_i}s,\nabla_{e_i}s) \\
&= -\sum_i g_Q(\triangle s,s) + \sum_i g_Q(\nabla_{e_i}s,\nabla_{e_i}s) \\
&= \sum_i g_Q(\nabla s,\nabla s) - \sum_i g_Q(\triangle s,s)
\end{aligned}
$$

Summing up Proposition (4.3) and (4.13), we have

Proposition 4.5. *For $Y \in V(\mathcal{F})$ and $s = \pi(Y)$,*

$$\frac{1}{2}\int_M \triangle|s|^2 = \frac{1}{2}||\theta(Y)J||^2 + \ll \rho_\nabla s, s \gg - \ll J\nabla_{J_T}s, s \gg - ||\nabla s||^2$$

Coroally 4.6. *If $Y \in V(\mathcal{F})$ is a transversally holomorphic, then*

$$-\frac{1}{2}\int_M \triangle|s|^2 = -\ll \rho_\nabla s, s \gg - \ll \nabla_T s, s \gg + ||\nabla s||^2$$

From this Coroally 4.6, we obtain

Theorem 4.7. *Let $\mathcal{F}$ be a non-harmonic Kähler foliation on a closed manifold M with transversal Ricci operator $\rho_\nabla \le 0$. If the transversally holomorphic infinitesimal automorphism is parallel in direction to the mean curvature vector, then $\nabla\pi(Y) = 0$. Moreover, if $\rho_\nabla < 0$ at some point $x \in M$ and $\nabla_\tau\pi(Y) = 0$, then $\pi(Y) = 0$, i.e., $Y \in \Gamma L$.*

REFERENCES

[1]. C. Ehesmann and G. Reeb, *Sur les champs d'elements complement integrables*, C. R. Ax. Sci. Paris **218** (1944), 955-957.

[2]. F. W. Kamber and Ph. Tondeur, *Harmonic foliations*, Proc. conference on Harmonic maps, Springer Lecture Notes **949**(1982), 87-121

[3]. F. W. Kamber and Ph. Tondeur, *Infinitesimal automorphisms and second variation of the energy for harmonic foliations*, Tohoku Math. J. **34**(1982), 525-538

[4]. F. W. Kamber and Ph. Tondeur and G. Toth, *Transversal jacobi fields for harmonic foliations*, Michigan Math. J. **34** (1987), 261-266

[5]. S. Nishikawa and Ph. Tondeur, *Transversal infinitesimal automorphisms for harmonic kähler foliations*, Tohoku Maht. J. **40**(1988), 599-611

[6]. B. O'Neill, *Thr fundamental equation of a submersion*, Michigan Math. J. **13**(1966), 459-469

[7]. J. S. Pak and S. Yorozu, *Transverse fields on foliated Riemannian manifolds*, J. Korean Math. Soc.**25**(1988), 83-92

[8]. B. L. Rienhart, *Foliated manifolds with bundle-like metrics*, Ann. of Math. **69**(1959), 119-132

[9]. Ph. Tondeur and G. Toth, *On transversal infinitesimal automorphisms for harmonic foliations*, Geometriae Dedicata.**24**(1987), 229-236

[10]. Ph. Tondeur, *Foliations on riemannian manifolds* Springer-Verlag, New York - Berlin - London - Paris - Tokyo (1988)

[11]. S. Yorozu and T. Tanemura, *Green's therocm on a foliated Riemannian manifold and its applications*, Acta Math. Hungar.**56**(1990),239-245

<國文抄綠>

Kähler 엽층구조를 갖는 리만다양체 상에서의 횡단적 무한소 자기동형

本 論文은 조화적인 Kähler 엽층구조 상에서의 S.Nishikawa 와 Ph.Tondeur([5])가 연구한 결과를 비조화적인 Kähler 엽층구조 를 갖는 다양체로 확장을 하여 다음과 같은 정리를 얻었다.

(정리). $(M, g_M, \mathcal{F})$를 비조화적인 Kähler 엽층구조 $\mathcal{F}$와 bundle - like 계량 g_M을 갖는 compact인 다양체라 하자. 만약 횡단적 곡률 ρ_∇이 양이 아닐 때 엽층 $\mathcal{F}$의 평균곡률 방향으로 평행인 정칙인 무한소 자기동형인 벡터는 모든 방향으로도 평행이다. 즉 $\nabla\pi(Y) = 0$이다. 더욱이, 어떤점에서 $\rho_\nabla < 0$ 일때 $\nabla_\tau\pi(Y) = 0$이면, $\pi(Y) = 0$. 즉 Y는 엽층 $\mathcal{F}$의 접벡터가된다.

더불어서, 횡단적 Killing 벡터와 Killing 벡터의 관계도 연구하였다.

감사의 글

먼저 지금까지 나를 인도하신 하나님께 감사와 영광을 돌려 드립니다.

본 논문이 완성되기까지 처음부터 끝까지 세심한 지도를 아끼지 않으시고 용기를 북돋워 주신 정승달교수님께 깊은 감사를 드립니다. 그리고 자세한 검토와 조언으로 애써주신 송석준교수님과 현진오교수님께도 감사드립니다. 대학원 과정동안 열심히 강의를 해주신 고봉수교수님, 고윤회교수님, 그리고 김철수교수님께도 고마음의 뜻을 전합니다.

서로 배우고 의지하며 같이 지낸 대학원 동기생들과 후배들에게도 감사의 마음을 전하고 여러 가지로 많은 도움을 준 이경은조교님에게도 감사 드립니다.

언제나 기도로써 격려하며 사랑으로 함께하는 양성군목사님과 행원교회 모든 성도님들에게도 감사와 사랑의 마음을 전합니다.

그리고 한국방송대학교 제주지역학습관 임홍선관장님을 비롯하여 임직원 모든분들께도 감사의 마음을 드립니다.

끝으로 말없이 지켜보시며 희생으로써 격려해주신 부모님을 비롯하여 큰형과 형수님, 작은형과 형수님 그리고 동생과 조카들, 사랑하는 희수와 함께 이 기쁨을 나누고 싶고 또한 저를 기억하는 모든분들과도 기쁨을 함께하겠습니다.

1997년 **6월**