博士學位論文

Transversal twistor spinors on a Riemannian foliation

濟州大學校 大學院

數 學 科

文 英 逢

2005年 6月

Transversal twistor spinors on a Riemannian foliation

Yeong Bong Moon
(Supervised by professor Seoung Dal Jung)

A thesis submitted in partial fulfillment of the requirement for the degree of Doctor of Science

2005. 6.

This thesis has been examined and approved.

Date Approved :

Department of Mathematics
GRADUATE SCHOOL
CHEJU NATIONAL UNIVERSITY

엽층구조를 가지는 리만다양체에서의 횡단적 twistor spinor에 관한 연구

指導教授 鄭 承 達

文 英 逢

이 論文을 理學 博士學位 論文으로 提出함

2005年 6月

文英逢의 理學 博士學位 論文을 認准함

審査委員長 ____________________

委 員 ____________________

委 員 ____________________

委 員 ____________________

委 員 ____________________

濟州大學校 大學院

2005年 6月

CONTENTS

〈 Abstract 〉

Transversal twistor spinors on a Riemannian foliation

We study the properties of the transversal Killing and twistor spinors for a Riemannian foliation with a transverse spin structure. Moreover, we investigate the relationship between them by means of two conformal invariants. And we study the properties of the transversal Weyl conformal curvature tensor which is invariant under any transversally conformal change of the metric. We give a new lower bound for the eigenvalues of the basic Dirac operator by using the transversal twistor operator.

1 Introduction

Twistor spinors were introduced by R. Penrose in General Relativity([25]). In [21], A. Lichnerowicz introduced the twistor operator on the spinors, which is a conformally invariant, and proved that the twistor spinors are zeroes of the twistor operator. Further, it is remarkable that the twistor spinors corrrespond to parallel sections in a certain bundle (see [3] or [6]). Any killing spinor is a twistor spinor. On the other hand, it is well known ([6],[20]) that for the twistor spinor Ψ, there are two interesting conformal scalar invariants $C(\Psi)$, $Q(\Psi)$ which are constant on a Riemannian spin manifold. And they proved that a non-vanishing twistor spinor Ψ is conformally equivalent to a real Killing spinor if and only if $C(\Psi) \neq 0$ and $Q(\Psi) = 0$. Similarly, we can define two invariants $C'(\Psi), Q'(\Psi)$ on a Riemannian spin foliation (see Chapter 6).

Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M. In [13], the author introduced the transversal Killing spinor which is given by the solution of the equation

$$\nabla_X \Psi + f\pi(X) \cdot \Psi = 0 \quad \text{for } X \in TM, \tag{1.1}$$

where f is a basic function and $\pi : TM \to Q$ is a projection (see (2.2)). It is well known [13] that any eigenvalue λ of the basic Dirac operator D_b satisfies the inequality

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M (\sigma^\nabla + |\kappa|^2), \tag{1.2}$$

where $q = codim\mathcal{F}$, σ^∇ is the transversal scalar curvature and κ is the mean curvature form of $\mathcal{F}$. And in the limiting case, M admits a transversal Killing spinor.

In this paper, we study the properties of the transversal Killing spinor which occurs in the limiting case in (1.2) and transversal twistor spinors. Moreover, we investigate the relationship between them by means of two conformal invariants.

The paper is organized as follows. In Chapter 2, we review the known facts on a foliated Riemannian manifold. In Chapter 3, we study the transversal twistor(resp. W-twistor) spinor, which satisfy the transversal twistor (resp. W-twistor) equation

$$\nabla_X \Psi + \frac{1}{q}\pi(X)\cdot D_{tr}\Psi = 0 \quad (\text{resp. } \nabla_X \Psi + \frac{1}{q}\pi(X)\cdot D'_{tr}\Psi = 0) \quad X \in TM.$$

Moreover, we prove that the transversal W-twistor spinors correspond to parallel basic sections in $E = S(\mathcal{F}) \oplus S(\mathcal{F})$. And we study the transversal Killing spinor. In Chapter 4, we consider a transversally conformal change of the Riemannian metric. Using the relation between the transversal Levi-Civita connections on the normal bundle Q corresponding to two transversally conformally related metrics, we relate the two canonical spinor connections acting on two isometric foliated spinor bundles. In Chapter 5, we define the transversal Weyl conformal curvature tensor W^{∇} and study the properties of W^{∇}. In Chapter 6, we define two transversally conformal invariants $C'(\Psi)$ and $Q'(\Psi)$, which are similar to ones on [6]. And we study the (transversally) conformal relation between transversal twistor spinors and transversal Killing spinors. In Chapter 7, any eigenvalue of the basic Dirac operator estimates.

2 Foliations

2.1 Riemannian foliations

Let M be a smooth manifold of dimension $p+q$.

Definition 2.1 A codimension q *foliation* $\mathcal{F}$ on M is given by an open cover $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ and for each α, a diffeomorphism $\varphi_\alpha : \mathbb{R}^{p+q} \to U_\alpha$ such that, on $U_\alpha \cap U_\beta \neq \emptyset$, the coordinate change $\varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(U_\alpha \cap U_\beta) \to \varphi_\beta^{-1}(U_\alpha \cap U_\beta)$ has the form

$$\varphi_\beta^{-1} \circ \varphi_\alpha(x,y) = (\varphi_{\alpha\beta}(x,y), \gamma_{\alpha\beta}(y)). \tag{2.1}$$

From Definition 2.1, the manifold M is decomposed into connected submanifolds of dimension p. Each of these submanifolds is called a *leaf* of $\mathcal{F}$. Coordinate patches $(U_\alpha, \varphi_\beta)$ are said to be *distinguished patches* for the foliation $\mathcal{F}$. The tangent bundle L of a foliation is the subbundle of TM, consisting of all vectors tangent to the leaves of $\mathcal{F}$. The normal bundle Q of a codimension q foliation $\mathcal{F}$ on M is the quotient bundle $Q = TM/L$. Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a Riemannian metric g_M with respect to $\mathcal{F}$. We recall the exact sequence

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0 \tag{2.2}$$

determined by the tangent bundle L and the normal bundle $Q = TM/L$ of $\mathcal{F}$. The *transversal geometry* of a foliation is the geometry infinitesimally modeled by Q, while the tangential geometry is infinitesimally modeled by L. A key fact is the existence of the *Bott connection* in Q defined by

$$\overset{\circ}{\nabla}_X s = \pi([X, Y_s]) \quad \text{for } X \in \Gamma L, \tag{2.3}$$

where $Y_s \in TM$ is any vector field projecting to s under $\pi : TM \to Q$. It is a partial connection along L. The right hand side in (2.3) is independent of the choice of Y_s. Namely, the difference of two such choices is a vector field $X' \in \Gamma L$ and $[X, X'] \in \Gamma L$ so that $\pi[X, X'] = 0$.

A Riemannian metric g_Q on the normal bundle Q of a foliation $\mathcal{F}$ is *holonomy invariant*, if

$$\theta(X) g_Q = 0 \quad \forall\, X \in \Gamma L, \tag{2.4}$$

where $\theta(X)$ is the transverse Lie derivative operator. Here we have by definition for $s, t \in \Gamma Q$,

$$(\theta(X) g_Q)(s, t) = X g_Q(s, t) - g_Q(\theta(X)s, t) - g_Q(s, \theta(X)t).$$

Definition 2.2 A *Riemannian foliation* is a foliation $\mathcal{F}$ with a holonomy invariant transversal metric g_Q. A metric g_M is *bundle-like,* if the induced metric g_Q on Q is holonomy invariant.

The study of Riemannian foliations was initiated by Reinhart in 1959. A simple example of a Riemannian foliation is given by a nonsingular Killing vector field X on (M, g_M). This means that $\theta(X) g_M = 0$.

An *adapted connection* in Q is a connection restricting along L to the partial Bott connection $\overset{\circ}{\nabla}$. To show that such connections exist, consider a Riemannian metric g_M on M. Then TM splits orthogonally as $TM = L \oplus L^{\perp}$. This means that there is a bundle map $\sigma : Q \to L^{\perp}$

splitting the exact sequence (2.2), i.e., satisfying $\pi \circ \sigma = identity$. This metric g_M on TM is then a direct sum

$$g_M = g_L \oplus g_{L^\perp}.$$

With $g_Q = \sigma^* g_{L^\perp}$, the splitting map $\sigma : (Q, g_Q) \to (L^\perp, g_{L^\perp})$ is a metric isomorphism.

For a distinguished chart $\mathcal{U} \subset M$ the leaves of $\mathcal{F}$ in $\mathcal{U}$ are given as the fibers of a Riemannian submersion $f : \mathcal{U} \to \mathcal{V} \subset N$ onto an open subset $\mathcal{V}$ of a model Riemannian manifold N.

For overlapping charts $U_\alpha \cap U_\beta$, the corresponding local transition functions $\gamma_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$ on N are isometries. Further, we denote by ∇ the canonical connection of the normal bundle Q of $\mathcal{F}$. It is defined by

$$\nabla_X s = \begin{cases} \pi([X, Y_s]) & \text{for } X \in \Gamma L, \\ \pi(\nabla^M_X Y_s) & \text{for } X \in \Gamma L^\perp, \end{cases} \tag{2.5}$$

where $s \in \Gamma Q$, and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $L^\perp \cong Q$. For any connection ∇ on Q, there is a torsion T_∇ defined by

$$T_\nabla(Y, Z) = \nabla_Y \pi(Z) - \nabla_Z \pi(Y) - \pi[Y, Z] \tag{2.6}$$

for any $Y, Z \in \Gamma TM$. Then we have the following proposition.

Proposition 2.3 *The connection ∇ is metric and torsion free.*

Proof. For $X \in \Gamma L$, $s, t \in \Gamma Q$,

$$X g_Q(s, t) = g_Q(\nabla_X s, t) + g_Q(s, \nabla_X t).$$

For $X \in \Gamma Q$ and $s, t \in \Gamma Q$, we have $Xg_Q(s,t) = Xg_M(Y_s, Y_t)$ for $Y_s = \sigma(s)$, $Y_t = \sigma(t)$.

$$\begin{aligned}
Xg_M(Y_s, Y_t) &= g_M(\nabla^M_X Y_s, Y_t) + g_M(Y_s, \nabla^M_X Y_t) \\
&= g_M(\sigma\pi(\nabla^M_X Y_s), \sigma(t)) + g_M(\sigma(s), \sigma\pi(\nabla^M_X Y_t)) \\
&= g_Q(\pi(\nabla^M_X Y_s), t) + g_Q(s, \pi(\nabla^M_X Y_t)) \\
&= g_Q(\nabla_X s, t) + g_Q(s, \nabla_X t).
\end{aligned}$$

Therefore, the connection ∇ is metric.

For $X \in \Gamma L, Y \in \Gamma TM$, we have $\pi(X) = 0$ and

$$\begin{aligned}
T_\nabla(X,Y) &= \nabla_X \pi(Y) - \nabla_Y \pi(X) - \pi[X,Y] \\
&= \nabla_X \pi(Y) - \pi[X,Y] = 0
\end{aligned}$$

by (2.3). For $Z, Z' \in \Gamma L^\perp$,

$$\begin{aligned}
T_\nabla(Z,Z') &= \pi(\nabla^M_Z Z') - \pi(\nabla^M_{Z'} Z) - \pi[Z,Z'] \\
&= \pi(T_{\nabla^M}(Z,Z')) = 0,
\end{aligned}$$

where T_{∇^M} is the (vanishing) torsion ∇^M. □

The connection ∇ corresponds to the Riemannian connection of the model space N. The curvature R^∇ of ∇ is defined by

$$R^\nabla(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} \quad \text{for } X,\ Y \in TM.$$

From an adapted connection ∇ in Q defined by (2.5), its curvature R^∇ coincides with $\overset{\circ}{R}$ for $X, Y \in \Gamma L$, hence $R^\nabla(X,Y) = 0$ for $X, Y \in \Gamma L$. And we have the following proposition ([18]).

Proposition 2.4 *Let* $(M, g_M, \mathcal{F})$ *be a* $(p+q)$*-dimensional Riemannian manifold with a foliation* $\mathcal{F}$ *of codimension* q *and bundle-like metric* g_M *with respect to* $\mathcal{F}$*. Let* ∇ *be a connection defined by (2.5) in* Q *with curvature* R^∇*. Then the following holds:*

$$i(X)R^\nabla = \theta(X)R^\nabla = 0 \quad \forall X \in \Gamma L. \tag{2.7}$$

Proof. (i) Let $Y \in \Gamma TM$ and $s \in \Gamma Q$. Then we have

$$\begin{aligned}
R^\nabla(X,Y)s &= \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s \\
&= \theta(X)\nabla_Y s - \nabla_Y\theta(X)s - \nabla_{\theta(X)Y}s \\
&= (\theta(X)\nabla)_Y s = 0.
\end{aligned}$$

(ii) Let $Y, Z \in \Gamma TM$ and $s \in \Gamma Q$. Then we have

$$\begin{aligned}
(\theta(X)R^\nabla)(Y,Z)s &= \theta(X)R^\nabla(Y,Z)s - R^\nabla(\theta(X)Y,Z)s \\
&\quad - R^\nabla(Y,\theta(X)Z)s - R^\nabla(Y,Z)\theta(X)s \\
&= \theta(X)\{\nabla_Y\nabla_Z s - \nabla_Z\nabla_Y s - \nabla_{[Y,Z]}s\} \\
&\quad -\{\nabla_{\theta(X)Y}\nabla_Z s - \nabla_Z\nabla_{\theta(X)Y}s - \nabla_{[\theta(X)Y,Z]}s\} \\
&\quad -\{\nabla_Y\nabla_{\theta(X)Z}s - \nabla_{\theta(X)Z}\nabla_Y s - \nabla_{[Y,\theta(X)Z]}s\} \\
&\quad -\{\nabla_Y\nabla_Z\theta(X)s - \nabla_Z\nabla_Y\theta(X)s - \nabla_{[Y,Z]}\theta(X)s\} \\
&= \nabla_Y(\theta(X)\nabla_Z s) - \nabla_Z(\theta(X)\nabla_Y s) - \nabla_{\theta(X)[Y,Z]}s \\
&\quad +\nabla_Z\nabla_{\theta(X)Y}s + \nabla_{[\theta(X)Y,Z]}s - \nabla_Y\nabla_{\theta(X)Z}s \\
&\quad +\nabla_{[Y,\theta(X)Z]}s - \nabla_Y\nabla_Z\theta(X)s + \nabla_Z\nabla_Y\theta(X)s \\
&= -\nabla_{\theta(X)[Y,Z]}s + \nabla_{[\theta(X)Y,Z]}s + \nabla_{[Y,\theta(X)Z]}s \\
&= (-\nabla_{[X,[Y,Z]]} + \nabla_{[[X,Y],Z]} + \nabla_{[Y,[X,Z]]})s = 0. \qquad \square
\end{aligned}$$

Since $i(X)R^{\nabla} = 0$ for any $X \in \Gamma L$, we can define the (transversal) Ricci curvature $\rho^{\nabla} : \Gamma Q \to \Gamma Q$ and the (transversal) scalar curvature σ^{∇} of $\mathcal{F}$ by

$$\rho^{\nabla}(s) = \sum_a R^{\nabla}(s, E_a)E_a, \quad \sigma^{\nabla} = \sum_a g_Q(\rho^{\nabla}(E_a), E_a),$$

where $\{E_a\}_{a=1,\cdots,q}$ is an orthonormal basic frame for Q.

Definition 2.5 The foliation $\mathcal{F}$ is said to be (transversally) *Einsteinian* if the model space N is Einsteinian, that is,

$$\rho^{\nabla} = \frac{1}{q}\sigma^{\nabla} \cdot id \tag{2.8}$$

with constant transversal scalar curvature σ^{∇}.

2.2 The basic cohomology

The *second fundamental form* α of $\mathcal{F}$ is given by

$$\alpha(X, Y) = \pi(\nabla_X^M Y) \quad \text{for } X, Y \in \Gamma L. \tag{2.9}$$

Proposition 2.6 *α is Q-valued, bilinear and symmetric.*

Proof. By definition, it is trivial that α is Q-valued and bilinear. Next, we have that for any $X, Y \in \Gamma L$,

$$\alpha(X, Y) = \pi(\nabla_X^M Y) = \pi(\nabla_Y^M X) - \pi([X, Y]).$$

By the torsion freeness of ∇^M and $\pi(X) = \pi(Y) = 0$, $\pi[X, Y] = 0$. Therefore, we have

$$\alpha(X, Y) = \pi(\nabla_Y^M X) = \alpha(Y, X). \qquad \square$$

Definition 2.7 The *mean curvature vector field* of $\mathcal{F}$ is defined by

$$\kappa^\sharp = \sum_i \alpha(E_i, E_i) = \sum_i \pi(\nabla^M_{E_i} E_i), \tag{2.10}$$

where $\{E_i\}_{i=1,\cdots,p}$ is an orthonormal basis of L. Its dual form κ, the *mean curvature form* for L, is then given by

$$\kappa(X) = g_Q(\kappa^\sharp, X) \quad \forall\ X \in \Gamma Q. \tag{2.11}$$

The foliation $\mathcal{F}$ is said to be *minimal* (or *harmonic*) if $\kappa = 0$.

Definition 2.8 Let $\mathcal{F}$ be an arbitrary foliation on a manifold M. A differential form $\omega \in \Omega^r(M)$ is *basic* if

$$i(X)\omega = 0,\ \theta(X)\omega = 0 \quad \forall X \in \Gamma L. \tag{2.12}$$

In a distinguished chart $(x_1, \ldots, x_p; y_1, \ldots, y_q)$ of $\mathcal{F}$, a basic form w is expressed by

$$\omega = \sum_{a_1 < \cdots < a_r} \omega_{a_1 \cdots a_r} dy_{a_1} \wedge \cdots \wedge dy_{a_r},$$

where the functions $\omega_{a_1 \cdots a_r}$ are independent of x, i.e. $\frac{\partial}{\partial x_i}\omega_{a_1 \cdots a_r} = 0$. Let $\Omega^r_B(\mathcal{F})$ be the set of all basic r-forms on M. Since for any $X \in \Gamma L$ and any basic form ω, $\theta(X)d\omega = d\theta(X)\omega = 0, i(X)d\omega = \theta(X)\omega - di(X)\omega = 0$, the exterior derivative d preserves basic forms. Hence $\Omega^r_B(\mathcal{F})$ constitutes a subcomplex

$$d : \Omega^r_B(\mathcal{F}) \to \Omega^{r+1}_B(\mathcal{F})$$

of the De Rham complex $\Omega^*(M)$ and the restriction $d_B = d|_{\Omega^*_B(\mathcal{F})}$ is well defined. Its cohomology

$$H_B(\mathcal{F}) = H(\Omega^*_B(\mathcal{F}), d_B)$$

is the *basic cohomology* of $\mathcal{F}$. It plays the role of the De Rham cohomology of the leaf space $M/\mathcal{F}$ of the foliation. Let δ_B be the formal adjoint operator of d_B. Then we have the following proposition ([1,13]).

Proposition 2.9 *On a Riemannian foliation $\mathcal{F}$, we have*

$$d_B = \sum_a \theta_a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_a i(E_a)\nabla_{E_a} + i(\kappa_B), \tag{2.13}$$

where κ_B is the basic component of κ, $\{E_a\}$ is a local orthonormal basic frame in Q and $\{\theta_a\}$ its g_Q-dual 1-form.

The foliation $\mathcal{F}$ is said to be *isoparametric* if $\kappa \in \Omega_B^1(\mathcal{F})$. We already know that κ is closed, i.e., $d\kappa = 0$ if $\mathcal{F}$ is isoparametric ([23]).

Definition 2.10 The *basic Laplacian* acting on $\Omega_B^*(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B. \tag{2.14}$$

The following theorem is proved in the same way as the corresponding usual result in De Rham-Hodge Theory.

Theorem 2.11 ([26]) *Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a closed oriented manifold (M, g_M). Assume g_M to be bundle-like metric with $\kappa \in \Omega_B^1(\mathcal{F})$. Then*

$$H_B^r(\mathcal{F}) \cong \mathcal{H}_B^r(\mathcal{F}),$$

where $\mathcal{H}_B^r(\mathcal{F}) = \{\omega \in \Omega^r(M) | \Delta_B\omega = 0\}$.

If $\mathcal{F}$ is the foliation by points of M, the basic Laplacian is the ordinary Laplacian.

3 The transverse spin structure

3.1 Clifford algebras

Definition 3.1 Let V be a vector space over a field $K = \{\mathbb{R}, \mathbb{C}\}$ of dimension n and g a non-degenerate bilinear form on V. The *Clifford algebra* $Cl(V, g)$ associated to g on V is the algebra over K generated by V with the relation

$$v \cdot w + w \cdot v = -2g(v, w) \tag{3.1}$$

for $v, w \in V$. The product "$\cdot$" is called the *Clifford multiplication.*

Remark. (1) If $(E_1, \cdots E_n)$ is a g-orthonormal basis of V, then

$$\{E_{i_1} \cdot \ldots \cdot E_{i_k} | 1 \leq i_1 < \ldots < i_k \leq n, \quad 0 \leq k \leq n\}$$

is a basis of $Cl(V, g)$, thus $\dim Cl(V, g) = 2^n$.

(2) There is a canonical isomorphism of vector spaces between the exterior algebra and the Clifford algebra of (V, g) which is given by :

$$\wedge^* V \xrightarrow{\simeq} Cl(V, g)$$

$$E_{i_1} \wedge \ldots \wedge E_{i_k} \longmapsto E_{i_1} \cdot \ldots \cdot E_{i_k}.$$

This isomorphism does not depend on the choice of the basis. Let us denote $Cl_n = Cl(\mathbb{R}^n, <\ ,\ >)$. Then we have the following proposition ([19]).

Proposition 3.2 *For all $v \in \mathbb{R}^n$ and all $\varphi \in Cl_n$, we have*

$$v \cdot \varphi \simeq v \wedge \varphi - i(v)\varphi \quad and \quad \varphi \cdot v \simeq (-1)^p (v \wedge \varphi + i(v)\varphi),$$

where $\wedge$ denotes the exterior, $i(v)$ the interior product and $\varphi \in \wedge^p \mathbb{R}^n \subset \wedge^* \mathbb{R}_n \simeq Cl_n$.

Proof. Let $v = E_j$ and $\varphi = E_{i_1} \cdot \ldots \cdot E_{i_p}$.

1. If there exists i_k such that $j = i_k$ then $v \wedge \varphi = 0$ and

$$\begin{aligned} i(v)\varphi &= (-1)^{k-1} E_{i_1} \wedge \ldots \wedge E_{i_{k-1}} \wedge E_{i_{k+1}} \wedge \ldots E_{i_p} \\ &\simeq (-1)^{k-1} E_{i_1} \cdot \ldots \cdot E_{i_{k-1}} \cdot E_{i_{k+1}} \cdot \ldots \cdot E_{i_p} \\ &= -v \cdot \varphi \\ &= (-1)^p \varphi \cdot v. \end{aligned}$$

2. If $j \notin \{i_1, \ldots, i_p\}$ then $i(v)\varphi = 0$ and

$$\begin{aligned} v \wedge \varphi &= E_j \wedge E_{i_1} \wedge \ldots \wedge E_{i_p} \simeq E_j \cdot E_{i_1} \cdot \ldots \cdot E_{i_p} \\ &= v \cdot \varphi \\ &= (-1)^p \varphi \cdot v. \end{aligned}$$

As the equalities of the assertion are bilinear, the proposition is proved. □

Definition 3.3 The *Pin group* $Pin(V)$ is defined by

$$Pin(V) = \{a \in Cl(V) | a = a_1 \cdots a_k, ||a_i|| = 1\}. \tag{3.2}$$

The *Spin group* is defined by

$$Spin(V) = \{a \in Pin(V) | aa^t = 1\}, \tag{3.3}$$

where $a^t = a_k \cdots a_1$ for any $a = a_1 \cdots a_k$. Equivalently, $Spin(V) = \{e_1 \cdots e_{2k} | \ |e_i| = 1\}$.

Let V be a real vector space. Then $Spin(V)$ is a compact and connected Lie group, and for $\dim V \geq 3$, it is also simply connected. Thus, for $\dim V \geq 3$, $Spin(V)$ is the universal cover of $SO(V)$ (for detail, see [19]).

3.2 Transversal Dirac operator

Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transversally oriented Riemannian foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. Let $SO(q) \to P_{SO} \to M$ be the principal bundle of (oriented) transverse orthonormal framings. Then a *transverse spin structure* is a principal $Spin(q)$-bundle P_{Spin} together with two sheeted covering $\xi : P_{Spin} \to P_{SO}$ such that $\xi(p \cdot g) = \xi(p)\xi_0(g)$ for all $p \in P_{Spin}$, $g \in Spin(q)$, where $\xi_0 : Spin(q) \to SO(q)$ is a covering. In this case, the foliation $\mathcal{F}$ is called a *transverse spin foliation*. We then define the *foliated spinor bundle* $S(\mathcal{F})$ associated with P_{Spin} by

$$S(\mathcal{F}) = P_{Spin} \times_{Spin(q)} S_q, \tag{3.4}$$

where S_q is the irreducible spinor space associated to Q. The Hermitian metric $\langle \cdot, \cdot \rangle$ on $S(\mathcal{F})$ induced from g_Q satisfies the following relation:

$$\langle \varphi, \psi \rangle = \langle v \cdot \varphi, v \cdot \psi \rangle \tag{3.5}$$

for every $v \in Q$, $g_Q(v, v) = 1$ and $\varphi, \psi \in S_q$. And the Riemannian connection ∇ on P_{SO} defined by (2.5) can be lifted to one on P_{Spin}, in particular, to one on $S(\mathcal{F})$, which will be denoted by the same letter.

Proposition 3.4 ([9, 12]) *The spinorial covariant derivative on $S(\mathcal{F})$ is given locally by:*

$$\nabla \Psi_\alpha = \frac{1}{4}\sum_{a,b} g_Q(\nabla E_a, E_b) E_a \cdot E_b \cdot \Psi_\alpha, \tag{3.6}$$

where Ψ_α is an orthonormal basis of S_q. And the curvature transform R^S on $S(\mathcal{F})$ is given as

$$R^S(X,Y)\Phi = \frac{1}{4}\sum_{a,b} g_Q(R^\nabla(X,Y)E_a, E_b) E_a \cdot E_b \cdot \Phi \tag{3.7}$$

for $X, Y \in TM$, where $\{E_a\}$ is an orthonormal basis of the normal bundle Q.

Proposition 3.5 ([19]) (Compatibility of ∇ with " $\cdot$ " and $< \cdot, \cdot >$)

$$(1) \quad X\langle\psi,\varphi\rangle = \langle\nabla_X\psi,\varphi\rangle + \langle\psi,\nabla_X\varphi\rangle, \quad \forall\, X \in \Gamma TM. \tag{3.8}$$

$$(2) \quad \nabla_X(Y\cdot\psi) = (\nabla_X Y)\cdot\psi + Y\cdot\nabla_X\psi, \quad \forall\, Y \in \Gamma Q. \tag{3.9}$$

Theorem 3.6 ([13,17]) *On the foliated spinor bundle $S(\mathcal{F})$, we have*

$$\sum_a E_a \cdot R^S(X, E_a)\Phi = -\frac{1}{2}\rho^\nabla(\pi(X))\cdot\Phi, \tag{3.10}$$

$$\sum_{a<b} E_a \cdot E_b \cdot R^S(E_a, E_b)\Phi = \frac{1}{4}\sigma^\nabla \Phi \tag{3.11}$$

for $X \in TM$.

Taking $\hat{\pi}$ to denote the projection

$$\hat{\pi} : C^\infty(T^*M \otimes S(\mathcal{F})) \to C^\infty(Q^* \otimes S(\mathcal{F})) \cong C^\infty(Q \otimes S(\mathcal{F})),$$

we define the *transversal Dirac Operator* D'_{tr}([4,7]) by

$$D'_{tr} = m \circ \hat{\pi} \circ \nabla^S,$$

where $m : Q \otimes S(\mathcal{F}) \to S(\mathcal{F})$ is the Clifford multiplication, ∇^S is a spinor derivation on $S(\mathcal{F})$ induced by (2.3).

If $\{E_a\}_{a=1,\cdots,q}$ is taken to be a local orthonormal basic frame in Q, then

$$D'_{tr} = \sum_a E_a \cdot \nabla_{E_a}. \tag{3.12}$$

In [4,7] it was shown that the formal adjoint ${D'_{tr}}^*$ is given by ${D'_{tr}}^* = D'_{tr} - \kappa \cdot$ and that therefore

$$D_{tr} = D'_{tr} - \frac{1}{2}\kappa \cdot = \sum_a E_a \cdot \nabla_{E_a} - \frac{1}{2}\kappa \cdot \tag{3.13}$$

is a symmetric, transversally elliptic differential operator. Then we have the Lichnerowicz-type formula on $S(\mathcal{F})$.

Theorem 3.7 ([7,13]) *On an isoparametric transverse spin foliation $\mathcal{F}$ with $\delta\kappa = 0$, we have*

$$D_{tr}^2 \Psi = \nabla_{tr}^* \nabla_{tr} \Psi + \frac{1}{4} K^\sigma \Psi, \tag{3.14}$$

where $K^\sigma = \sigma^\nabla + |\kappa|^2$ and

$$\nabla_{tr}^* \nabla_{tr} \Psi = -\sum_a \nabla^2_{E_a, E_a} \Psi + \nabla_{\kappa^\sharp} \Psi. \tag{3.15}$$

The operator $\nabla_{tr}^* \nabla_{tr}$ is non-negative and formally self-adjoint([13]). In fact, we have the following proposition.

Proposition 3.8 ([13]) *Let $(M, g_M, \mathcal{F}, S(\mathcal{F}))$ be a compact Riemannian manifold with the transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then*

$$\ll \nabla_{tr}^* \nabla_{tr} \Phi, \Psi \gg = \ll \nabla_{tr} \Phi, \nabla_{tr} \Psi \gg$$

for all $\Phi, \Psi \in \Gamma E$*, where* $\ll \Phi, \Psi \gg = \int_M \langle \Phi, \Psi \rangle$ *is the inner product on* $S(\mathcal{F})$.

Proposition 3.9 *Let* $(M, g_M, \mathcal{F})$ *be a compact Riemannian manifold with an isoparametric transverse spin foliation* $\mathcal{F}$ *and a bundle-like metric* g_M *such that* $\delta\kappa = 0$*. Then it holds*

$$D_{tr}^2 \Psi = D_{tr}'^2 \Psi + \frac{1}{4}|\kappa|^2 \Psi + \nabla_{\kappa^\sharp} \Psi. \tag{3.16}$$

Proof. By (3.13), we have

$$D_{tr}^2 \Psi = D_{tr}'^2 \Psi - \frac{1}{2}\{\kappa \cdot D'_{tr} \Psi + D'_{tr}(\kappa \cdot \Psi)\} - \frac{1}{4}|\kappa|^2 \Psi. \tag{3.17}$$

Moreover, on an isoparametric transverse spin foliation $\mathcal{F}$ with $\delta\kappa = 0$, we have

$$\begin{aligned}
D'_{tr}(\kappa \cdot \Psi) &+ \kappa \cdot D'_{tr} \Psi \\
&= \sum_a E_a \cdot \nabla_{E_a}(\kappa \cdot \Psi) + \kappa \cdot D'_{tr} \Psi \\
&= \sum_a \{E_a \cdot (\nabla_{E_a} \kappa) \cdot \Psi + E_a \cdot \kappa \cdot \nabla_{E_a} \Psi\} + \kappa \cdot D'_{tr} \Psi \\
&= \sum_a \{(E_a \wedge \nabla_{E_a} \kappa - i(E_a) \nabla_{E_a} \kappa) \Psi \\
&\quad - (\kappa \cdot E_a \nabla_{E_a} \Psi + 2\nabla_\kappa \Psi)\} + \kappa \cdot D'_{tr} \Psi \\
&= (d_B \kappa + \delta_B \kappa - |\kappa|^2) \Psi - 2\nabla_{\kappa^\sharp} \Psi \\
&= -|\kappa|^2 \Psi - 2\nabla_{\kappa^\sharp} \Psi.
\end{aligned}$$

From (3.17), the proof is completed. □

We define the subspace $\Gamma_B(S(\mathcal{F}))$ of *basic* or *holonomy invariant* sections of $S(\mathcal{F})$ by

$$\Gamma_B(S(\mathcal{F})) = \{\Psi \in \Gamma S(\mathcal{F}) | \nabla_X \Psi = 0 \quad \text{for } X \in \Gamma L\}.$$

Then $D_b = D_{tr}|_{\Gamma_B(S(\mathcal{F}))} : \Gamma_B(S(\mathcal{F})) \to \Gamma_B(S(\mathcal{F}))$ preserves the basic sections if $\mathcal{F}$ is isoparametric, i.e., $\kappa \in \Omega^1_B(\mathcal{F})$. This operator D_b is called the *basic Dirac operator* on (smooth) basic sections. It is well known that D_b and D_b^2 have a discrete spectrum, respectively.

3.3 Transversal twistor operator

Let $p : Q \otimes S(\mathcal{F}) \to \mathrm{Ker}\ m$ onto $\mathrm{Ker}\ m$ be a projection given by the formula

$$p(X \otimes \Psi) = X \otimes \Psi + \frac{1}{q}\sum_{a=1}^{q} E_a \otimes E_a \cdot X \cdot \Psi. \tag{3.18}$$

We define the *transversal twistor operator* P'_{tr} on $S(\mathcal{F})$ by

$$P'_{tr} = p \circ \hat{\pi} \circ \nabla^S.$$

If it does not cause any confusion, we will henceforward use $\nabla = \nabla^S$. Locally, it is given by

$$P'_{tr}\Psi = \sum_a E_a \otimes P'_{E_a}\Psi, \tag{3.19}$$

where $P'_X\Psi = \nabla_X\Psi + \frac{1}{q}\pi(X) \cdot D'_{tr}\Psi$ for any $X \in TM$. Similarly, we put

$$P_{tr}\Psi = \sum_a E_a \otimes P_{E_a}\Psi, \tag{3.20}$$

where $P_X\Psi = \nabla_X\Psi + \frac{1}{q}\pi(X) \cdot D_{tr}\Psi$. Trivially, we have the following lemma.

Lemma 3.10 *On the transverse spin foliation $\mathcal{F}$, we have that for any $X \in TM$*

$$P_X\Psi = P'_X\Psi - \frac{1}{2q}\pi(X) \cdot \kappa \cdot \Psi. \tag{3.21}$$

Definition 3.11 *A spinor field of kernel of P_{tr} (resp. kernel of P'_{tr}) is called the transversal twistor* (resp. *W-twistor*) *spinor, which satisfies the so-called transversal twistor* (resp. *W-twistor*) *equation*

$$\nabla_X\Psi + \frac{1}{q}\pi(X)\cdot D_{tr}\Psi = 0(\text{resp.}\nabla_X\Psi + \frac{1}{q}\pi(X)\cdot D'_{tr}\Psi = 0). \quad (3.22)$$

Theorem 3.12 *If M admits a non-vanishing transversal twistor spinor Ψ, then $\mathcal{F}$ is minimal.*

Proof. Let $(0 \neq)\Psi \in \text{Ker}P_{tr}$. Then we have

$$\begin{aligned} 0 = \sum_a E_a \cdot P_{E_a}\Psi &= \sum_a E_a \cdot \nabla_{E_a}\Psi + \frac{1}{q}\sum_a E_a \cdot E_a \cdot D_{tr}\Psi \\ &= D_{tr}\Psi + \frac{1}{2}\kappa \cdot \Psi - D_{tr}\Psi = \frac{1}{2}\kappa \cdot \Psi, \end{aligned}$$

which implies that $\kappa = 0$. Therefore $\mathcal{F}$ is minimal. □

Remark. From Theorem 3.12, we know that there does not exist a solution of the transversal twistor equation (3.22) if $\mathcal{F}$ is not minimal. So we use the operator P'_{tr} for much information of the foliation $\mathcal{F}$ when it is not minimal. Note that any transversal twistor spinor is the transversal W-twistor spinor. But the converse is not true in general.

Theorem 3.13 *Let $\Psi \in \text{Ker}P'_{tr}$ be a transversal W-twistor spinor. Then for all vector fields X, $Y \in TM$, we have*

$$\pi(X)\cdot\nabla_Y\Psi + \pi(Y)\cdot\nabla_X\Psi = \frac{2}{q}g_Q(\pi(X),\pi(Y))D'_{tr}\Psi. \quad (3.23)$$

Also, the converse holds.

Proof. Let Ψ be a transversal W-twistor spinor. Then Ψ satisfies the transversal W-twistor equation

$$\nabla_X\Psi + \frac{1}{q}\pi(X)\cdot D'_{tr}\Psi = 0 \quad \text{for any } X \in TM.$$

Multiplying the above equation by a vector field on M we have

$$\pi(Y)\cdot\nabla_X\Psi + \frac{1}{q}\pi(Y)\cdot\pi(X)\cdot D'_{tr}\Psi = 0,$$
$$\pi(X)\cdot\nabla_Y\Psi + \frac{1}{q}\pi(X)\cdot\pi(Y)\cdot D'_{tr}\Psi = 0.$$

By summing the above equations, we have

$$\pi(X)\cdot\nabla_Y\Psi + \pi(Y)\cdot\nabla_X\Psi = \frac{2}{q}g_Q(\pi(X),\pi(Y))D'_{tr}\Psi.$$

Conversely, let (3.23) be valid. Then we have

$$\sum_a E_a\cdot\pi(X)\cdot\nabla_{E_a}\Psi + \sum_a E_a\cdot E_a\cdot\nabla_X\Psi = \frac{2}{q}\sum_a g_Q(\pi(X),E_a)E_a\cdot D'_{tr}\Psi.$$

By the properties of the Clifford multiplication, we have

$$\nabla_X\Psi + \frac{1}{q}\pi(X)\cdot D'_{tr}\Psi = 0.$$

This implies that Ψ is a transversal W-twistor spinor. □

Theorem 3.14 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with an isoparametric transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M such that $\delta\kappa = 0$. Then every transversal W-twistor spinor $\Psi \in \mathrm{Ker}P'_{tr}$ satisfies*

$$D'^2_{tr}\Psi = \frac{q}{4(q-1)}\sigma^\nabla\Psi, \tag{3.24}$$

$$\nabla_X D'_{tr}\Psi = \frac{q}{2(q-2)}\{\frac{\sigma^\nabla}{2(q-1)}\pi(X) - \rho^\nabla(\pi(X))\}\Psi \tag{3.25}$$

for all $X \in \Gamma TM$.

Proof. Let $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ with the property that $(\nabla E_a)_x = 0$ for all a. From (3.22), we have at x that for any transversal W-twistor spinor Ψ

$$\sum_a \nabla_{E_a}\nabla_{E_a}\Psi + \frac{1}{q}D_{tr}'^2\Psi = 0. \tag{3.26}$$

On the other hand, from (3.14), (3,15) and (3.16) we have

$$D_{tr}'^2\Psi = -\sum_a \nabla_{E_a}\nabla_{E_a}\Psi + \frac{1}{4}\sigma^\nabla\Psi. \tag{3.27}$$

From (3.26) and (3.27), the first equation (3.24) is proved.

Next, let X be a local vector field arising from a vector in T_xM by parallel displacement along transversal geodesics. Then we have from (3.15) that at x,

$$\nabla_{E_a}\nabla_X\Psi + \frac{1}{q}\pi(X)\cdot\nabla_{E_a}D'_{tr}\Psi = 0.$$

Therefore, we have

$$R^S(X,E_a)\Psi = \frac{1}{q}\{\pi(X)\cdot\nabla_{E_a}D'_{tr}\Psi - E_a\cdot\nabla_X D'_{tr}\Psi\}. \tag{3.28}$$

Since $i(X)R^S = 0$ for $X \in \Gamma L$, if $\Psi \in \Gamma_B S(\mathcal{F})$, then $D'_{tr}\Psi \in \Gamma_B S(\mathcal{F})$. From (3.10) and (3.28), we have

$$\begin{aligned}
\rho^\nabla(\pi(X))\cdot\Psi &= -2\sum_a E_a\cdot R^S(X,E_a)\Psi \\
&= -\frac{2}{q}\sum_a E_a\cdot\{\pi(X)\cdot\nabla_{E_a}D'_{tr}\Psi - E_a\cdot\nabla_X D'_{tr}\Psi\} \\
&= -\frac{2}{q}\{(q-2)\nabla_X D'_{tr}\Psi - \pi(X)\cdot D_{tr}'^2\Psi\}.
\end{aligned}$$

From the above equation with (3.24), we obtain the second equation. □

Now we prove a further condition for Ψ being the transversal W-twistor spinor.

Definition 3.15 *We define the bundle map $K : TM \to Q$ by*

$$K(X) = \frac{1}{q-2}\{\frac{\sigma^\nabla}{2(q-1)}\pi(X) - \rho^\nabla(\pi(X))\}. \tag{3.29}$$

From (3.25), it is trivial that for any transversal W-twistor spinor Ψ

$$\nabla_X D'_{tr}\Psi = \frac{q}{2}K(X)\cdot\Psi. \tag{3.30}$$

We consider the bundle $E = S(\mathcal{F}) \oplus S(\mathcal{F})$ and the covariant derivative ∇^E in E defined by

$$\nabla^E_X \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \begin{pmatrix} \nabla_X\Phi + \frac{1}{q}\pi(X)\cdot\Psi \\ \nabla_X\Psi - \frac{q}{2}K(X)\cdot\Phi \end{pmatrix}. \tag{3.31}$$

Then we have the following theorem.

Theorem 3.16 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with an isoparametric transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M such that $\delta\kappa = 0$. Then every transversal W-twistor spinor Φ satisfies*

$$\nabla^E \begin{pmatrix} \Phi \\ D'_{tr}\Phi \end{pmatrix} \equiv 0.$$

Conversely, if $\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \in \Gamma_B E$ *is ∇^E-parallel, then Φ is a transversal W-twistor spinor and $\Psi = D'_{tr}\Phi$.*

Proof. Let $\Phi \in \mathrm{Ker}P'_{tr}$ be a transversal W-twistor spinor. From (3.31), we have

$$\nabla^E_X \begin{pmatrix} \Phi \\ D'_{tr}\Phi \end{pmatrix} = \begin{pmatrix} \nabla_X\Phi + \frac{1}{q}\pi(X)\cdot D'_{tr}\Phi \\ \nabla_X D'_{tr}\Phi - \frac{q}{2}K(X)\cdot\Phi \end{pmatrix}.$$

From (3.22) and (3.30), we have

$$\nabla^E \begin{pmatrix} \Phi \\ D'_{tr}\Phi \end{pmatrix} \equiv 0.$$

Conversely, let $\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \in \Gamma_B E$ be a ∇^E-parallel section:

$$\nabla^E \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = 0.$$

Then by definition of ∇^E, we have

$$\nabla_X \Phi + \frac{1}{q}\pi(X) \cdot \Psi = 0 \quad \text{for any } X \in TM$$

and then

$$\sum_a E_a \cdot \nabla_{E_a}\Phi + \sum_a \frac{1}{q} E_a \cdot E_a \cdot \Psi = 0,$$

where $\{E_a\}$ is an orthonormal baisc frame of Q. Hence $D'_{tr}\Phi = \Psi$. This implies that Φ is a solution of the transversal W-twistor equation. □

Remark. By Theorem 3.16, the transversal W-twistor spinors correspond to the ∇^E-parallel basic sections of the bundle E. Hence the transversal W-twistor spinor Φ is defined by its values $\Phi(x_0), D'_{tr}\Phi(x_0)$ at some point $x_0 \in M$. So the dimension of the space of the transversal W-twistor spinors is less than or equal to $2^{[\frac{q}{2}]+1}$. Moreover, if Φ is the transversal W-twistor spinor on M such that Φ and $D'_{tr}\Phi$ vanish at some point $x_0 \in M$, then Φ is trivial, i.e., $\Phi \equiv 0$.

3.4 Transversal Killing spinor

Definition 3.17 For a basic function f, the spinor field $\Psi \in \Gamma S(\mathcal{F})$ satisfies the *transversal Killing equation* if for any $X \in TM$

$$\nabla_X^f \Psi \equiv \nabla_X \Psi + f\pi(X) \cdot \Psi = 0. \tag{3.32}$$

In this case, Ψ is called a transversal Killing spinor on $\mathcal{F}$.

Lemma 3.18 *If Ψ is a transversal Killing spinor, then the associate vector field X_Ψ defined by*

$$X_\Psi = i\sum_a \langle \Psi, E_a \cdot \Psi \rangle E_a$$

is a transversal Killing vector field, i.e., $\theta(X_\Psi)g_Q = 0$.

Proof. Generally, we have that for any $Y, Z \in \Gamma Q$

$$(\theta(X)g_Q)(Y, Z) = g_Q(\nabla_Y \pi(X), Z) + g_Q(Y, \nabla_Z \pi(X)).$$

Let $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ with the property that $(\nabla E_a)_x = 0$ for all a. Then we have at x that for any transversal Killing spinor Ψ with $\nabla_X \Psi = -f\pi(X) \cdot \Psi$

$$\begin{aligned}
\nabla_Y X_\Psi &= i\sum_a Y\langle \Psi, E_a \cdot \Psi \rangle E_a \\
&= i\sum_a \{\langle \nabla_Y \Psi, E_a \cdot \Psi \rangle + \langle \Psi, E_a \cdot \nabla_Y \Psi \rangle\} E_a \\
&= -if\sum_a \{\langle Y \cdot \Psi, E_a \cdot \Psi \rangle + \langle \Psi, E_a \cdot Y \cdot \Psi \rangle\} E_a.
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
g_Q(\nabla_Y X_\Psi, Z) &= -if\{g_Q(\sum_a(\langle Y\cdot\Psi, E_a\cdot\Psi\rangle E_a, Z) \\
&\quad + \sum_a(\langle\Psi, E_a\cdot Y\cdot\Psi\rangle E_a, Z))\} \\
&= -if\{\langle Y\cdot\Psi, Z\cdot\Psi\rangle + \langle\Psi, Z\cdot Y\cdot\Psi\rangle\}.
\end{aligned}
$$

Similarly,

$$g_Q(Y, \nabla_Z X_\Psi) = -if\{\langle Z\cdot\Psi, Y\cdot\Psi\rangle + \langle\Psi, Y\cdot Z\cdot\Psi\rangle\}.$$

Therefore, we have

$$(\theta(X_\Psi)g_Q)(Y, Z) = g_Q(\nabla_Y X_\Psi, Z) + g_Q(Y, \nabla_Z X_\Psi) = 0.$$

This implies that X_Ψ is a transversal Killing vector field. □

Lemma 3.19 *If Ψ is a transversal Killing spinor, then $|\Psi|^2$ is constant.*

Proof. Let Ψ be a transversal Killing spinor, i.e., for some basic function f, $\nabla_X\Psi = -f\pi(X)\cdot\Psi$. For any $X \in TM$

$$
\begin{aligned}
X|\Psi|^2 &= \langle\nabla_X\Psi, \Psi\rangle + \langle\Psi, \nabla_X\Psi\rangle \\
&= -f\{\langle\pi(X)\cdot\Psi, \Psi\rangle + \langle\Psi, \pi(X)\cdot\Psi\rangle\} \\
&= 0.
\end{aligned}
$$

So $|\Psi|^2$ is constant. □

Theorem 3.20 ([13]) *If M admits a non-vanishing transversal Killing spinor Ψ with $\nabla^f_{tr}\Psi = 0$, then*

(1) *f is constant and $f^2 = \frac{\sigma^\nabla}{4q(q-1)}$.*

(2) *$\mathcal{F}$ is transversally Einsteinian with constant transversal scalar curvature $\sigma^\nabla > 0$.*

Proof. By a direct calculation, we have

$$\sum_a E_a \cdot R^f_{XE_a}\Psi = -\frac{1}{2}\rho^\nabla(X)\cdot\Psi + 2(q-1)f^2X\cdot\Psi - qX(f)\Psi - grad_\nabla(f)\cdot X\cdot\Psi$$

for $X \in \Gamma Q$. Since $\nabla^f\Psi = 0$, we have

$$0 = -\frac{1}{2}\rho^\nabla(X)\cdot\Psi + 2(q-1)f^2X\cdot\Psi - qX(f)\Psi - grad_\nabla(f)\cdot X\cdot\Psi. \quad (3.33)$$

If we put $X = grad_\nabla(f)$, then

$$\begin{aligned} -\frac{1}{2}\rho^\nabla(X)\cdot\Psi + 2(q-1)f^2X\cdot\Psi &= q|grad_\nabla(f)|^2\Psi - |grad_\nabla(f)|^2\Psi \\ &= (q-1)|grad_\nabla(f)|^2\Psi. \end{aligned}$$

Therefore, we have

$$\langle -\frac{1}{2}\rho^\nabla(X)\cdot\Psi + 2(q-1)f^2X\cdot\Psi, \Psi\rangle = (q-1)|grad_\nabla(f)|^2|\Psi|^2. \quad (3.34)$$

Since the left hand side is pure imaginary and right hand side is real, we have

$$|grad_\nabla(f)| = 0.$$

Since f is a basic function, f is constant. Hence from (3.33) we have

$$-\frac{1}{2}\rho^\nabla(X)\cdot\Psi + 2(q-1)f^2X\cdot\Psi = 0.$$

Therefore, we have

$$\rho^\nabla(X) = 4(q-1)f^2X.$$

Thus there exists $f^2 = \frac{\sigma^\nabla}{4q(q-1)}$ such that $\rho^\nabla(X) = \frac{1}{q}\sigma^\nabla X$. This implies that $\mathcal{F}$ is transversally Einsteinian. From (2.8), we have $\sigma^\nabla = 4q(q-1)f^2$.
□

Theorem 3.21 *If Ψ is a transversal Killing spinor with $\nabla^f_{tr}\Psi = 0$, then*

$$|D_{tr}\Psi|^2 = \frac{1}{4}(\frac{q}{q-1}\sigma^\nabla + |\kappa|^2)|\Psi|^2, \tag{3.35}$$

$$\mathrm{Re}\langle D_{tr}\Psi, \kappa\cdot\Psi\rangle = -\frac{1}{2}|\kappa|^2|\Psi|^2. \tag{3.36}$$

Proof. Let Ψ be a transversal Killing spinor with $\nabla^f_X\Psi = 0$. From (3.32), we have

$$\sum_a E_a \cdot \nabla_{E_a}\Psi = -f\sum_a E_a \cdot E_a \cdot \Psi,$$

i.e., $D'_{tr}\Psi = fq\Psi$. Thus

$$D_{tr}\Psi = fq\Psi - \frac{1}{2}\kappa\cdot\Psi, \tag{3.37}$$

where $f^2 = \frac{\sigma^\nabla}{4q(q-1)}$. From (3.37), we get

$$\begin{aligned}\langle D_{tr}\Psi, D_{tr}\Psi\rangle &= \langle fq\Psi - \frac{1}{2}\kappa\cdot\Psi, fq\Psi - \frac{1}{2}\kappa\cdot\Psi\rangle \\ &= (f^2q^2 + \frac{1}{4}|\kappa|^2)\langle\Psi,\Psi\rangle.\end{aligned}$$

Therefore, we have

$$|D_{tr}\Psi|^2 = \frac{1}{4}(\frac{q}{q-1}\sigma^\nabla + |\kappa|^2)|\Psi|^2.$$

From (3.37), we get

$$\begin{aligned}\langle D_{tr}\Psi, \kappa\cdot\Psi\rangle &= \langle fq\Psi - \frac{1}{2}\kappa\cdot\Psi, \kappa\cdot\Psi\rangle \\ &= fq\langle\Psi, \kappa\cdot\Psi\rangle - \frac{1}{2}|\kappa|^2|\Psi|^2.\end{aligned}$$

Since $\langle\Psi, \kappa\cdot\Psi\rangle$ is pure imaginary,

$$Re\langle D_{tr}\Psi, \kappa\cdot\Psi\rangle = -\frac{1}{2}|\kappa|^2|\Psi|^2. \quad \square$$

Corollary 3.22 *If there exists an eigenspinor* Ψ *of* D_b *with* $\nabla_{tr}^f \Psi = 0$, *then* $\mathcal{F}$ *is minimal.*

Proof. From (3.37),

$$D_b\Psi = fq\Psi - \frac{1}{2}\kappa \cdot \Psi.$$

There exists an eigenvalue λ such that $D_b\Psi = \lambda\Psi$. Put $\lambda = fq$. Then

$$-\frac{1}{2}\kappa \cdot \Psi = 0,$$

i.e. $\kappa = 0$. Thus $\mathcal{F}$ is minimal. □

Corollary 3.23 *On the minimal foliation* $\mathcal{F}$, *every transversal Killing spinor is an eigenspinor of* D_b.

Proof. Let Ψ be the transversal Killing spinor. From (3.37), if $\mathcal{F}$ is minimal, then

$$D_b\Psi = fq\Psi.$$

From Theorem 3.20, f is constant. Hence Ψ is an eigenspinor. □

4 Transversally conformal change

Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M. Now, we consider, for any real basic function u on M, the transversally conformal metric $\bar{g}_Q = e^{2u} g_Q$. Let $\bar{P}_{so}(\mathcal{F})$ be the principal bundle of $\bar{g}_Q$ -orthogonal frames. Locally, the section $\bar{s}$ of $\bar{P}_{so}(\mathcal{F})$ corresponding a section $s = (E_1, \cdots, E_q)$ of $P_{so}(\mathcal{F})$ is $\bar{s} = (\bar{E}_1, \cdots, \bar{E}_q)$, where $\bar{E}_a = e^{-u} E_a$ $(a = 1, \cdots, q)$. This isometry will be denoted by I_u. Thanks to the isomorphism I_u one can define a transverse spin structure $\bar{P}_{spin}(\mathcal{F})$ on $\mathcal{F}$ in such a way that the diagram

$$\begin{array}{ccc} P_{spin}(\mathcal{F}) & \xrightarrow{\tilde{I}_u} & \bar{P}_{spin}(\mathcal{F}) \\ \downarrow & & \downarrow \\ P_{so}(\mathcal{F}) & \xrightarrow{I_u} & \bar{P}_{so}(\mathcal{F}) \end{array}$$

commutes.

Let $\bar{S}(\mathcal{F})$ be the foliated spinor bundles associated with $\bar{P}_{spin}(\mathcal{F})$. For any section Ψ of $S(\mathcal{F})$, we write $\bar{\Psi} \equiv I_u \Psi$. If $\langle \cdot, \cdot \rangle_{g_Q}$ and $\langle \cdot, \cdot \rangle_{\bar{g}_Q}$ denote respectively the natural Hermitian metrics on $S(\mathcal{F})$ and $\bar{S}(\mathcal{F})$, then for any $\Phi, \ \Psi \in \Gamma S(\mathcal{F})$

$$\langle \Phi, \Psi \rangle_{g_Q} = \langle \bar{\Phi}, \bar{\Psi} \rangle_{\bar{g}_Q} \tag{4.1}$$

and the Clifford multiplication in $\bar{S}(\mathcal{F})$ is given by

$$\bar{X} \bar{\cdot} \bar{\Psi} = \overline{X \cdot \Psi} \quad \text{for } X \in \Gamma Q. \tag{4.2}$$

Let $\bar{\nabla}$ be the metric and torsion free connection corresponding to $\bar{g}_Q$. Then we have the following lemma.

Lemma 4.1 ([17]) *Let* ∇ *and* $\bar{\nabla}$ *be the transversal Levi-Civita connections of* g_Q *and* $\bar{g}_Q = e^{2u}g_Q$ *on* Q*, respectively. Then for any* $X, Y \in TM$

$$\bar{\nabla}_X\pi(Y) = \nabla_X\pi(Y) + X(u)\pi(Y) + Y(u)\pi(X) - g_Q(\pi(X), \pi(Y))grad_\nabla(u),$$

where $grad_\nabla(u) = \sum_a E_a(u)E_a$ *is a transversal gradient of* u *and* $X(u)$ *is the Lie derivative of the function* u *in the direction of* X.

Proof. Since $\bar{\nabla}$ is the metric and torsion free connection with respect to $\bar{g}_Q$ on Q, we have

$$\begin{aligned} 2\bar{g}_Q(\bar{\nabla}_X s, t) &= X\bar{g}_Q(s,t) + Y\bar{g}_Q(\pi(X), t) - Z_t\bar{g}_Q(\pi(X), s) \\ &+ \bar{g}_Q(\pi[X, Y_s], t) + \bar{g}_Q(\pi[Z_t, X], s) - \bar{g}_Q(\pi[Y_s, Z_t], \pi(X)), \end{aligned}$$

where $\pi(Y_s) = s$ and $\pi(Z_t) = t$. From this formula, the proof is completed. □

The transversal Ricci curvature $\rho^{\bar{\nabla}}$ of $\bar{g}_Q = e^{2u}g_Q$ and the transversal scalar curvature $\sigma^{\bar{\nabla}}$ of $\bar{g}_Q$ are related to the transversal Ricci curvature ρ^{∇} of g_Q and the transversal scalar curvature σ^{∇} of g_Q by the following lemma.

Lemma 4.2 *On a Riemannian foliation* $\mathcal{F}$*, we have that for any* $X \in Q$*,*

$$\begin{aligned} e^{2u}\rho^{\bar{\nabla}}(X) =& \rho^{\nabla}(X) + (2-q)\nabla_X grad_\nabla(u) + (2-q)|grad_\nabla(u)|^2 X \\ &+ (q-2)X(u)grad_\nabla(u) + \{\Delta_B u - \kappa(u)\}X, \end{aligned} \tag{4.3}$$

$$e^{2u}\sigma^{\bar{\nabla}} = \sigma^{\nabla} + (q-1)(2-q)|grad_\nabla(u)|^2 + 2(q-1)\{\Delta_B u - \kappa(u)\}. \tag{4.4}$$

Proof. Let $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ with the property that $(\nabla E_a)_x = 0$ for all a. Then

$$\begin{aligned}\rho^{\bar\nabla}(X) &= \sum_a R^{\bar\nabla}(X, \bar E_a)\bar E_a \\ &= \sum_a \bar\nabla_X \bar\nabla_{\bar E_a} \bar E_a - \sum_a \bar\nabla_{\bar E_a} \bar\nabla_X \bar E_a - \sum_a \bar\nabla_{[X, \bar E_a]} \bar E_a.\end{aligned}$$

By a direct calculation, we have

$$\begin{aligned}e^{2u} \sum_a \bar\nabla_X \bar\nabla_{\bar E_a} \bar E_a &= (1-q)\{\nabla_X grad_\nabla(u) + |grad_\nabla(u)|^2 X \\ &\quad -2X(u) grad_\nabla(u)\} + \sum_a \nabla_X \nabla_{E_a} E_a.\end{aligned}$$

Similarly,

$$\begin{aligned}e^{2u} \sum_a \bar\nabla_{\bar E_a} \bar\nabla_X \bar E_a &= \sum_a \nabla_{E_a} \nabla_X E_a + \sum_a E_a E_a(u) X \\ &\quad + \nabla_{grad_\nabla(u)} X - \sum_a g(\nabla_{E_a} X, E_a) grad_\nabla(u) \\ &\quad - \nabla_X grad_\nabla(u) - |grad_\nabla(u)|^2 X - X(u) grad_\nabla(u),\end{aligned}$$

and

$$\begin{aligned}e^{2u} \sum_a \bar\nabla_{[X, \bar E_a]} \bar E_a &= \sum_a \nabla_{[X, E_a]} E_a + X(u)(q-1) grad_\nabla(u) \\ &\quad - \nabla_{grad_\nabla(u)} X + \sum_a g(\nabla_{E_a} X, E_a) grad_\nabla(u).\end{aligned}$$

Since $\Delta_B u = \delta_B d_B u = -\sum_a E_a E_a(u) + i(\kappa) d_B u$, the above equations give (4.3). On the other hand,

$$\sigma^{\bar\nabla} = \sum_a \bar g_Q(\rho^{\bar\nabla}(\bar E_a), \bar E_a) = \sum_a g_Q(\rho^{\bar\nabla}(E_a), E_a).$$

From (4.3), we have

$$\begin{aligned} e^{2u}\sigma^{\bar\nabla} &= \sum_a g_Q(e^{2u}\rho^{\bar\nabla}(E_a), E_a) \\ &= \sigma^{\nabla} + (2-q)\sum_a g_Q(\nabla_{E_a} grad_\nabla(u), E_a) \\ &\quad +(q-1)(2-q)|grad_\nabla(u)|^2 + q\{\Delta_B u - \kappa(u)\}. \end{aligned}$$

Since $\sum_a g_Q(\nabla_{E_a} grad_\nabla(u), E_a) = \sum_a E_a E_a(u) = -\Delta_B u + \kappa(u)$, we have

$$e^{2u}\sigma^{\bar\nabla} = \sigma^{\nabla} + (q-1)(2-q)|grad_\nabla(u)|^2 + 2(q-1)\{\Delta_B u - \kappa(u)\},$$

which proves (4.4). □

On the other hand, for $q \geq 3$, if we choose the positive function h by $u = \frac{2}{q-2}\ln h$, then we have

$$\Delta_B u = \frac{2}{q-2}\{h^{-2}|grad_\nabla(h)|^2 + h^{-1}\Delta_B h\}, \tag{4.5}$$

$$|grad_\nabla(u)|^2 = (\frac{2}{q-2})^2 h^{-2}|grad_\nabla(h)|^2. \tag{4.6}$$

Proposition 4.3 ([17]) *The connection ∇ and $\bar\nabla$ acting repectively on the sections of $S(\mathcal{F})$ and $\bar S(\mathcal{F})$, are related, for any vector field X and any spinor field Ψ by*

$$\bar\nabla_X\bar\Psi = \overline{\nabla_X\Psi - \frac{1}{2}\pi(X)\cdot grad_\nabla(u)\cdot\Psi} - \frac{1}{2}g_Q(grad_\nabla(u), \pi(X))\bar\Psi. \tag{4.7}$$

Proof. Let $\{E_a\}$ be an orthonormal basis of Q and denote by ω and $\bar\omega$, the connection forms corresponding to g_Q and $\bar g_Q$. That is, for any vector field $X \in TM$,

$$\nabla_X E_b = \sum_c \omega_{bc}(\pi(X))E_c, \quad \bar\nabla_X \bar E_b = \sum_c \bar\omega_{bc}(\pi(X))\bar E_c. \tag{4.8}$$

From Lemma 4.1, we have

$$\bar{\omega}_{bc}(\pi(X)) = \omega_{bc}(\pi(X)) + g_Q(\pi(X), E_c)E_b(u) - g_Q(\pi(X), E_b)E_c(u). \quad (4.9)$$

Let $\{\Psi_A\}(A = 1, \cdots, 2^{[\frac{q}{2}]})$ be a local frame field of $S(\mathcal{F})$. Then the spinor covariant derivative of Ψ_A is given ([14]) by

$$\nabla_X \Psi_A = \frac{1}{2}\sum_{b<c} \omega_{bc}(\pi(X)) E_b \cdot E_c \cdot \Psi_A. \quad (4.10)$$

With respect to $\bar{g}_Q$, we have

$$\begin{aligned}
\bar{\nabla}_X \bar{\Psi}_A &= \frac{1}{2}\sum_{b<c} \bar{\omega}_{bc}(\pi(X)) \bar{E}_b \bar{\cdot} \bar{E}_c \bar{\cdot} \bar{\Psi}_A \\
&= \frac{1}{2}\sum_{b<c} \{\omega_{bc}(\pi(X)) + g_Q(\pi(X), E_c)E_b(u) \\
&\quad - g_Q(\pi(X), E_b)E_c(u)\} \bar{E}_b \bar{\cdot} \bar{E}_c \bar{\cdot} \bar{\Psi}_A \\
&= \overline{\nabla_X \Psi_A} - \frac{1}{2}\sum_{b \neq c} g_Q(\pi(X), E_c)E_b(u) \bar{E}_c \bar{\cdot} \bar{E}_b \bar{\cdot} \bar{\Psi}_A \\
&= \overline{\nabla_X \Psi_A} - \overline{\frac{1}{2}\pi(X) \cdot grad_\nabla(u) \cdot \Psi_A} - \frac{1}{2} g_Q(grad_\nabla(u), \pi(X)) \bar{\Psi}_A.
\end{aligned}$$

□

Let $\bar{D}_{tr}$ be the transversal Dirac operator associated with the metric $\bar{g}_Q = e^{2u} g_Q$ and acting on the sections of the foliated spinor bundle $\bar{S}(\mathcal{F})$. Let $\{E_a\}$ be a local frame of $P_{so}(\mathcal{F})$ and $\{\bar{E}_a\}$ a local frame of $\bar{P}_{so}(\mathcal{F})$. Locally, $\bar{D}_{tr}$ is expressed by

$$\bar{D}_{tr}\bar{\Psi} = \bar{D}'_{tr}\bar{\Psi} - \frac{1}{2}\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi}, \quad (4.11)$$

where $\bar{D}'_{tr}\bar{\Psi} = \sum_a \bar{E}_a \bar{\cdot} \bar{\nabla}_{\bar{E}_a} \bar{\Psi}$ and $\kappa_{\bar{g}}$ is the mean curvature form associated with $\bar{g}_Q$.

Lemma 4.4 *The mean curvature forms $\kappa_{\bar g}$ and κ with respect to $\bar g_Q = e^{2u} g_Q$ and g_Q, respectively, are related by $\kappa_{\bar g} = e^{-2u}\kappa$.*

Proof. Let $\{E_i\}$ be an orthonormal basis of L. Then we have

$$g_M(\kappa^\sharp, X) = g_M(\nabla_{E_i} E_i, X), \quad X \in \Gamma Q.$$

Using the fact that $g_M(X,Y) = 0$ for $X \in L$, $Y \in Q$ and $g_L = \bar g_L$, we have

$$\begin{aligned} \bar g_M(\bar\kappa^\sharp, X) &= \bar g_M(\bar\nabla_{E_i} E_i, X) \\ &= \frac{1}{2}\{E_i \bar g_M(E_i, X) + E_i \bar g_M(X, E_i) - X \bar g_M(E_i, E_i) \\ &\quad - \bar g_M([E_i, X], E_i) - \bar g_M([E_i, X], E_i) + \bar g_M([E_i, E_i], X)\} \\ &= g_M(\nabla_{E_i} E_i, X) = g_M(\kappa^\sharp, X). \end{aligned}$$

Since $\bar g_M(\bar\kappa^\sharp, X) = e^{2u} g_Q(\bar\kappa^\sharp, X) = g_Q(\kappa^\sharp, X)$, we have that $\bar\kappa^\sharp = e^{-2u}\kappa^\sharp$. □

Using (4.7), we have that for any Ψ,

$$\bar D'_{tr}\bar\Psi = e^{-u}\{\overline{D'_{tr}\Psi} + \frac{q-1}{2}\overline{grad_\nabla(u)\cdot\Psi}\}. \tag{4.12}$$

Now, for any function f, we have $\bar D_{tr}(f\bar\Psi) = e^{-u}\overline{grad_\nabla(f)\cdot\Psi} + f\bar D_{tr}\bar\Psi$. Hence we have

$$\bar D'_{tr}(f\bar\Psi) = e^{-u}\overline{grad_\nabla(f)\cdot\Psi} + f\bar D'_{tr}\bar\Psi. \tag{4.13}$$

It is well known [14] that the transverse Dirac operators D_{tr} and $\bar D_{tr}$ satisfy

$$\bar D_{tr}(e^{-\frac{q-1}{2}u}\bar\Psi) = e^{-\frac{q+1}{2}u}\overline{D_{tr}\Psi} \tag{4.14}$$

for any spinor field $\Psi \in S(\mathcal{F})$. From (4.11) and (4.14), we have the following proposition.

Proposition 4.5 *Let $\mathcal{F}$ be the transverse spin foliation of codimension q. Then we have that for any spinor field $\Psi \in S(\mathcal{F})$*

$$\bar{D}'_{tr}(e^{-\frac{q-1}{2}u}\bar{\Psi}) = e^{-\frac{q+1}{2}u}\overline{D'_{tr}\Psi}. \tag{4.15}$$

From (4.14) and Proposition 4.5, we have the following proposition.

Corollary 4.6 *On the transverse spin foliation $\mathcal{F}$, the dimensions of the kernel of D_{tr} and D'_{tr} are transversally conformal invariant, respectively.*

Let $\bar{P}'_{tr}$ be the transversal W-twistor operator of $\bar{g}_M = g_L \oplus \bar{g}_Q$, where $\bar{g}_Q = e^{2u}g_Q$. Then we have the following theorem.

Theorem 4.7 *For any spinor field $\Psi \in \Gamma_B S(\mathcal{F})$, we have*

$$\bar{P}'_{tr}(e^{\frac{u}{2}}\bar{\Psi}) = e^{-\frac{u}{2}}\overline{P'_{tr}\Psi}. \tag{4.16}$$

In particular, $\Psi \in \Gamma_B S(\mathcal{F})$ is a transversal W-twistor spinor on (M, g_M) iff $e^{\frac{u}{2}}\bar{\Psi} \in \Gamma_B \bar{S}$ is a transversal W-twistor spinor on $(M, \bar{g}_M)$.

Proof. From (3.19), (4.7), (4.13), we have

$$\begin{aligned}
\bar{P}'_{\bar{E}_a}(e^{\frac{u}{2}}\bar{\Psi}) &= \bar{\nabla}_{\bar{E}_a}(e^{\frac{u}{2}}\bar{\Psi}) + \frac{1}{q}\bar{E}_a\bar{\cdot}\bar{D}'_{tr}(e^{\frac{u}{2}}\bar{\Psi}) \\
&= \frac{1}{2}e^{\frac{u}{2}}\bar{E}_a(u)\bar{\Psi} + e^{\frac{u}{2}}\bar{\nabla}_{\bar{E}_a}\bar{\Psi} + \frac{1}{q}\bar{E}_a\bar{\cdot}\bar{D}'_{tr}(e^{\frac{u}{2}}\bar{\Psi}) \\
&= \frac{1}{2}e^{-\frac{u}{2}}E_a(u)\bar{\Psi} + e^{-\frac{u}{2}}\{\overline{\nabla_{E_a}\Psi - \frac{1}{2}E_a\cdot grad_\nabla(u)\cdot\Psi} \\
&\quad -\frac{1}{2}E_a(u)\bar{\Psi}\} + \frac{1}{2q}e^{-\frac{u}{2}}\overline{E_a\cdot grad_\nabla(u)\cdot\Psi} \\
&\quad + e^{-\frac{u}{2}}\{\frac{1}{q}\overline{E_a\cdot D'_{tr}\Psi} + \frac{q-1}{2q}\overline{E_a\cdot grad_\nabla(u)\cdot\Psi}\} \\
&= e^{-\frac{u}{2}}\overline{P'_{E_a}\Psi}.
\end{aligned}$$

The second statement is trivial from (4.16). □

5 The transversal Weyl conformal curvature tensor

Let $(M, g_M, \mathcal{F}, S(\mathcal{F}))$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$, a bundle-like metric g_M and a foliated spinor bundle $S(\mathcal{F})$.

Definition 5.1 *For any vectors $X, Y \in TM$ and $s \in \Gamma Q$, the transversal Weyl conformal curvature tensor W^∇ is defined by*

$$\begin{aligned} W^\nabla(X,Y)s &= R^\nabla(X,Y)s \\ &+\frac{1}{q-2}\{g_Q(\rho^\nabla(\pi(X),s)\pi(Y) - g_Q(\rho^\nabla(\pi(Y),s)\pi(X) \\ &+g_Q(\pi(X),s)\rho^\nabla(\pi(Y)) - g_Q(\pi(Y),s)\rho^\nabla(\pi(X))\} \\ &-\frac{\sigma^\nabla}{(q-1)(q-2)}\{g_Q(\pi(X),s)\pi(Y) - g_Q(\pi(Y),s)\pi(X)\}. \end{aligned} \tag{5.1}$$

By a direct calculation, the transversal Weyl conformal curvature tensor W^∇ vanishes identically for $q = 3$, where $q = codim\mathcal{F}$. Moreover, we have the following theorem.

Theorem 5.2 *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then the transversal Weyl conformal curvature tensor is invariant under any transversal conformal change of g_M.*

Proof. By (5.1), we have

$$\begin{aligned} W^{\bar\nabla}(X,Y)s &= R^{\bar\nabla}(X,Y)s \\ &+ \frac{1}{q-2}\{\bar g_Q(\rho^{\bar\nabla}(\pi(X),s)\pi(Y) - \bar g_Q(\rho^{\bar\nabla}(\pi(Y),s)\pi(X) \\ &+\bar g_Q(\pi(X),s)\rho^{\bar\nabla}(\pi(Y)) - \bar g_Q(\pi(Y),s)\rho^{\bar\nabla}(\pi(X))\} \\ &- \frac{\sigma^{\bar\nabla}}{(q-1)(q-2)}\{\bar g_Q(\pi(X),s)\pi(Y) - \bar g_Q(\pi(Y),s)\pi(X)\}. \end{aligned}$$

For any vector $X, Y \in TM$, we get from Lemma 4.1

$$
\begin{aligned}
\bar{\nabla}_X \bar{\nabla}_Y s = {} & \nabla_X \nabla_Y s + X(u) \nabla_Y s + (\nabla_Y s)(u) \pi(X) \\
& - g_Q(\pi(X), \nabla_Y s) grad_\nabla(u) + (\nabla_X Y(u)) s + Y(u) \nabla_X s \\
& + X(u) Y(u) s + Y(u) s(u) \pi(X) - g_Q(\pi(X), Y(u) s) grad_\nabla(u) \\
& + s(u) X(u) \pi(Y) + s(u) \nabla_X Y + X(u) s(u) \pi(Y) + s(u) Y(u) \pi(X) \\
& - g_Q(\pi(X), s(u) \pi(Y)) grad_\nabla(u) - g_Q(\nabla_X Y, s) grad_\nabla(u) \\
& - g_Q(\pi(Y), \nabla_X s) grad_\nabla(u) - g_Q(\pi(Y), s) \nabla_X grad_\nabla(u) \\
& - X(u) g_Q(\pi(Y), s) grad_\nabla(u) - g_Q(\pi(Y), s) |grad_\nabla(u)|^2 \pi(X) \\
& + g_Q(\pi(Y), s) X(u) grad_\nabla(u).
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
R^{\bar{\nabla}}(X, Y) s = {} & R^{\nabla}(X, Y) s \\
& + g_Q(\pi(X), s) |grad_\nabla(u)|^2 \pi(Y) - g_Q(\pi(Y), s) |grad_\nabla(u)|^2 \pi(X) \\
& - s(u) X(u) \pi(Y) + s(u) Y(u) \pi(X) \\
& + g_Q(\nabla_X grad_\nabla(u), s) \pi(Y) - g_Q(\nabla_Y grad_\nabla(u), s) \pi(X) \\
& + g_Q(\pi(X), s) \nabla_Y grad_\nabla(u) - g_Q(\pi(Y), s) \nabla_X grad_\nabla(u) \\
& - g_Q(\pi(X), s) Y(u) grad_\nabla(u) + g_Q(\pi(Y), s) X(u) grad_\nabla(u).
\end{aligned}
$$

From (4.3), (4.4), we obtain

$$
\begin{aligned}
& \bar{g}_Q(\rho^{\bar{\nabla}}(\pi(X), s) \pi(Y) \\
& = g_Q(\rho^{\nabla}(\pi(X), s) \pi(Y) - (q-2) g_Q(\nabla_X grad_\nabla(u), s) \pi(Y) \\
& - (q-2)\{g_Q(|grad_\nabla(u)|^2 \pi(X), s) \pi(Y) + g_Q(X(u) grad_\nabla(u), s) \pi(Y)\} \\
& - g_Q(\sum_a E_a E_a(u) \pi(X), s) \pi(Y),
\end{aligned}
$$

$$
\begin{aligned}
&\sigma^{\bar{\nabla}}\bar{g}_Q(\pi(X),s)\pi(Y)\\
&\quad = \sigma^{\nabla} g_Q(\pi(X),s)\pi(Y) - (q-1)(q-2)|grad_{\nabla}(u)|^2 g_Q(\pi(X),s)\pi(Y)\\
&\quad - 2(q-1)\{g_Q(\pi(X),s)\sum_a E_a E_a(u)\pi(Y)\}.
\end{aligned}
$$

From the above equations, $W^{\bar{\nabla}} = W^{\nabla}$ □

On the other hand, the Weyl tensor W^S on $S(\mathcal{F})$ is similarly(cf.(3.7)) given by

$$
W^S(X,Y)\Psi = \frac{1}{4}\sum_{a,b} g_Q(W^{\nabla}(X,Y)E_a, E_b)E_a\cdot E_b\cdot\Psi \tag{5.2}
$$

for any $X, Y \in TM$ and $\Psi \in \Gamma S(\mathcal{F})$.

Proposition 5.3 *For any $X, Y \in TM$ and spinor $\Psi \in \Gamma S(\mathcal{F})$,*

$$
W^S(X,Y)\Psi = R^S(X,Y)\Psi + \frac{1}{2}\{K(Y)\cdot\pi(X) - K(X)\cdot\pi(Y)\}\Psi. \tag{5.3}
$$

Proof. From (3.1), (5.1) and (5.2), we have

$$
\begin{aligned}
W^S(X,Y)\Psi &= \frac{1}{4}\sum_{a,b} g_Q(R^{\nabla}(X,Y)E_a,E_b)E_a\cdot E_b\cdot\Psi\\
&\quad + \frac{1}{4(q-2)}\{\rho^{\nabla}(\pi(X))\cdot\pi(Y) - \rho^{\nabla}(\pi(Y))\cdot\pi(X)\\
&\quad + \pi(X)\cdot\rho^{\nabla}(\pi(Y)) - \pi(Y)\cdot\rho^{\nabla}(\pi(X))\}\\
&\quad - \frac{\sigma^{\nabla}}{4(q-1)(q-2)}\{\pi(X)\cdot\pi(Y) - \pi(Y)\cdot\pi(X)\}\\
&= R^S(X,Y)\Psi\\
&\quad - \frac{1}{2(q-2)}\{\rho^{\nabla}(\pi(X))\cdot\pi(Y) - \rho^{\nabla}(\pi(Y))\cdot\pi(X)\\
&\quad - \frac{\sigma^{\nabla}}{4(q-1)(q-2)}\{\pi(X)\cdot\pi(Y) - \pi(Y)\cdot\pi(X)\}\\
&= R^S(X,Y)\Psi + \frac{1}{2}\{K(Y)\cdot\pi(X) - K(X)\cdot\pi(Y)\}\Psi. \quad \square
\end{aligned}
$$

Now we recall the covariant derivative ∇^E on $E = S(\mathcal{F}) \oplus S(\mathcal{F})$ which is defined in (3.31). Then we have the following proposition.

Proposition 5.4 *The curvature tensor R^E on the vector bundle $E = S(\mathcal{F}) \oplus S(\mathcal{F})$ is given by the following;*

$$R^E(X,Y)\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} = \begin{pmatrix} W^S(X,Y)\Phi \\ W^S(X,Y)\Psi + \frac{q}{2}\{(\nabla_Y K)(X) - (\nabla_X K)(Y)\}\Phi\end{pmatrix}.$$

Proof. By definition (3.31) of ∇^E, we have

$$\begin{aligned}\nabla^E_X\nabla^E_Y\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} &= \nabla^E_X\begin{pmatrix}\nabla_Y\Phi + \frac{1}{q}\pi(Y)\cdot\Psi)\\ \nabla_Y\Psi - \frac{q}{2}K(Y)\cdot\Phi\end{pmatrix}\\ &= \begin{pmatrix}\nabla_X\bar{\Phi} + \frac{1}{q}\pi(X)\cdot\bar{\Psi}\\ \nabla_X\bar{\Psi} - \frac{q}{2}K(X)\cdot\bar{\Phi}\end{pmatrix},\end{aligned}$$

where $\bar{\Phi} = \nabla_Y\Phi + \frac{1}{q}\pi(Y)\cdot\Psi$ and $\bar{\Psi} = \nabla_Y\Psi - \frac{q}{2}K(Y)\cdot\Phi$.

On the other hand,

$$\begin{aligned}\nabla_X\bar{\Phi} + \frac{1}{q}\pi(X)\cdot\bar{\Psi} &= \nabla_X\nabla_Y\Phi + \frac{1}{q}\nabla_X\pi(Y)\cdot\Psi + \frac{1}{q}\pi(Y)\cdot\nabla_X\Psi\\ &\quad + \frac{1}{q}\pi(X)\cdot\nabla_Y\Psi - \frac{1}{2}\pi(X)\cdot K(Y)\cdot\Phi,\\ \nabla_X\bar{\Psi} - \frac{q}{2}K(X)\cdot\bar{\Phi} &= \nabla_X\nabla_Y\Psi - \frac{q}{2}(\nabla_X K(Y))\cdot\Phi - \frac{q}{2}K(Y)\cdot\nabla_X\Phi\\ &\quad - \frac{q}{2}K(X)\cdot\nabla_Y\Phi - \frac{1}{2}K(X)\cdot\pi(Y)\cdot\Psi.\end{aligned}$$

From Proposition 5.3, the proof is completed. □

From Proposition 5.4, we obtain the following theorem.

Theorem 5.5 *For any transversal W-twistor spinor $\Psi \in \Gamma_B S(\mathcal{F})$, it holds that for any vector X, Y and Z*

$$\begin{aligned} W^S(Y,Z)\Psi &= 0, && (5.4)\\ W^S(Y,Z)D'_{tr}\Psi &= -\frac{q}{2}\{(\nabla_Z K)(Y) - (\nabla_Y K)(Z)\}\cdot\Psi, && (5.5)\\ (\nabla_X W^S)(Y,Z)\Psi &= -\frac{q}{2}\pi(X)\cdot\{(\nabla_Z K)(Y) - (\nabla_Y K)(Z)\}\cdot\Psi \\ &\quad +(W^\nabla(Y,Z)\pi(X)\cdot D'_{tr}\Psi. && (5.6)\end{aligned}$$

Proof. Let $\Psi \in \mathrm{Ker}P'_{tr}$. First two equations (5.4), (5.5) follow from the Proposition 5.4 for the curvature tensor R^E and Theorem 3.16. From (3.22) and (5.4), we have

$$\begin{aligned} (\nabla_X W^S)(Y,Z)\Psi &= -W^S(Y,Z)\nabla_X\Psi \\ &= \frac{1}{q}W^S(Y,Z)(\pi(X)\cdot D'_{tr}\Psi). \end{aligned}$$

From (5.2) and the properties of the Clifford multiplication,

$$\begin{aligned} W^S(Y,Z)(\pi(X)\cdot D'_{tr}\Psi) &= \frac{1}{4}\sum_{a,b} g_Q(W^\nabla(X,Y)E_a,E_b)E_a\cdot E_b\cdot\pi(X)\cdot D'_{tr}\Psi \\ &= \pi(X)\cdot W^S(Y,Z)D'_{tr}\Psi + (W^\nabla(Y,Z)\pi(X))\cdot D'_{tr}\Psi. \end{aligned}$$

From the second equation, the third equation holds. □

Definition 5.6 *A Riemannian foliation $\mathcal{F}$ is called transversally conformally symmetric if*

$$\nabla W^\nabla = 0. \qquad (5.7)$$

Now we study the relation between the transversal Weyl conformal curvature tensor W^∇ and the tensor K.

Proposition 5.7 *For any $X, Y \in TM$ and $Z \in \Gamma Q$, we have*

$$\sum_a g_Q((\nabla_{E_a} W^\nabla)(X,Y)Z, E_a) = (q-3)\{g_Q((\nabla_{\pi(Y)}K)(\pi(X)) - (\nabla_{\pi(X)}K)(\pi(Y)), Z)\}. \tag{5.8}$$

Proof. From (5.1), we have

$$\begin{aligned}
\sum_a g_Q((\nabla_{E_a} W^\nabla)(X,Y)Z, E_a) &= \sum_a g_Q((\nabla_{E_a} R^\nabla)(X,Y)Z, E_a) \\
&+ \frac{1}{q-2}\{g_Q((\nabla_{\pi(Y)}\rho^\nabla)(\pi(X)), Z) - g_Q((\nabla_{\pi(X)}\rho^\nabla)(\pi(Y)), Z)\} \\
&+ \frac{q-3}{2(q-1)(q-2)}\{\pi(Y)(\sigma^\nabla)g_Q(\pi(X), Z) - \pi(X)(\sigma)g_Q(\pi(Y), Z)\},
\end{aligned}$$

where $\{E_a\}$ is an orthonormal basic frame of Q. Here we used the fact that $Y(\sigma^\nabla) = 2\sum_a g_Q((\nabla_{E_a}\rho^\nabla)(Y), E_a)$ for any vector $Y \in \Gamma Q$. From the secomd Bianchi identity, we have

$$\begin{aligned}
\sum_a g_Q((\nabla_{E_a} R^\nabla)(X,Y)Z, E_a) &= \sum_a \{g_Q(-\nabla_{\pi(X)}R^\nabla)(Y, E_a)Z, E_a) - g_Q(\nabla_{\pi(Y)}R^\nabla)(E_a, X)Z, E_a)\} \\
&= g_Q((\nabla_{\pi(X)}\rho^\nabla)(\pi(Y)), Z) - g_Q((\nabla_{\pi(Y)}\rho^\nabla)(\pi(X)), Z).
\end{aligned}$$

Hence, we have that for any $X, Y \in TM$ and $Z \in \Gamma Q$

$$\begin{aligned}
\sum_a g_Q((\nabla_{E_a} W^\nabla)(X,Y)Z, E_a) &= \frac{q-3}{q-2}\{g_Q((\nabla_{\pi(X)}\rho^\nabla)(\pi(Y)), Z) - g_Q((\nabla_{\pi(Y)}\rho^\nabla)(\pi(X)), Z)\} \\
&+ \frac{q-3}{2(q-1)(q-2)}\{\pi(Y)(\sigma^\nabla)g_Q(\pi(X), Z) - \pi(X)(\sigma)g_Q(\pi(Y), Z)\}.
\end{aligned}$$

From the definition of the tensor K, the proof is completed. □

From (5.7), we have the following corollary.

Corollary 5.8 *Let $\mathcal{F}$ be a transversally conformally symmetric Riemannian foliation of codimension $q > 3$. Then the following equation*

$$(\nabla_{\pi(X)}K)(\pi(Y)) - (\nabla_{\pi(Y)}K)(\pi(X)) = 0, \quad \forall\ X, Y \in \Gamma Q$$

holds.

Proof. Since $\nabla W^\nabla = 0$, the above equation holds. □

Theorem 5.9 *Let $\mathcal{F}$ be a transversal conformal symmetric Riemannian foliation with a non-trivial transversal W-twistor spinor Φ and suppose that $D'_{tr}\Phi$ vanishes on a discrete set only. Then $\mathcal{F}$ is a transversally conformally flat space. i.e. $W^\nabla = 0$. In particular, if $\mathcal{F}$ is codimension 3 with a non-trivial transversal W-twistor spinor, then $\mathcal{F}$ is a transversally conformally flat space.*

Proof. Let Φ be a non-trivial transversal W-twistor spinor. From Theorem 5.5 and Theorem 5.8, we have that for any vector fields X, Y and Z

$$(\nabla_X W^S)(Y, Z)\Psi = W^\nabla(Y, Z)\pi(X) \cdot D'_{tr}\Psi.$$

By a direct calculation, we have

$$(\nabla_X W^S)(Y, Z)\Psi = \frac{1}{4}\sum_{a,b} g_Q((\nabla_X W^\nabla)(Y, Z)E_a, E_b)E_a \cdot E_b \cdot \Psi,$$

where $\{E_a\}$ is an orthonormal basic frame of Q. Hence $\nabla W^\nabla = 0$ implies $W^\nabla(Y, Z)\pi(X) \cdot D'_{tr}\Psi = 0$. Since $D'_{tr}\Phi$ vanishes only on a discrete set, this yields that $W^\nabla = 0$. If $\mathcal{F}$ is codimension $q = 3$ with a non-trivial transversal W-twistor spinor, then $W^\nabla = 0$. □

6 The conformal relation between transversal twistor and transversal Killing spinors

Let $(M, g_M, \mathcal{F})$ be a compact connected Riemannian manifold with an isoparametric transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M such that $\delta\kappa = 0$.

Definition 6.1 *On the vector space* $\mathrm{Ker}P'_{tr}$*, a quadratic form* C' *and a form* Q' *are defined by*

$$C'(\Psi) = (D'_{tr}\Psi, \Psi) = Re\langle D'_{tr}\Psi, \Psi\rangle, \tag{6.1}$$

$$Q'(\Psi) = |\Psi|^2 |D'_{tr}\Psi|^2 - C'(\Psi)^2 - \sum_a (D'_{tr}\Psi, E_a \cdot \Psi)^2, \tag{6.2}$$

where $\{E_a\}$ *is an orthonormal basic frame on* Q *(see* [6] *for ordinary case).*

Remark. Since $\langle X \cdot \Psi, \Psi\rangle$ is pure imaginary for any spinor Ψ, it is trivial that $C'(\Psi) = (D_{tr}\Psi, \Psi)$.

Lemma 6.2 *For any transversal W-twistor spinor* $\Psi \in \Gamma_B S(\mathcal{F})$*, it holds that*

$$\bar{C}'(e^{u/2}\bar{\Psi}) = C'(\Psi), \quad \bar{Q}'(e^{u/2}\bar{\Psi}) = Q'(\Psi). \tag{6.3}$$

Proof. From (4.15), (6.1) and (6.2), we have

$$\begin{aligned} \bar{C}'(e^{u/2}\bar{\Psi}) &= (\bar{D}'_{tr}(e^{u/2}\bar{\Psi}), e^{u/2}\bar{\Psi}) = (e^{-u/2}\overline{D'_{tr}\Psi}, e^{u/2}\bar{\Psi}) \\ &= \overline{(D'_{tr}\Psi, \Psi)} = C'(\Psi), \end{aligned}$$

$$
\begin{aligned}
&\bar{Q}'(e^{u/2}\bar{\Psi}) \\
&= |e^{u/2}\bar{\Psi}|^2|\bar{D}'_{tr}(e^{u/2}\bar{\Psi}|^2 - \bar{C}'(e^{u/2}\bar{\Psi})^2 - \sum_a(\bar{D}'_{tr}(e^{u/2}\bar{\Psi}), \bar{E}_a \cdot e^{u/2}\bar{\Psi})^2 \\
&= |e^{u/2}\bar{\Psi}|^2|e^{-u/2}\overline{D'_{tr}\Psi}|^2 - C'(\Psi)^2 - \sum_a(e^{-u/2}\overline{D'_{tr}\Psi}, \bar{E}_a \cdot e^{u/2}\bar{\Psi})^2 \\
&= |\bar{\Psi}|^2|\overline{D'_{tr}\Psi}|^2 - C'(\Psi)^2 - \sum_a \overline{(D'_{tr}\Psi, E_a \cdot \Psi)}^2 \\
&= Q'(\Psi). \quad \square
\end{aligned}
$$

Hence, we have the following theorem.

Theorem 6.3 *Let $(M, g_M, \mathcal{F})$ be a compact connected Riemannian manifold with an isoparametric transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M such that $\delta\kappa = 0$. Then for any transversal W-twistor spinor Ψ, $C'(\Psi)$ and $Q'(\Psi)$ are transversally conformal invariants with respect to $\Psi \to e^{u/2}\bar{\Psi}$. Moreover they are constant.*

Proof. The first statement follows from (6.3). Next, if we differentiate $C'(\Psi)$ with respect to $X \in \Gamma TM$, then

$$\nabla_X C'(\Psi) = (\nabla_X D'_{tr}\Psi, \Psi) + (D'_{tr}\Psi, \nabla_X \Psi).$$

From (3.22) and Theorem 3.14, we have

$$
\begin{aligned}
\nabla_X C'(\Psi) \;=\; & \frac{q\sigma^{\nabla}}{4(q-1)(q-2)}(\pi(X)\cdot\Psi, \Psi) - \frac{q}{2(q-2)}(\rho^{\nabla}(\pi(X))\cdot\Psi, \Psi) \\
& -\frac{1}{q}(D'_{tr}\Psi, \pi(X)\cdot D'_{tr}\Psi).
\end{aligned}
$$

Since $\langle X \cdot \Psi, \Psi\rangle$ is pure imaginary for any $X \in \Gamma Q$, we know that $\nabla_X C'(\Psi) = 0$. That is, $C'(\Psi)$ is constant. Since $C'(\Psi)$ is constant, we have from (3.22)

$$\begin{aligned}\nabla_X Q'(\Psi) &= 2(\nabla_X\Psi,\Psi)|D'_{tr}\Psi|^2+2|\Psi|^2(\nabla_X D'_{tr}\Psi, D'_{tr}\Psi)\\ &\quad -2\sum_a (D'_{tr}\Psi, E_a\cdot\Psi)(\nabla_X D'_{tr}\Psi, E_a\cdot\Psi)\\ &\quad -\frac{2}{q}\sum_a (D'_{tr}\Psi, E_a\cdot\Psi)(E_a\cdot D'_{tr}\Psi, \pi(X)\cdot D'_{tr}\Psi).\end{aligned}$$

Since $(X\cdot\Psi, Y\cdot\Psi)=g_Q(X,Y)|\Psi|^2$ for $X,Y\in\Gamma Q$, Theorem 3.14 implies $\nabla_X Q'(\Psi)=0$. □

Definition 6.4 *For any spinor Ψ, we define the associated vector field T^Ψ by*

$$T^\Psi = 2\sum_a (\Psi, E_a\cdot D'_{tr}\Psi)E_a. \tag{6.4}$$

Proposition 6.5 *Any non-vanishing transversal W-twistor spinor Ψ satisfies*

(1) $T^\Psi = -q\ grad_\nabla w,$

(2) $|C'(\Psi)\Psi - wD'_{tr}\Psi + \frac{q}{2}grad_\nabla w\cdot\Psi|^2 = wQ'(\Psi),$

where $w=|\Psi|^2$.

Proof. The equation (1) is trivial from (6.4). Now we prove the second equation (2). By a direct calculation, we have

$$\begin{aligned}&|C'(\Psi)\Psi - wD'_{tr}\Psi - \frac{1}{2}T^\Psi\cdot\Psi|^2\\ &\qquad = wC'(\Psi)^2 - 2wC'(\Psi)^2 + w^2|D'_{tr}\Psi|^2 - w\sum_a (D'_{tr}\Psi, E_a\cdot\Psi)^2.\end{aligned}$$

Therefore, we have

$$|C'(\Psi)\Psi - wD'_{tr}\Psi - \frac{1}{2}T^\Psi\cdot\Psi|^2 = wQ'(\Psi). \quad \square$$

From Proposition 6.5 (2), we have the following corollary.

Corollary 6.6 *If Ψ is a non-vanishing transversal W-twistor spinor such that $C'(\Psi) = Q'(\Psi) = 0$, then we have*

$$wD'_{tr}\Psi = \frac{q}{2}grad_{\nabla}(w)\cdot\Psi.$$

Theorem 6.7 *Under the same condition as in Theorem 6.3, if M admits a non-vanishing transversal W-twistor spinor Ψ such that $C'(\Psi) = 0 = Q'(\Psi)$, then $\mathcal{F}$ is transversally conformally equivalent to a transversally Ricci-flat foliation on $(M, \bar{g}_M)$ with parallel basic spinor.*

Proof. Consider the metric $\bar{g}_M = g_L + \bar{g}_Q$, where $\bar{g}_Q = e^{2u}g_Q$, $u = -\ln w$. From Proposition 4.3, we have

$$\bar\nabla_X(w^{-\frac{1}{2}}\bar\Psi) = \overline{\nabla_X(w^{-\frac{1}{2}}\Psi)} - \frac{1}{2}w^{-\frac{1}{2}}\overline{\pi(X)\cdot grad_{\nabla}(u)\cdot\Psi} - \frac{1}{2}w^{-\frac{1}{2}}X(u)\bar\Psi.$$

Since $grad_{\nabla}(u) = -\frac{1}{w}grad_{\nabla}(w)$, the above equation is the following:

$$\begin{aligned}\bar\nabla_X(w^{-\frac{1}{2}}\bar\Psi) &= X(w^{-\frac{1}{2}})\bar\Psi + w^{-\frac{1}{2}}\overline{\nabla_X\Psi} + \frac{1}{2}w^{-\frac{3}{2}}\overline{\pi(X)\cdot grad_{\nabla}(w)\cdot\Psi}\\ &\quad + \frac{1}{2}w^{-\frac{3}{2}}X(w)\bar\Psi\\ &= w^{-\frac{1}{2}}\overline{(\nabla_X\Psi + \frac{1}{2w}\pi(X)\cdot grad_{\nabla}(w)\cdot\Psi)}\end{aligned}$$

for $X \in \Gamma TM$. From (3.22) and Corollary 6.6, it follows that

$$\bar\nabla_X(w^{-\frac{1}{2}}\bar\Psi) = 0.$$

That is, $\bar\Phi \equiv w^{-\frac{1}{2}}\bar\Psi$ is a parallel basic spinor with respect to the metric $\bar{g}_M$. From (3,10), $\rho^{\bar\nabla}(\pi(X))\bar\cdot\bar\Phi = \sum_a \bar{E}_a\bar\cdot R^{\bar{S}}(X,\bar{E}_a)\bar\Phi = 0$. Hence, the foliation $\mathcal{F}$ on $(M, \bar{g}_M)$ is transversally Ricci-flat. □

Now, we study the relation between the transversal W-twistor spinor and transversal Killing spinor.

Proposition 6.8 *Under the same condition as in Theorem 6.3, every transversal W-twistor spinor* Ψ *satisfies*

$$\frac{q}{2}\Delta_B|\Psi|^2 = \frac{q}{4(q-1)}\sigma^\nabla|\Psi|^2 + \frac{1}{4}|\kappa|^2|\Psi|^2 - |D_{tr}\Psi|^2. \tag{6.5}$$

Proof. By (2.13) and (2.14), we have

$$\Delta_B|\Psi|^2 = \delta_B d_B|\Psi|^2 = -\sum_a \nabla_{E_a}\nabla_{E_a}|\Psi|^2 + \nabla_{\kappa^\sharp}|\Psi|^2. \tag{6.6}$$

This equation with (3.14), (3.15) and (3.22) gives

$$\begin{aligned}
-2\sum_a \nabla_{E_a}\langle\nabla_{E_a}\Psi,\Psi\rangle + 2\langle\nabla_{\kappa^\sharp}\Psi,\Psi\rangle \\
&= 2\langle-\nabla_{E_a}\nabla_{E_a}\Psi + \nabla_{\kappa^\sharp}\Psi,\Psi\rangle - 2\sum_a\langle\nabla_{E_a}\Psi,\nabla_{E_a}\Psi\rangle \\
&= 2\langle\nabla_{tr}^*\nabla_{tr}\Psi,\Psi\rangle - \frac{2}{q}|D_{tr}'^2\Psi|^2 \\
&= 2\langle D_{tr}^2\Psi - \frac{1}{4}(\sigma^\nabla + |\kappa|^2)\Psi,\Psi\rangle - \frac{2}{q}|D_{tr}'^2\Psi|^2.
\end{aligned}$$

By (3.13), (3.16) and (3.22), we have

$$\Delta_B|\Psi|^2 = \frac{2}{4(q-1)}\sigma^\nabla|\Psi|^2 + \frac{1}{2q}|\kappa|^2|\Psi|^2 - \frac{2}{q}|D_{tr}\Psi|^2. \quad \square$$

Proposition 6.9 *Under the same condition as in Theorem 6.3, every non-vanishing transversal W-twistor spinor* Ψ *satisfies the following equation*

$$\begin{aligned}
\frac{q}{2}\{\omega\Delta_B(\ln\omega) - \kappa^\sharp(\omega)\} + \frac{q(q-2)}{4\omega}|grad_\nabla\omega|^2 \\
&= \frac{q}{4(q-1)}\sigma^\nabla\omega - \frac{1}{\omega}\{Q'(\Psi) + C'(\Psi)^2\},
\end{aligned}$$

where $w = |\Psi|^2$.

Proof. Let $(0 \neq)\Psi \in \mathrm{Ker}P'_{tr}$ be a transversal W-twistor spinor. From (6.2), we have

$$Q'(\Psi) = \omega|D'_{tr}\Psi|^2 - C'(\Psi)^2 - \frac{1}{4}|T^{\Psi}|^2. \tag{6.7}$$

Since $\Delta_B(\ln\omega) = \frac{1}{\omega^2}|grad_\nabla \omega|^2 + \frac{1}{\omega}\Delta_B\omega$, Proposition 6.5 implies

$$\Delta_B(\ln\omega) = \frac{1}{q^2\omega^2}|T^{\Psi}|^2 + \frac{1}{\omega}\Delta_B\omega. \tag{6.8}$$

On the other hand, we have

$$\begin{aligned}\langle\kappa\cdot D'_{tr}\Psi,\Psi\rangle+&\langle\Psi,\kappa\cdot D'_{tr}\Psi\rangle\\ &= -q\{\langle\nabla_{\kappa^\sharp}\Psi,\Psi\rangle + \langle\Psi,\nabla_{\kappa^\sharp}\Psi\rangle\}\\ &= -q\kappa^\sharp(w)\end{aligned}$$

and thus

$$|D_{tr}\Psi|^2 = |D'_{tr}\Psi|^2 + \frac{1}{4}|\kappa|^2\omega - \frac{q}{2}\kappa^\sharp(\omega). \tag{6.9}$$

From (6.5), (6.8), (6.9), we have

$$\Delta_B(\ln\omega) = \frac{1}{q^2\omega^2}|T^{\Psi}|^2 + \frac{1}{2(q-1)}\sigma^\nabla + \frac{1}{\omega}\kappa^\sharp(\omega) - \frac{2}{q\omega}|D'_{tr}\Psi|^2, \tag{6.10}$$

which completes the proof by using Proposition 6.5 and (6.10). □

Definition 6.10 *Any spinor field Ψ is transversally conformally equivalent to a transversal Killing spinor if there exists a transversally conformal change of the metric $\bar g_M = g_L + e^{2u}g_Q$ such that $e^{\frac{u}{2}}\bar\Psi$ is a transversal Killing spinor with respect to $\bar g_M$. Equivalently, for any $X \in TM$*

$$\bar\nabla_X(e^{\frac{u}{2}}\bar\Psi) + a\pi(X)\bar\cdot(e^{\frac{u}{2}}\bar\Psi) = 0, \tag{6.11}$$

where $a \neq 0$ is a real number.

Then we have the following theorem.

Theorem 6.11 *Let $\mathcal{F}$ be a transverse spin foliation and $\Psi \in \mathrm{Ker}P'_{tr}$ a non-vanishing transversal W-twistor spinor. Then Ψ is transversally conformally equivalent to a transversal Killing spinor if and only if $C'(\Psi) \neq 0$ and $Q'(\Psi) = 0$.*

Proof. Let $\Psi \in \mathrm{Ker}P'_{tr}$ be transversally conformally equivalent to a transversal Killing spinor with respect to $\bar{g}_Q = e^{2u}g_Q$. Since

$$\begin{aligned}
\bar{\nabla}_X(e^{\frac{u}{2}}\bar{\Psi}) &= \frac{1}{2}e^{\frac{u}{2}}X(u)\bar{\Psi} + e^{\frac{u}{2}}\bar{\nabla}_X\bar{\Psi} \\
&= \frac{1}{2}e^{\frac{u}{2}}X(u)\bar{\Psi} + e^{\frac{u}{2}}\{\overline{\nabla_X\Psi} - \frac{1}{2}\overline{\pi(X)\cdot grad_\nabla(u)\cdot\Psi} \\
&\quad -\frac{1}{2}X(u)\bar{\Psi}\} \\
&= e^{\frac{u}{2}}\{\overline{\nabla_X\Psi} - \frac{1}{2}\overline{\pi(X)\cdot grad_\nabla(u)\cdot\Psi}\},
\end{aligned}$$

the equation (6.11) is equivalent to

$$\nabla_X\Psi = \frac{1}{2}\pi(X)\cdot grad_\nabla u\cdot\Psi - ae^u\pi(X)\cdot\Psi \tag{6.12}$$

for $X \in \Gamma TM$, where $a(\neq 0)$ is a real number. Now if we choose $u = -\ln w$, then (6.12) implies

$$\frac{1}{q}\omega D'_{tr}\Psi = \frac{1}{2}grad_\nabla w\cdot\Psi + a\Psi. \tag{6.13}$$

Since $\langle X\cdot\Psi,\Psi\rangle$ is pure imaginary, from (6.1) and (6.13) we have

$$\begin{aligned}
C'(\Psi) &= Re\langle D'_{tr}\Psi,\Psi\rangle \\
&= Re\langle\frac{q}{2\omega}\,grad_\nabla(\omega)\cdot\Psi + \frac{qa}{w}\cdot\Psi,\Psi\rangle \\
&= qa\frac{1}{w}|\Psi|^2 = qa.
\end{aligned} \tag{6.14}$$

This implies that $C'(\Psi) \neq 0$ for any $a \neq 0$. On the other hand, from (6.13) we have

$$\omega |D'_{tr}\Psi|^2 = \frac{q^2}{4}|grad_\nabla w|^2 + q^2a^2. \tag{6.15}$$

From (6.2), (6.14), (6.15) and Proposition 6.5, we obtain $Q'(\Psi) = 0$. Conversely, we assume $C'(\Psi) \neq 0$ and $Q'(\Psi) = 0$. From Proposition 6.5, we have

$$C'(\Psi)\Psi - wD'_{tr}\Psi + \frac{q}{2}grad_\nabla w \cdot \Psi = 0. \tag{6.16}$$

If we choose u satisfying $w = \frac{C'(\Psi)}{qa}e^{-u}$, then we have

$$\pi(X)\cdot\Psi - \frac{1}{qa}e^{-u}\pi(X)\cdot D'_{tr}\Psi - \frac{1}{2a}e^{-u}\pi(X)\cdot grad_\nabla(u)\cdot\Psi = 0.$$

Therefore, we have

$$\nabla_X\Psi = \frac{1}{2}\pi(X)\cdot grad_\nabla(u)\cdot\Psi - ae^u\pi(X)\cdot\Psi.$$

This means that $e^{\frac{u}{2}}\bar{\Psi}$ is a transversal Killing spinor with respect to $\bar{g}_Q = e^{2u}g_Q$. So Ψ is transversally conformally equivalent to a transversal Killing spinor. □

7 Eigenvalue estimates

Let $(M, g_M, \mathcal{F})$ be a closed connected Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M such that the mean curvature form κ satisfies $\Delta_B\kappa = 0$. The existence of a bundle-like g_M for $(M, \mathcal{F})$ such that $\kappa \in \Omega_B^1(\mathcal{F})$ is proved in [22]. Moreover it is assured [23] that there exists another bundle-like metric whose mean curvature form is basic harmonic, i.e., $\Delta_B\kappa = 0$. For any spinor field $\Psi \in \Gamma S(\mathcal{F})$, we have from (3.19)

$$
\begin{aligned}
|P'_{tr}\Psi|^2 &= \sum_a \langle P'_{E_a}\Psi, P'_{E_a}\Psi\rangle \\
&= \sum_a \langle \nabla_{E_a}\Psi + \frac{1}{q}E_a\cdot D'_{tr}\Psi, \nabla_{E_a}\Psi + \frac{1}{q}E_a\cdot D'_{tr}\Psi\rangle \\
&= |\nabla_{tr}\Psi|^2 - \frac{1}{q}\langle E_a\cdot\nabla_{E_a}\Psi, D'_{tr}\Psi\rangle \\
&\quad -\frac{1}{q}\langle D'_{tr}\Psi, E_a\cdot\nabla_{E_a}\Psi\rangle + \frac{1}{q}|D'_{tr}\Psi|^2 \\
&= |\nabla_{tr}\Psi|^2 - \frac{1}{q}|D'_{tr}\Psi|^2. \qquad (7.1)
\end{aligned}
$$

By integrating (7.1) together with (3.14), we have

$$\int_M |P'_{tr}\Psi|^2 = \int_M |D_{tr}\Psi|^2 - \frac{1}{4}K^\sigma|\Psi|^2 - \frac{1}{q}|D'_{tr}\Psi|^2. \qquad (7.2)$$

Since $D_{tr}\Psi = D'_{tr}\Psi - \frac{1}{2}\kappa\cdot\Psi$, we have

$$|D_{tr}\Psi|^2 = |D'_{tr}\Psi|^2 - \frac{1}{4}|\kappa|^2|\Psi|^2 - Re\langle D_{tr}\Psi, \kappa\cdot\Psi\rangle. \qquad (7.3)$$

From (7.2) and (7.3), we have

$$
\begin{aligned}
\int_M |P'_{tr}\Psi|^2 &= \frac{q-1}{q}\int_M \{|D_{tr}\Psi|^2 - \frac{q}{4(q-1)}(K^\sigma + \frac{1}{q}|\kappa|^2)|\Psi|^2\} \\
&\quad + \int_M Re\langle D_{tr}\Psi, \kappa\cdot\Psi\rangle. \qquad (7.4)
\end{aligned}
$$

Let $D_b\Psi = \lambda\Psi$. Since $\langle\Psi, \kappa\cdot\Psi\rangle$ is pure imaginary, we have

$$\int_M |P'_{tr}\Psi|^2 = \frac{q-1}{q}\int_M \{\lambda^2 - \frac{q}{4(q-1)}(K^\sigma + \frac{1}{q}|\kappa|^2)\}|\Psi|^2. \qquad (7.5)$$

Thus we have the following theorem.

Theorem 7.1 *Let $(M, g_M, \mathcal{F})$ be a closed connected Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 2$ and bundle-like metric g_M such that $\Delta_B\kappa = 0$. Then any eigenvalue λ of the basic Dirac operator D_b satisfies*

$$\lambda^2 \geq \frac{q}{4(q-1)}\inf_M(K^\sigma + \frac{1}{q}|\kappa|^2). \qquad (7.6)$$

In the limiting case, $\mathcal{F}$ is minimal and transversally Einsteianian with constant transversal scalar curvature σ^∇.

Proof. The inequality (7.6) is trivial from (7.5). Now we study the limiting case. Let $D_b\Psi = \lambda\Psi$ with $\lambda^2 = \frac{q}{4(q-1)}\inf_M(K^\sigma + \frac{1}{q}|\kappa|^2)$. From (7.5), we see $P'_{tr}\Psi = 0$. Since $D'_{tr}\Psi = \lambda\Psi + \frac{1}{2}\kappa\cdot\Psi$. we have from (3.22)

$$\nabla_X\Psi = -\frac{\lambda}{q}\pi(X)\cdot\Psi - \frac{1}{2q}\pi(X)\cdot\kappa\cdot\Psi. \qquad (7.7)$$

for $X \in \Gamma TM$. From (3.16) and (3.24) we have that

$$\begin{aligned}\nabla_{\kappa^\sharp}\Psi &= D_b^2\Psi - D_{tr}'^2\Psi - \frac{1}{4}|\kappa|^2\Psi \\ &= \lambda^2\Psi - \frac{q}{4(q-1)}\sigma^\nabla\Psi - \frac{1}{4}|\kappa|^2\Psi.\end{aligned}$$

Therefore, we have

$$\{\frac{q}{4(q-1)}\sigma^\nabla - \lambda^2 + \frac{q+2}{4q}|\kappa|^2\}\Psi = \frac{\lambda}{q}\kappa\cdot\Psi. \qquad (7.8)$$

It follows from (7.8) that $\mathcal{F}$ is minimal. Hence (7.7) implies that Ψ is transversal Killing spinor. From Theorem 3.20, the foliation $\mathcal{F}$ is transversally Einsteinian with constant transversal scalar curvature $\sigma^{\nabla} > 0$. □

References

[1] J. A. Alvarez López, *The basic component of the mean curvature of Riemannian foliations*, Ann. Global Anal. Geom. 10 (1992), 179-194.

[2] C. Bär, *Real Killing spinors and holonomy*, Comm. Math. Phys. 154(1993), 509-521.

[3] H. Baum, T. Friedrich, R. Grunewald and I. Kath, *Twistor and Killing Spinors on Riemannian Manifolds*, Seminarbericht Nr. 108, Humboldt-Universität zu Berlin, 1990.

[4] J. Brüning and F. W. Kamber, *Vanishing theorems and index formulas for transversal Dirac operators*, A.M.S Meeting 845, Special Session on operator theory and applications to Geometry, Lawrence, KA; A.M.S. Abstracts, October 1988.

[5] D. Domínguez, *A tenseness theorem for Riemannian foliations*, C. R. Acad. Sci. Sér. I 320(1995), 1331-1335.

[6] T. Friedrich, *On the conformal relation between twistors and Killing spinors*, Suppl. Rend. Circ. Mat. Palermo (1989), 59-75.

[7] J. F. Glazebrook and F. W. Kamber, *Transversal Dirac families in Riemannian foliations*, Comm. Math. Phys. 140 (1991), 217-240.

[8] K. Habermann, *Twistor spinors and their zeros*, J. Geom. Physics 14(1994), 1-24.

[9] K. Habermann, *The Twistor equation on Riemannian manifolds*, J. Geom. Physics 7(1990), 469-488.

[10] J. J. Hebda, *Curvature and focal points in Riemannian foliation*, Indiana Univ. Math. J. 35(1986), 321-331.

[11] O. Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*, Comm. Math. Phys. 104 (1986), 151-162.

[12] O. Hijazi, *Lower bounds for eigenvalues of the Dirac operator* , J. Geom. Phys. 16(1995), 27-38.

[13] S. D. Jung, *The first eigenvalue of the transversal Dirac operator*, J. Geom. Phys. 39(2001), 253-264.

[14] S. D. Jung and Y. S. Ko, *Eigenvalue estimates of the basic Dirac operator on a Riemannian foliation*, to appear in Taiwanese J. Math.

[15] S. D. Jung and Y. B. Moon, *The propertiets of the transversal Killing sponors and transversal twistor sponors on Riemannian folintion*, to appear in J. Korean Math. Soc.

[16] S. D. Jung and Y. B. Moon, *The transversal Weyl conformal curvature tensor on a Riemannian foliation*, in preprint.

[17] S. D. Jung, B. H. Kim and J. S. Pak, *Lower bounds for the eigenvalues of the basic Dirac operator on a Riemannian foliation*, J. Geom. Phys. 51(2004), 166-182.

[18] F. W. Kamber and Ph. Tondeur, *Harmonic foliations*, Proc. National Science Foundation Conference on Harmonic Maps, Tulane, Dec. 1980, Lecture Notes in Math. 949, Springer-Verlag, New York, 1982, 87-121.

[19] H. B. Lawson, Jr. and M. L. Michelsohn, *Spin geometry,* Princeton Univ. Press, Princeton, New Jersey, 1989.

[20] A. Lichnerowicz, *Killing spinors, twistor-spinors and Hijazi inequality,* J. Geom. Phys, 5(1988), 2-18.

[21] A. Lichnerowicz, *On the twistor-spinors*, Lett. Math. Physics 18(1989), 333-345.

[22] P. March, M. Min-Oo and E. A. Ruh, *Mean curvature of Riemannian foliations*, Canad. Math. Bull. 39(1996), 95-105.

[23] A. Mason, *An application of stochastic flows to Riemannian foliations*, Houston J. Math. 26(2000), 481-515.

[24] E. Park and K. Richardson, *The basic Laplacian of a Riemannian foliation*, Amer. J. Math. 118(1996), 1249-1275.

[25] R. Penrose and W. Rindler, *Spinors and Space Time*, Vol. 2, Cambr. Mono. in Math. Physics (1986).

[26] Ph. Tondeur, *Foliations on Riemannian manifolds*, Springer-Verlag, New-York, 1988.

〈국 문 초 록〉

엽층구조를 가지는 리만 다양체에서의 횡단적 twistor spinor에 관한 연구

엽층구조를 가지는 리만다양체의 횡단적 구조가 spin 구조를 가질 때, 횡단적 Killing spinor와 twistor spinor의 성질을 알아보고, 엽층 다양체 상에서 어떤 공형 불변량을 사용하여 그들 사이의 관계를 조사한다. 그리고 공형 변환에 불변인 횡단적 Weyl 공형 곡률을 정의하여 횡단적 twistor spinor와의 관계를 연구하는 한편, 횡단적 twistor 연산자를 이용하여 basic Dirac 연산자의 고유치를 구한다.