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WEYL AND BROWDER SPECTRA OF A LINEAR OPERATOR ON A BANACH SPACE

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康　銀　希

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指導敎授　梁　永　五

康　銀　希

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WEYL AND BROWDER SPECTRA
OF A LINEAR OPERATOR
ON A BANACH SPACE

Eun-Hui Kang
(Supervised by professor Young-Oh Yang)

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Department of Mathematics
GRADUATE SCHOOL
CHEJU NATIONAL UNIVERSITY

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Abstract (Korean)

Acknowledgements (Korean)

< **Abstract** >

WEYL AND BROWDER SPECTRA OF A LINEAR OPERATOR ON A BANACH SPACE

In this thesis, we study several properties of Weyl operator, Browder operator and their spectra on an infinite dimensional Banach space and investigate the systematic relations between Weyl's(a-Weyl's) theorem and Browder's(a-Browder's) theorem, respectively. The followings are the main results of this thesis.

(1) Fredholm, Weyl and Browder operators are stable under compact perturbation and open with the norm topology. Also we give equivalent conditions of Weyl and Browder operator, respectively.

(2) The essential spectrum, Weyl spectrum and Browder spectrum are upper semi-continuous and their spectral radius are also upper semi-continuous. Also these spectra are invariant under similarity.

(3) The spectral mapping theorem holds for the Browder spectrum of a bounded linear operator. Also we extends this result as follows : If $T \in \mathcal{L}(X)$ and f is a holomorphic function defined in a neighborhood of $\sigma(T)$, then $f(\sigma_b(T)) = \sigma_b(f(T))$.

(4) We show that a-Weyl's theorem implies Weyl's theorem and Weyl's theorem implies Browder's theorem for bounded linear operators. Also we obtain that a-Weyl's theorem implies a-Browder's theorem, and that a-Browder's theorem implies Browder's theorem for bounded linear operators.

1 Introduction

Let X be an infinite-dimensional Banach space and let $\mathcal{L}(X)$ be the set of all bounded linear operators on X and $\mathcal{K}(X)$ the set of all compact operators on X. If $T \in \mathcal{L}(X)$, we shall write $N(T)(= \ker(T))$ and $R(T)$ for the *null space* and *range* of T, respectively. We note that $X/\overline{R(T)} = N(T^*)$. Also $\alpha(T) = \dim N(T) = \dim\ker(T)$ and $\beta(T) = \dim N(T^*) = \dim(X/\overline{R(T)})$. Here X^* denotes the *dual space* of X and $T^* \in \mathcal{L}(X^*)$ is the *adjoint operator* of T.

In this paper, we will study several properties of Fredholm, Weyl and Browder operators and the inclusion relations between Weyl spectrum, Browder spectrum and other spectra.

The organization of this thesis is as follows:

In section 2, we introduce topological properties(openness, the stability under compact perturbation, etc) of Fredholm, Weyl and Browder operators on an infinite-dimensional Banach space X.

In section 3, we introduce various spectra(essential spectrum, Weyl spectrum, Browder spectrum, etc) of a bounded linear operator on X and the inclusion relations among them. Also we study the upper semi-continuities of various spectra and their spectral radius. In particular we show that these spectra are invariant under similarity, and that the spectral mapping theorem does not hold for the Weyl spectrum of a bounded linear operator in general. But the spectral mapping theorem hold for the Browder spectrum of a bounded linear operator.

In section 4, we determine whether T obeys Weyl's theorem, Browder's theorem, a-Weyl's theorem and a-Browder's theorem respectively. We study the relations between Weyl's(a-weyl's) theorem and Browder's(a-Browder's) theorem. Also we give necessary and sufficient conditions of Weyl's theorem and Browder's theorem.

2 Fredholm, Weyl and Browder operators

Definition 2.1. An operator $T \in \mathcal{L}(X)$ is called a *Fredholm operator* if $N(T) = \ker(T)$ is finite dimensional, $R(T)$ is closed and $N(T^*)$ is finite dimensional.

The following properties of compact operators are well-known([1],[2]): For all $K,\ K' \in \mathcal{K}(X)$, $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$

(1) $K + K' \in \mathcal{K}(X)$, $\lambda K \in \mathcal{K}(X)$ and so $\mathcal{K}(X)$ is a linear space over $\mathbb{C}$.

(2) $TK,\ KT \in \mathcal{K}(X)$ and so $\mathcal{K}(X)$ becomes an ideal in $\mathcal{L}(X)$.

(3) If $T_n \to T$ in the norm and $T_n \in \mathcal{K}(X)$, then $T \in \mathcal{K}(X)$ and so $\mathcal{K}(X)$ is closed in $\mathcal{L}(X)$.

From (1), (2) and (3), the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$ is a C^*-algebra since $\mathcal{L}(X)$ is a C^*-algebra. We call this quotient algebra the *Calkin algebra* of X. Let $\pi : \mathcal{L}(X) \longrightarrow \mathcal{L}(X)/\mathcal{K}(X)$ denote the *natural projection* of $\mathcal{L}(X)$ by $T \longmapsto \pi(T) = \widehat{T} = T + \mathcal{K}(X)$.

Theorem 2.2. (Atkinson's theorem [1]) An operator $T \in \mathcal{L}(X)$ is a Fredholm operator if and only if $\pi(T)$ is an invertible operator in $\mathcal{L}(X)/\mathcal{K}(X)$.

Proof. Suppose that $\widehat{T}$ is invertible. There is an operator $S \in \mathcal{L}(X)$ such that $\widehat{S} = \widehat{T}^{-1}$. Then $\widehat{T}\widehat{S} = I$ and $\widehat{S}\widehat{T} = I$ and so there exist $K_1, K_2 \in \mathcal{K}(X)$ such that $I - ST = K_1$ and $I - TS = K_2$. We have to show that $N(T)$ is finite dimensional and that $R(T)$ is a closed subspace of finite codimension. Since $ST = I - K_1$,

$N(T) \subseteq N(ST) = N(I - K_1)$. Since $N(I - K_1)$ is finite dimensional, $N(T)$ is finite dimensional. Consider $R(T)$. Since $TS = I - K_2$, we have $R(T) \supseteq R(TS) = R(I - K_2)$, and $R(I - K_2)$ is a closed subspace of X of finite codimension. We can make an obvious inductive argument to find a finite set of vectors $v_1, v_2, \cdots, v_r$ such that $R(T) = R(I - K_2) + [v_1, v_2, \cdots, v_r]$. Thus $R(T)$ is a closed subspace of finite codimension in X.

Conversely, suppose that $T \in \Phi(X)$. Then $N(T)$ is finite dimensional and $R(T)$ is a closed subspace of finite codimension. There exist projections $P,\ Q \in \mathcal{L}(X)$ such that $P^2 = P,\ Q^2 = Q,\ R(P) = N(T)$ and $R(Q) = R(T)$. Since P and $I - Q$ are finite-rank, there exists $S \in \mathcal{L}(X)$ such that

$$ST = I - P,\ TS = Q = I - (I - Q)$$

Thus $\widehat{S}\widehat{T} = I = \widehat{T}\widehat{S}$ in $\mathcal{L}(X)/\mathcal{K}(X)$. □

Corollary 2.3. (Atkinson's theorem [1]) The following conditions are equivalent:

(1) T is a Fredholm operator.

(2) There is an operator $S \in \mathcal{L}(X)$ such that $I - ST$ and $I - TS$ are compact.

(3) There is an operator $S \in \mathcal{L}(X)$ such that $I - ST$ and $I - TS$ are finite-rank operators.

Definition 2.4. An operator $T \in \mathcal{L}(X)$ is called a *left(right)-Fredholm operator* if $\pi(T) = \widehat{T}$ is left(right)-invertible in $\mathcal{L}(X)/\mathcal{K}(X)$. T is called a *Fredholm operator* if $\pi(T) = \widehat{T}$ is invertible in $\mathcal{L}(X)/\mathcal{K}(X)$. Let $\Phi_l(X), \Phi_r(X)$ and $\Phi(X)$ denote the set of all left-Fredholm, right-Fredholm and Fredholm operators respectively. Operators in the set $\mathcal{S}\Phi(X) = \Phi_l(X) \cup \Phi_r(X)$ are said to be *semi-Fredholm*.

By the definition, we see that $\Phi(X) = \Phi_l(X) \cap \Phi_r(X)$.

Definition 2.5. The *index* of $T \in \Phi(X)$, denoted by $i(T)$, is defined by $i(T) = \alpha(T) - \beta(T) = \dim N(T) - \dim N(T^*)$.

For examples, if T is normal operator(i.e., $T^*T = TT^*$) then $i(T) = 0$, because $N(T) = N(T^*)$. And if T is hyponormal(i.e., $T^*T \geq TT^*$) then $i(T) \leq 0$, because $N(T) \leq N(T^*)$ and so dim $N(T) \leq N(T^*)$.

Lemma 2.6. ([1]) Let $T \in \mathcal{K}(X)$ be any compact operator. Then $i(I-T) = 0$, i.e., $\dim N(I-T) = \dim N((I-T)^*)$.

Proof. Let us assume that T is of finite-rank. Then there are closed subspaces M and Z of X such that M is finite-dimensional, $X = M \oplus Z$, $TM \subset M$ and $TZ = \{0\}$. Let $(I-T)|_M$ be the restriction of $I - T$ to M. Then $R(I-T) = R((I-T)|_M) \oplus Z$ and $N(I-T) = N((I-T)|_M)$.

For all $x \in X$ there are $u \in M$ and $z \in Z$ such that $x = u + z$. Hence

$$(I-T)x = (I-T)(u+z) = (I-T)u + (I-T)z = (I-T)u + z$$

and $(I-T)u \in M$.

If $(I-T)x = 0$, then $z = Tu - u = -u + Tu \in M \cap Z = \{0\}$ and so $z = 0$. Then $x = u \in M$. Thus $(I-T)x = (I-T)|_M x$.

Since $(I-T)|_M \in \mathcal{L}(X)$ and M is finite-dimensional,

$$
\begin{aligned}
\infty &> \dim N(I-T) = \dim N((I-T)|_M) \\
&= \operatorname{codim} R((I-T)|_M) = \operatorname{codim} R(I-T) = \dim N(I-T)^*
\end{aligned}
$$

Thus $i(I-T) = 0$. Since $\dim R((I-T)|_M) < \infty$ and Z is closed, $R(I-T)$ is closed. □

Lemma 2.7. If $T \in \mathcal{L}(X)$ is invertible, then T is a Fredholm operator in X and $i(T) = 0$.

Proof. Since T is invertible, $N(T) = \{0\}$ and so $\alpha(T) = \dim\, N(T) = 0 < \infty$. Since X is closed and $R(T) = X$, $\beta(T) = \dim N(T^*) = \dim(X/R(T)) = 0 < \infty$. Thus $T \in \Phi(X)$ and $i(T) = \alpha(T) - \beta(T) = 0$. □

Let $\mathcal{I}(X)$ be the set of all invertible operators. Then $\mathcal{I}(X) \subseteq \Phi(X)$ by the above Lemma 2.7.

Theorem 2.8. (The index product theorem [1]) Let T, S be a Fredholm operator. Then TS is also a Fredholm operator and $i(TS) = i(T) + i(S)$.

Proof. Since T and S are Fredholm, $\pi(T)$ and $\pi(S)$ are invertible. Then $\pi(T)\pi(T)^{-1} = \pi(T)^{-1}\pi(T) = I$ and $\pi(S)\pi(S)^{-1} = \pi(S)^{-1}\pi(S) = I$. Thus

$$
\begin{aligned}
\pi(TS)[\pi(TS)]^{-1} &= \pi(TS)[\pi(TS)^{-1}] = \pi(TS)\pi(S^{-1}T^{-1}) \\
&= \pi(T)\pi(S)[\pi(S)]^{-1}[\pi(T)]^{-1} = I.
\end{aligned}
$$

Similarly $[\pi(TS)]^{-1}\pi(TS) = I$. Therefore $\pi(TS)$ is invertible and hence TS is Fredholm. □

Theorem 2.9. If $T, S \in \mathcal{L}(X)$ with $TS = ST$ and if ST is Fredholm, then S and T are Fredholm.

Proof. Since $N(T) \cup N(S) \subset N(TS)$, we have $\dim N(T) \leq \dim N(TS) < \infty$ and $\dim N(S) \leq \dim N(TS) < \infty$. Similarly,

$$N(T^*) \cup N(S^*) \subset N(T^*S^*) = N((ST)^*) = N((TS)^*).$$

Then $\dim N(T^*) \leq \dim N((TS)^*) < \infty$ and $\dim N(S^*) \leq \dim N((TS)^*) < \infty$. Finally, we show that $R(T)$ and $R(S)$ are closed. If $R(T)$ is not closed, then there is $z \in X$ such that $z = \lim_{n\to\infty} z_n$, $z_n \in R(T)$ and $z \notin R(T)$. Since $z_n \in R(T)$, $Sz_n \in S(R(T)) = R(ST)$. Since S is continuous, $Sz = S(\lim_{n\to\infty} z_n) = \lim_{n\to\infty} Sz_n$ and $Sz_n \in R(ST)$. Thus $z \in R(T)$. This is a contradiction. Hence $R(T)$ is closed. Similarly, $R(S)$ is closed. □

Theorem 2.10. ([2]) If $T \in \Phi(X)$, then $T^* \in \Phi(X^*)$ and $i(T^*) = -i(T)$.

Theorem 2.11. (The stability under compact perturbation) If T is a Fredholm operator and K is a compact operator, then $T+K$ is also Fredholm and $i(T+K) = i(T)$.

Proof. By Atkinson's theorem, there exists $S \in \mathcal{L}(X)$ such that $ST = I - K_1$ and $TS = I - K_2$ where $K_1, K_2 \in \mathcal{K}(X)$. Thus

$$S(T+K) = ST + SK = I - K_1 + SK = I - (K_1 - SK) = I - F_1,$$

$$(T+K)S = TS + KS = I - K_2 + KS = I - (K_2 - KS) = I - F_2$$

where $F_1, F_2 \in \mathcal{K}(X)$. Hence $T + K \in \Phi(X)$ by Atkinson's theorem.

By Lemma 2.6, $i(I-K_1) = 0 = i(I-F_1)$ and so $i(ST) = i(S(T+K))$. Thus $i(S)+i(T+K) = i(S(T+K)) = i(ST) = i(S)+i(T)$ and so $i(T+K) = i(T)$. The proof is complete. $\square$

Theorem 2.12. ([17]) Let T be a Fredholm operator. Then there is an $\eta > 0$ such that $T+S \in \Phi(X)$ and $i(T+S) = i(T)$ for any $S \in \mathcal{L}(X)$ satisfying $\|S\| < \eta$. Hence $\Phi(X)$ is open in $\mathcal{L}(X)$ with the norm topology.

Proof. By Atkinson's theorem, there exists $T_1 \in \mathcal{L}(X)$ such that $T_1T = I-K_1$ and $TT_1 = I-K_2$ where $K_1, K_2 \in \mathcal{K}(X)$. Let S be any operator in $\mathcal{L}(X)$ satisfying $\|S\| < \eta$. Then

$$T_1(T+S) = T_1T + T_1S = I - K_1 + T_1S \ ,$$

$$(T+S)T_1 = TT_1 + ST_1 = I - K_2 + ST_1 \ .$$

Take $\eta = \frac{1}{\|T_1\|}$. Then $\|T_1S\| \le \|T_1\|\|S\| = \frac{\|S\|}{\eta} < 1$. Similarly $\|ST_1\| < 1$. Thus $I+T_1S$ and $I+ST_1$ have bounded inverse. Consequently

$$(I+T_1S)^{-1}T_1(T+S) = (I+T_1S)^{-1}(I+T_1S-K_1) = I-(I+T_1S)^{-1}K_1 \ ,$$

$$(T+S)T_1(I+ST_1)^{-1} = (I+ST_1-K_2)(I+ST_1)^{-1} = I-K_2(I+ST_1)^{-1} \ .$$

Hence $T+S \in \Phi(X)$. Moreover

$$i((I+T_1S)^{-1}) + i(T_1) + i(T+S) = i(I-(I+T_1S)^{-1}K_1) = 0$$

Since $i((I+T_1S)^{-1}) = 0$ and $i(T_1)+i(T) = i(T_1T) = i(I-K_1) = 0$, we have $i(T+S) = -i(T_1) = i(T)$. $\square$

Corollary 2.13. ([1]) Let T be a Fredholm operator and let $\{T_n\}$ be a sequence in $\mathcal{L}(X)$ that converges to T in norm topology, i.e., $\lim_{n\to\infty} \|T_n - T\| = 0$. There is a positive integer n_0 such that for any positive integer $n \geq n_0$, $T_n \in \Phi(X)$ with $i(T_n) = i(T)$.

Theorem 2.14. The sets $\Phi_l(X), \Phi_r(X)$ and $\Phi(X)$ are all open in $\mathcal{L}(X)$ with the norm, and $T \in \Phi_l(X)$ if and only if $T^* \in \Phi_r(X)$.

Proof. If $\widehat{G} = \{\ \pi(T) : \pi(T) \text{ is invertible}\} \subseteq \mathcal{L}(X)/\mathcal{K}(X)$, then $\widehat{G}$ is open since the set of all invertible operators is open in $\mathcal{L}(X)$. Since the natural projection $\pi : \mathcal{L}(X) \longrightarrow \mathcal{L}(X)/\mathcal{K}(X)$ is continuous and onto, $\pi^{-1}(\widehat{G})$ is open. Thus $\Phi(X) = \{\ T : \pi(T) \text{ is invertible}\} = \{\ T : T \in \pi^{-1}(\widehat{G})\} = \pi^{-1}(\widehat{G})$ and so $\Phi(X)$ is open. Similarly, $\widehat{G}_l = \{\pi(T) : \pi(T) \text{ is left-invertible}\}$ and $\widehat{G}_r = \{\pi(T) : \pi(T) \text{ is right-invertible}\}$ are open. Then $\pi^{-1}(\widehat{G}_l)$ and $\pi^{-1}(\widehat{G}_r)$ are open. Thus $\Phi_l(X) = \{T : T \in \pi^{-1}(\widehat{G}_l)\} = \pi^{-1}(\widehat{G}_l)$ and $\Phi_r(X) = \pi^{-1}(\widehat{G}_r)$ are open.

Since $T \in \Phi_l(X)$, there is $S \in \mathcal{L}(X)$ such that $ST = I$. Then $I = I^* = (ST)^* = T^*S^*$, i.e., $T^*S^* = I$. Hence $T^* \in \Phi_r(X)$. □

Definition 2.15. An operator $T \in \mathcal{L}(X)$ is called a *Weyl operator* if T is Fredholm and $i(T) = 0$.

We write $\Phi_0(X) = \{T \in \mathcal{L}(X) : T \in \Phi(X) \text{ and } i(T) = 0\}$ for the set of all Weyl operators. Clearly $\mathcal{I}(X) \subseteq \Phi_0(X) \subseteq \Phi(X)$ by Lemma 2.7.

Theorem 2.16. If K is compact, then $I - K$ is a Weyl operator.

Theorem 2.17. If T is a Weyl operator, then there exists $K \in \mathcal{L}(X)$ of finite-rank such that $T+K$ is invertible.

Proof. Since $T \in \Phi_0(X)$, $i(T)=0$ and dim $N(T) =$ dim $N(T^*) < \infty$. Since $X = (X/N(T)) \oplus N(T) = (X/R(T)) \oplus R(T)$, there exists an invertible operator $F_0 : N(T) \to N(T^*)$ defined by $F = F_0(I-P)$ where P is projection of X onto $X/N(T)$. Then F is finite-rank. First we show that $T+F$ is injective, i.e., $(T+F)x = 0$ implies $x=0$. If $x \in N(T)$ then $0 = (T+F)x = Fx$. Since $Fx = F_0(I-P)x = F_0(x-Px) = F_0x = 0$ and F_0 is injective, $x=0$. If $x \in X/N(T)$ then $Fx = F_0(I-P)x = F_0(x-Px) = F_0(x-x) = F_0(0) = 0$. Thus $0 = (T+F)x = Tx+Fx = Tx$, i.e., $x \in N(T)$. Since $N(T) \cap X/N(T) = \{0\}$, $x=0$.

Secondly we show that $T+F$ is onto. If $x \in X$, then there are $u \in R(T)$ and $v \in X/R(T)$ such that $x = u+v$. Then $u = Tp$ for some $p \in X/N(T)$ and $v = F_0q$ for some $q \in N(T)$. Thus $x = u+v = Tp+F_0q$. Put $h = p+q \in X/N(T) \oplus N(T) = X$. Then $Fq = F_0(I-P)q = F_0q$, $\quad Fp = F_0(I-P)p = F_0(p-p) = F_0(0) = 0$ and so $Fh = Fp+Fq = F_0q$. Thus $x = Tp+F_0q = Th+Fh = (T+F)h$. Hence $T+F$ is onto and hence $T+F$ is invertible. □

Theorem 2.18. If T is Weyl and K is compact, then $T+K$ is a Weyl operator.

Proof. Since $T \in \Phi_0(X)$, $i(T)=0$ and $T \in \Phi(X)$. Then $T+K \in \Phi(X)$ and $i(T+K) = i(T)$ by Theorem 2.11. Thus $T+K \in \Phi(X)$ and $i(T+K)=0$. Hence $T+K \in \Phi_0(X)$. □

Corollary 2.19. If S is invertible and K is compact, then $S+K$ is a Weyl operator.

Proof. If S is invertible, then S is Weyl. Since K is compact, by theorem 2.18, $S+K$ is Weyl. □

Corollary 2.20. Let $T \in \mathcal{L}(X)$ be any operator. The following conditions are equivalent:

(1) T is a Weyl operator

(2) $T = S + F$ with S invertible and F finite-rank

(3) $T = S + K$ with S invertible and K compact

Proof. (1)⇒ (2) : From Theorem 2.17, if $T \in \Phi_0(X)$ then there is K of finite-rank such that $T+K$ is invertible. So $T = (T+K) - K = (T+K) + (-K)$. If we take $T + K = S$ and $-K = F$, then $T = S + F$.

(2) ⇒ (3) : All finite-rank operator is compact.
(3) ⇒ (1) : It is clear from Corollary 2.19. □

Recall that $a(T)(d(T)$ respectively), the *ascent*(*descent* respectively) of T, is the smallest non-negative integer n such that $N(T^n) = N(T^{n+1})(R(T^n) = R(T^{n+1})$ respectively). If no such n exists, then $a(T) = \infty$ and $d(T) = \infty$. If $a(T) < \infty$ and $d(T) < \infty$, then $a(T) = d(T)$([5]).

Example. If T is invertible, then $a(T) = 1$.

Proof. If T is invertible, then T is injective and so $N(T) = \{0\}$. Clearly $N(T) \subset N(T^2)$. We show that $N(T^2) \subset N(T)$. If $x \in N(T^2)$ then $T^2x = 0$. Since T is invertible, there exists T^{-1}. Then $T^{-1}T^2x = 0$ and so $Tx = 0$. Hence $x \in N(T)$ and so $N(T) = N(T^2)$. Thus $a(T) = 1$.

Definition 2.21. An operator T is called *upper semi-Browder* if $T \in \Phi_l(X)$ and $a(T) < \infty$. An operator T is called *lower semi-Browder* if $T \in \Phi_r(X)$ and $d(T) < \infty$. An operator T is called *Browder* if T is both upper semi-Browder and lower semi-Browder.

Let $\mathcal{B}_+(X), \mathcal{B}_-(X)$ and $\mathcal{B}(X)$ denote the set of all upper semi-Browder, lower semi-Browder and Browder operators, respectively.

Clearly, $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$.

Remark 2.1. ([3]) The following conditions are equivalent:

(1) T is a Browder operator.

(2) T is Fredholm of finite ascent and descent.

(3) T is Fredholm and $T - \lambda$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$.

Theorem 2.22. ([5]) The sets $\mathcal{B}_+(X), \mathcal{B}_-(X)$ and $\mathcal{B}(X)$ are open subsets of $\mathcal{L}(X)$.

Theorem 2.23. ([13]) If $S, T \in \mathcal{B}(X)$ and $ST = TS$, then $ST \in \mathcal{B}(X)$.

Theorem 2.24. ([13]) If $T \in \mathcal{L}(X)$, $K \in \mathcal{K}(X)$ and $TK = KT$, then $T \in \mathcal{B}(X)$ implies $T + K \in \mathcal{B}(X)$.

3 Properties of several spectra

Definition 3.1. ([2],[8]) For any $T \in \mathcal{L}(X)$, we define various spectra as follows:

(1) $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not invertible $\}$ is called the *spectrum* of T.

(2) $\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not injective $\} = \{\lambda \in \mathbb{C} : N(T - \lambda) \neq \{0\}\}$ is called the *point spectrum* of T.

(3) $\sigma_{com}(T) = \{\lambda \in \mathbb{C} : R(T - \lambda)$ is not dense in X $\}$ is called the *compression spectrum* of T.

(4) $\sigma_{ap}(T) = \{\lambda \in \mathbb{C}$: there exists a sequence $\{x_n\}$ in X with $\|x_n\| = 1$ for all n, such that $\|(T - \lambda)x_n\| \to 0\}$ is called an *approximate spectrum* of T.

We recall that $\sigma(T)$ is a non-empty compact subset of $\mathbb{C}$. Also if T is self-adjoint operator then $\sigma(T) \subseteq \mathbb{R}$. In particular, if $T \in \mathcal{K}(X)$, then $0 \in \sigma(T)$, each nonzero point of $\sigma(T)$ is an eigenvalue of T whose eigenspace is finite dimensional. Also $\sigma(T)$ is either a finite set or it is a sequence which converges to zero ([2]).

The *spectral radius* of T, $r(T)$, is defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$.

Lemma 3.2. ([2]) For any $T \in \mathcal{L}(X)$, we have the following properties:

(1) $|\lambda| \leq \|T\|$ for any $\lambda \in \sigma(T)$.

(2) $\sigma_p(T) \subset \sigma(T)$ and $\sigma_{com}(T) \subset \sigma(T)$.

(3) $\sigma_{ap}(T)$ is a nonempty closed compact subset of $\sigma(T)$.

(4) $\sigma_p(T) \subset \sigma_{ap}(T) \subset \sigma(T)$ and $\partial\sigma(T) \subset \sigma_{ap}(T)$.

Remark 3.1. $\sigma_p(T)$ need not a non-empty.

For example, if T is a unilateral shift operator on l^2, then $\sigma_p(T) = \phi$. For, suppose that $x = (x_1, x_2, \cdots) \in l^2$. If $Tx = \lambda x$ with $\lambda \neq 0$, then

$$T(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots) = \lambda x = (\lambda x_1, \lambda x_2, \cdots).$$

Then $0 = \lambda x_1,\ x_1 = \lambda x_2,\ x_2 = \lambda x_3,\ \cdots$. Since $\lambda \neq 0,\ x_1 = 0,\ x_2 = 0,\ \cdots$. Thus $N(T - \lambda) = \{0\}$. If $\lambda = 0$, then $Tx = (0, x_1, x_2, \cdots) = (0, 0, \cdots)$. Then $x_1 = x_2 = \cdots = 0$ and so $x = (0, 0, \cdots) = 0$. Hence $\lambda = 0 \notin \sigma_p(T)$, i.e., $\sigma_p(T) = \phi$.

Definition 3.3. For any $T \in \mathcal{L}(X)$, $\sigma_e(T) = \sigma(\pi(T)) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi(X)\}$ is called the *essential spectrum* of T. Similarly, $\sigma_{le}(T) = \sigma_l(\pi(T))$ and $\sigma_{re}(T) = \sigma_r(\pi(T))$ are called the *right* and *left essential spectrum* of T, respectively.

Theorem 3.4. ([1]) For any $T \in \mathcal{L}(X)$, $\sigma_e(T)$ is a non-empty compact subset of $\sigma(T)$ and $\sigma_e(T) \subseteq \cap\{\sigma(T + K) : K \in \mathcal{K}(X)\}$.

Proof. If $\lambda \in \sigma_e(T)$, then $T - \lambda \notin \Phi(X)$ and so $T - \lambda \notin \mathcal{I}(X)$ since $\mathcal{I}(X) \subset \Phi(X)$. Thus $\lambda \in \sigma(T)$ and hence $\sigma_e(T) \subset \sigma(T)$. Since $\sigma_e(T) = \sigma(\pi(T))$ and the spectrum of every operator is a non-empty compact subset of $\mathbb{C}$, $\sigma_e(T)$ is a non-empty compact subset of $\sigma(T)$. Moreover, if $\lambda \notin \sigma(T + K)$ for any $K \in \mathcal{K}(X)$, then $(T + K) - \lambda$ is invertible. Since $\mathcal{I}(X) \subset \Phi(X)$, $(T + K) - \lambda \in \Phi(X)$. By the stability under compact perturbation of a Fredholm operator, $T - \lambda \in \Phi(X)$ and so $\lambda \notin \sigma_e(T)$. Hence $\sigma_e(T) \subseteq \bigcap_{K \in \mathcal{K}(X)} \sigma(T + K)$. $\square$

Theorem 3.5. ([2]) For any $T \in \mathcal{L}(X)$, we have the following properties:

(1) $\sigma_{le}(T) \cup \sigma_{re}(T) = \sigma_e(T)$ and $\sigma_{le}(T) = \sigma_{re}(T^*)^*$.

(2) $\sigma_{le}(T) \subseteq \sigma_l(T)$, $\sigma_{re}(T) \subseteq \sigma_r(T)$ and $\sigma_e(T) \subseteq \sigma(T)$

(3) If $K \in \mathcal{K}(X)$ then $\sigma_{le}(T) = \sigma_{le}(T+K), \sigma_{re}(T) = \sigma_{re}(T+K)$ and $\sigma_e(T) = \sigma_e(T+K)$

Proof. (1) Note that

$$\begin{aligned} \lambda \in \sigma_e(T) &\Leftrightarrow T-\lambda \notin \Phi(X) \Leftrightarrow T-\lambda \notin \Phi_l(X) \text{ or } T-\lambda \notin \Phi_r(X) \\ &\Leftrightarrow \lambda \in \sigma_{le}(T) \text{ or } \lambda \in \sigma_{re}(T) \\ &\Leftrightarrow \lambda \in \sigma_{le}(T) \cup \sigma_{re}(T). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \lambda \in \sigma_{le}(T) &\Leftrightarrow \lambda \in \sigma_l(\pi(T)) \Leftrightarrow \pi(T-\lambda) \text{ is not left invertible} \\ &\Leftrightarrow \pi(T^*-\lambda) \text{ is not right invertible} \Leftrightarrow \lambda \in \sigma_r(\pi(T^*))^* \\ &\Leftrightarrow \lambda \in \sigma_{re}(T^*)^*. \end{aligned}$$

(2) If $\lambda \notin \sigma_l(T)$, then $T-\lambda$ is left invertible and so there is $S \in \mathcal{L}(X)$ such that $S(T-\lambda) = I$. Thus $I = \pi(I) = \pi(S(T-\lambda)) = \pi(S)\pi(T-\lambda)$. Therefore $\pi(T)-\lambda$ is left invertible, i.e., $\lambda \notin \sigma_l(\pi(T)) = \sigma_{le}(T)$. Hence $\sigma_{le}(T) \subseteq \sigma_l(T)$. Similarly, $\sigma_{re}(T) \subseteq \sigma_r(T)$.

(3) Note that

$$\begin{aligned} \lambda \notin \sigma_e(T) &\Leftrightarrow T-\lambda \in \Phi(X) \\ &\Leftrightarrow (T-\lambda)+K \in \Phi(X) \text{ where } K \in \mathcal{K}(X) \text{ by Theorem 2.11.} \\ &\Leftrightarrow (T+K)-\lambda \in \Phi(X) \Leftrightarrow \lambda \notin \sigma_e(T+K). \end{aligned}$$

Hence $\sigma_e(T) = \sigma_e(T+K)$.

Similarly, $\sigma_{le}(T) = \sigma_{le}(T+K)$ and $\sigma_{re}(T) = \sigma_{re}(T+K)$. □

Definition 3.6. For each $T \in \mathcal{L}(X)$, $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_0(X)\}$ is called the *Weyl spectrum* of T.

Theorem 3.7. Let $T \in \mathcal{L}(X)$ be any operator. Then

(1) $\sigma_w(T) = \bigcap_{K \in \mathcal{K}(X)} \sigma(T+K)$

(2) $\sigma_w(T)$ is a non-empty compact subset of $\sigma(T)$ and $\sigma_e(T) \subseteq \sigma_w(T)$.

(3) $\sigma_w(T+K) = \sigma_w(T)$ for any $K \in \mathcal{K}(X)$.

(4) $\sigma_w(T) = \sigma_e(T)$ if T is normal.

(5) $\partial \sigma_w(T) \subset \sigma_e(T)$.

Proof. (1) If $\lambda \notin \sigma_w(T)$, then $T - \lambda$ is a Weyl operator. By Theorem 2.19, there exists a compact operator K such that $T - \lambda + K$ is invertible and so $\lambda \notin \sigma(T+K)$ for some $K \in \mathcal{K}(X)$. Thus $\lambda \notin \bigcap_{K \in \mathcal{K}(X)} \sigma(T+K)$.

Conversely, if $\lambda \notin \bigcap_{K \in \mathcal{K}(X)} \sigma(T+K)$, then $T + K - \lambda$ is invertible and so $(T + K - \lambda) - K = T - \lambda$ is a Weyl operator by Theorem 2.19. Thus $\lambda \notin \sigma_w(T)$.

(2) If $\lambda \in \sigma_w(T)$, then $T - \lambda \notin \Phi_0(X)$. Since $\mathcal{I}(X) \subset \Phi_0(X)$, $T - \lambda \notin \mathcal{I}(X)$. Then $\lambda \in \sigma(T)$. Thus $\sigma_w(T) \subseteq \sigma(T)$ and so $\sigma_w(T)$ is bounded. Since $\sigma(T+K)$ is bounded and closed for any $K \in \mathcal{K}(X)$, $\bigcap_{K \in \mathcal{K}(X)} \sigma(T+K)$ is closed. Thus $\sigma_w(T)$ is a compact. Since $\Phi_0(X) \subset \Phi(X)$, $\sigma_e(T) \subset \sigma_w(T)$. Hence $\sigma_w(T)$ is a non-empty.

(3) Since $K, K' \in \mathcal{K}(X)$, $K + K' \in \mathcal{K}(X)$. Then

$$\sigma_w(T+K) = \bigcap_{K' \in \mathcal{K}(X)} \sigma(T+K+K') = \bigcap_{K \in \mathcal{K}(X)} \sigma(T+K) = \sigma_w(T).$$

(4) If T is normal, then $T-\lambda$ is also normal for any $\lambda \in \mathbb{C}$, i.e., for any $\lambda \in \mathbb{C}$, $\|(T-\lambda)x\| = \|(T-\lambda)^* x\|$ for any $x \in X$, and so $N(T-\lambda) = N(T-\lambda)^*$. Thus $i(T-\lambda) = 0$ and so $\{\lambda \in \mathbb{C} : T-\lambda \in \Phi(X) \text{ and } i(T-\lambda) \neq 0\} = \phi$. If $\lambda \notin \sigma_e(T)$, then $T-\lambda \in \Phi(X)$. Since $i(T-\lambda) = 0$, $T-\lambda \in \Phi_0(X)$ and so $\lambda \notin \sigma_w(T)$. Hence $\sigma_w(T) \subset \sigma_e(T)$. Therefore $\sigma_w(T) = \sigma_e(T)$.

(5) Suppose that $\lambda \in \partial\sigma_w(T) - \sigma_e(T)$, then $\lambda \notin \sigma_e(T)$ and so $T-\lambda \in \Phi(X)$. Also $\lambda \in \partial\sigma_w(T)$, there is a sequence $\{\lambda_n\}$ in $\sigma_w(T)^c$ such that $\lambda_n \to \lambda$ and $T-\lambda_n \in \Phi_0(X)$ for all n. By the continuity of the index, $T-\lambda \in \Phi_0(X)$ and so $\lambda \notin \sigma_w(T)$. But $\sigma_w(T)$ is closed since $\sigma_w(T)$ is compact. Then $\lambda \in \sigma_w(T)$. This is contradiction. Hence $\lambda \notin \partial\sigma_w(T) - \sigma_e(T)$ for any $\lambda \in \mathbb{C}$. Then $\partial\sigma_w(T) - \sigma_e(T) = \phi$. Hence $\partial\sigma_w(T) \subset \sigma_e(T)$. □

Definition 3.8. ([5]) Let $T \in \mathcal{L}(X)$ be any operator. $\sigma_b(T) = \{\lambda \in \mathbb{C} : T-\lambda \notin \mathcal{B}(X)\} = \cap\{\sigma(T+K) : TK = KT,\ K \in \mathcal{K}(X)\}$ is called the *Browder spectrum* of T where $\mathcal{B}(X)$ the set of all Browder operators.

Let acc$K$ denote the set of all accumulation points of $K \subseteq \mathbb{C}$. Then $\sigma_b(T) = \sigma_e(T) \cup \text{acc}\,\sigma(T)$ ([3],[4]).

For example, let S_r be the unilateral shift operator on l^2. Then $\sigma_e(S_r) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, $\sigma_w(S_r) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\sigma_b(S_r) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

Theorem 3.9. For any $T \in \mathcal{L}(X)$, we have the following properties:

(1) $\sigma_w(T) \subseteq \sigma_b(T)$, and hence $\sigma_b(T) \neq \phi$.

(2) $\sigma_b(T)$ is a compact subset of $\sigma(T)$.

(3) $\sigma_b(T+K) = \sigma_b(T)$ for any $K \in \mathcal{K}(X)$ with $TK = KT$.

Proof. (1) If $\lambda \notin \sigma_b(T)$, then $T-\lambda \in \mathcal{B}(X)$, i.e., $T-\lambda \in \Phi(X)$ and $a(T-\lambda) = d(T-\lambda) < \infty$ where $a(T)$= ascent T and $d(T)$= descent T. Say $a(T-\lambda) = d(T-\lambda) = n$. Thus

$$\begin{aligned} 0 &\leq i(T-\lambda) = \dim N(T-\lambda) - \dim(X/R(T-\lambda)) \\ &\leq \dim N(T-\lambda)^n - \dim(X/R(T-\lambda)^n) = 0 \end{aligned}$$

and so $T-\lambda \in \Phi_0(X)$. Hence $\lambda \notin \sigma_w(T)$.

(2) Since $\sigma_b(T) = \sigma_e(T) \cup \mathrm{acc}\sigma(T)$, $\sigma_e(T)$ is closed and $\mathrm{acc}\sigma(T)$ is closed. Hence $\sigma_b(T)$ is also closed. And since $\sigma_b(T) \subset \sigma(T)$, $\sigma_b(T)$ is bounded. Thus $\sigma_b(T)$ is a compact subset of $\sigma(T)$.

(3) If $\lambda \notin \sigma_b(T)$, then $T-\lambda \in \mathcal{B}(X)$ and

$$(T-\lambda)K = TK - \lambda K = KT - K\lambda = K(T-\lambda)$$

for any $K \in \mathcal{K}(X)$ with $TK = KT$. Thus by Theorem 2.24, $(T-\lambda)+K \in \mathcal{B}(X)$, i.e., $(T+K)-\lambda \in \mathcal{B}(X)$. Thus $\lambda \notin \sigma_b(T+K)$. Hence $\sigma_b(T+K) \subseteq \sigma_b(T)$.

If $\lambda \notin \sigma_b(T+K)$ for any $K \in \mathcal{K}(X)$ with $TK = KT$, then $(T+K)-\lambda \in \mathcal{B}(X)$ and

$$\begin{aligned} \{(T+K)-\lambda\}(-K) &= (T+K)(-K) - \lambda(-K) = -TK - KK + \lambda K \\ &= -KT + (-K)K - (-K)\lambda = (-K)(T+K-\lambda). \end{aligned}$$

Thus by Theorem 2.24, $(T+K-\lambda)+(-K) \in \mathcal{B}(X)$, i.e., $T-\lambda \in \mathcal{B}(X)$. Thus $\lambda \notin \sigma_b(T)$. Hence $\sigma_b(T) \subseteq \sigma_b(T+K)$. Therefore $\sigma_b(T+K) = \sigma_b(T)$. □

A mapping p, defined on $\mathcal{L}(X)$, whose values are compact subsets of $\mathbb{C}$, is said to be *upper* (respectively *lower*) *semi-continuous at* T, provided that if $T_n \to T$ then $\limsup p(T_n) \subset p(T)$(respectively $p(T) \subset \liminf p(T_n)$). If p is both upper and lower semi-continuous at T, then it is said to be *continuous at* T and in this case $\lim p(T_n) = p(T)$ ([11]).

Theorem 3.10. Let $T \in \mathcal{L}(X)$ be any operator.

(1) The mapping $T \to \sigma(T)$ is upper semi-continuous ([8]).

(2) The mapping $T \to \sigma_e(T)$ is upper semi-continuous.

(3) The mapping $T \to \sigma_w(T)$ is upper semi-continuous.

(4) The mapping $T \to \sigma_b(T)$ is upper semi-continuous.

Proof. (1) Let A be the set of all singular operators(=non-invertible operators) and let $\varphi(\lambda) = d(T-\lambda, A)$ for any $T \in \mathcal{L}(X)$. Then φ is continuous. If Λ_0 is an open set containing $\sigma(T)$, if $\Delta = B_{1+\|T\|}(0)$ is a closed ball with center 0 and radius $1+\|T\|$ and if $\lambda \in \Delta - \Lambda_0$, then $\lambda \notin \sigma(T)$ and so $T-\lambda$ is invertible. Then $T-\lambda \notin A$. Since A is closed, $\varphi(\lambda) > 0$. Since $\Delta - \Lambda_0 = \Delta \cap \Lambda_0^c$ is a closed subset of Δ and Δ is compact, $\Delta - \Lambda_0$ is compact. Since $\varphi(\lambda)$ is continuous on $\Delta - \Lambda_0$ and $\varphi(\lambda) > 0$ for all $\lambda \in \Delta - \Lambda_0$, there exists $\varepsilon > 0$ such that $\varphi(\lambda) \geq \varepsilon$. Suppose that $\|T-S\| < \varepsilon < 1$. If $\lambda \in \Delta - \Lambda_0$, then $\|(T-\lambda)-(S-\lambda)\| < \varepsilon \leq \varphi(\lambda) = d(T-\lambda, A)$.

Thus $S-\lambda \notin A$, i.e., $S-\lambda$ is invertible. Then $\lambda \notin \sigma(S)$, i.e., if $\lambda \in \Delta - \Lambda_0$ then $\lambda \notin \sigma(S)$. If $\lambda \in \sigma(S)$, then

$$|\lambda| \leq \|S\| = \|-S\| = \|T-S-T\| \leq \|T-S\| + \|T\| < 1 + \|T\|$$

and so $\lambda \in \Delta$. Thus $\sigma(S) \subset \Delta$. Hence $\sigma(S) \subset \Lambda_0$.

(2) Suppose that $\lambda \notin \sigma_e(T)$ then $T-\lambda \in \Phi(X)$. Then there exists $\varepsilon > 0$ such that $\|(T-\lambda)-S\| < \varepsilon \to S \in \mathcal{S}\Phi(X)$ and $i(S) = i(T-\lambda)$. Since $T-\lambda \in \Phi(X)$, $\alpha(T-\lambda) < \infty$ and $\beta(T-\lambda) < \infty$. Since $i(S) = i(T-\lambda)$, $\alpha(S)-\beta(S) = \alpha(T-\lambda)-\beta(T-\lambda)$. Then $\alpha(S) < \infty$ and $\beta(S) < \infty$. Also $R(S)$ is closed. Thus $S \in \Phi(X)$. Therefore we have shown that $\|(T-\lambda)-S\| < \varepsilon \to S \in \Phi(X)$. Since $T_n \to T$, there is $N > 0$ such that for all $n \geq N \to \|T_n - T\| < \frac{\varepsilon}{2}$. For all $\mu \in B_{\frac{\varepsilon}{2}}(\lambda)$ with $|\mu - \lambda| < \frac{\varepsilon}{2}$ and for all $n \geq N$,

$$\begin{aligned} \|(\lambda - T) - (\mu - T_n)\| &\leq \|(\lambda-\mu)I\| + \|T_n - T\| \\ &= |\lambda - \mu| + \|T_n - T\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Then $\mu - T_n \in \Phi(X)$. Thus $\mu \notin \sigma_e(T_n)$ for all $n \geq N$ for all $\mu \in B_{\frac{\varepsilon}{2}}(\lambda)$. Hence $\lambda \notin \limsup \sigma_e(T_n)$ and so $\limsup \sigma_e(T_n) \subset \sigma_e(T)$, i.e., $\sigma_e(T)$ is upper semi-continuous.

(3) Let $\lambda \notin \sigma_w(T)$ so that $T-\lambda \in \Phi_0(X)$. There exists an $\varepsilon > 0$ such that if $S \in \mathcal{L}(X)$ and $\|(T-\lambda)-S\| < \varepsilon$, then $S \in \Phi_0(X)$. Since $T_n \to T$, there exists an integer N such that for any $n \geq N$

$$\|(T_n - \lambda) - (T-\lambda)\| < \frac{\varepsilon}{2}.$$

Let V be an open $\frac{\varepsilon}{2}$-neighborhood of λ. We have for all $\mu \in V$ and $n \geq N$,

$$\begin{aligned}
\|(T_n - \mu) - (T - \lambda)\| &= \|(T_n - \mu) - (T - \lambda) + (T_n - \lambda) - (T_n - \lambda)\| \\
&\leq \|(T_n - \lambda) - (T - \lambda)\| + \|(T_n - \mu) - (T_n - \lambda)\| \\
&= \|T_n - T\| + \|\lambda - \mu\| \\
&= \|T_n - T\| + |\lambda - \mu| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Thus $T_n - \mu \in \Phi_0(X)$, i.e., $\mu \notin \sigma_w(T_n)$ for all $n \geq N$ and for all $\mu \in V$. Thus $V \cap \sigma_w(T_n) = \phi$. Hence $\lambda \notin \limsup \sigma_w(T_n)$ and so $\limsup \sigma_w(T_n) \subseteq \sigma_w(T)$.

(4) Let $T_n \to T$, we show that $\limsup \sigma_b(T_n) \subset \sigma_b(T)$. Suppose that $\lambda \notin \sigma_b(T)$. If $\lambda \notin \sigma(T)$, then $\limsup \sigma_b(T_n) \subset \limsup \sigma(T_n) \subset \sigma(T)$ and so $\lambda \notin \limsup \sigma_b(T_n)$. Let $\lambda \in \sigma(T) \backslash \sigma_b(T)$. Then $\lambda \notin \sigma_e(T)$ and $\lambda \notin \mathrm{acc}\sigma(T)$. Thus $T - \lambda \in \Phi(X)$ and λ is an isolated point of $\sigma(T)$. Then there exists $\varepsilon_1 > 0$ such that

$$\|(T - \lambda) - S\| < \varepsilon_1 \Rightarrow S \in \Phi(X)$$

Since $T_n \to T$, there is N_1 such that $n \geq N$, $\|T_n - T\| < \varepsilon_1$. Then

$$\|T_n - T\| = \|(T_n - \lambda) - (T - \lambda)\| < \varepsilon_1, \text{ for all } n \geq N_1$$

Thus $T_n - \lambda \in \Phi(X)$, for all $n \geq N_1$. Since λ is an isolated point of $\sigma(T)$, there is $\varepsilon_2 > 0$ such that $\sigma(T) \cap \{\mu : |\mu - \lambda| < \varepsilon_2\} = \{\lambda\}$. Put $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, for all μ with $|\mu - \lambda| < \varepsilon$, $\mu \notin \sigma(T)$. Then $\mu \notin \limsup \sigma(T_n) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \sigma(T_k)$ and so $\mu \notin \bigcup_{k=m}^{\infty} \sigma(T_k)$ for some m, i.e., $\mu \notin \sigma(T_k)$ for all $k \geq m$. Let $N =$

$\max\{m,\ N_1\}$. If $\lambda \notin \limsup \sigma(T_n)$, then $\lambda \notin \limsup \sigma_b(T_n)$. If $\lambda \in \limsup \sigma(T_n)$, then $\lambda \in \bigcup_{k=n}^{\infty} \sigma(T_k)$, for all n. Thus $\lambda \in \bigcup_{k=N}^{\infty} \sigma(T_k)$ and so $\lambda \in \sigma(T_{k_1})$ for some $k_1 \geq N$. And $\lambda \in \bigcup_{k=N+1}^{\infty} \sigma(T_k)$, then $\lambda \in \sigma(T_{k_2})$ for some $k_2 \geq k_1 \geq N$. There is a sequence $\{k_n\}$ such that $\lambda \in \sigma(T_{k_n})$ for all n, $k_n \geq N$. Thus $T_{k_n} - \lambda \in \Phi(X)$ and λ is an isolated point of $\sigma(T_{k_n})$ for all n. Hence $\lambda \notin \sigma_b(T_{k_n})$ for all n and so $\lambda \notin \limsup \sigma_b(T_n)$. Therefore $\limsup \sigma_b(T_n) \subset \sigma_b(T)$. □

Corollary 3.11.

(1) The spectral radius $r(T)$ is upper semi-continuous, i.e., for each $T \in \mathcal{L}(X)$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T - S\| < \delta$ implies $r(S) < r(T) + \varepsilon$.

(2) The essential spectral radius $r_e(T)$, the Weyl spectral radius $r_w(T)$ and the Browder spectral radius $r_b(T)$ are upper semi-continuous, i.e., for each $T \in \mathcal{L}(X)$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T - S\| < \delta$ implies $r_i(S) < r_i(T) + \varepsilon$ for $i = e, w, b$.

Proof. (1) Let $\varepsilon > 0$ and let $r_\varepsilon = r(T) + \varepsilon$. If $\lambda \in \sigma(T)$, then $|\lambda| \leq r(T)$ and so $|\lambda| \leq r(T) + \varepsilon = r_\varepsilon$. Thus $\lambda \in B_{r_\varepsilon}(0)$, i.e., $\sigma(T) \subset B_{r_\varepsilon}(0)$. Since $\sigma(T)$ is semi-continuous, there is $\delta > 0$ such that $\|S - T\| < \delta$ implies $\sigma(S) \subset B_{r_\varepsilon}(0)$. For all $\lambda \in \sigma(S)$, $|\lambda| < r_\varepsilon$. Thus $r(S) \leq r(T) + \varepsilon$.

(2) Let $\varepsilon > 0$ be given and let $r_\varepsilon = r_i(T) + \varepsilon > r_i(T)$ for $i = e, w, b$. For all $\lambda \in \sigma_i(T)$, $|\lambda| \leq r_i(T) < r_i(T) + \varepsilon$. Then $\lambda \in B_{r_\varepsilon}(0)$. Thus $\sigma_i(T) \subset B_{r_\varepsilon}(0)$. Since $\sigma_i(T)$ is upper semi-continuous, there is $\delta > 0$ such that $\|S - T\| < \delta$

implies $\sigma_i(S) \subset B_{r_\varepsilon}(0)$. For all $\lambda \in \sigma_i(S)$, $|\lambda| < r_\varepsilon$, i.e., $|\lambda| < r_i(T) + \varepsilon$. Thus $\sup\{|\lambda| : \lambda \in \sigma_i(S)\} \leq r_i(T) + \varepsilon$. Hence $r_i(S) \leq r_i(T) + \varepsilon$. □

Theorem 3.12. ([11]) Let $T_n \to T$. If $\lim \sigma_e(T_n) = \sigma_e(T)$, then $\lim \sigma_w(T_n) = \sigma_w(T)$.

Proof. By Theorem 3.10.(3), $\limsup \sigma_w(T_n) \subseteq \sigma_w(T)$. It is sufficient to show that $\sigma_w(T) \subseteq \liminf \sigma_w(T_n)$. Suppose that $\lambda \notin \liminf \sigma_w(T_n)$. Then there is a neighborhood V of λ that does not intersect infinitely many $\sigma_w(T_n)$. Since $\sigma_e(T_n) \subset \sigma_w(T_n)$, V does not intersect infinitely many $\sigma_e(T_n)$. Then $\lambda \notin \lim \sigma_e(T_n) = \sigma_e(T)$. Thus $T - \lambda \in \Phi(X)$. Also λ does not belong to $\sigma_w(T_n)$ for infinitely many n and so $T_n - \lambda \in \Phi_0(T)$. Then $i(T_n - \lambda) = 0$. By the index of Fredholm is continuous and $T_n \to T$, $i(T - \lambda) = 0$. Thus $T - \lambda \in \Phi_0(X)$. Hence $\lambda \notin \sigma_w(T)$ and so $\sigma_w(T) \subseteq \liminf \sigma_w(T_n)$. □

We recall that a space in which all components are one-point sets is called *totally disconnected*. A space X is totally disconnected if and only if, for any two elements x and y of X, there exist disjoint open neighborhoods U of x and V of y such that X is the union of U and V ([2]).

Corollary 3.13. ([11]) Let $T_n \to T$. Then $\lim \sigma_w(T_n) = \sigma_w(T)$ in each one of the following cases holds.

(1) $T_nT = TT_n$ for all n.

(2) $\sigma(T)$ is totally disconnected.

(3) T_n and T are normal operators.

Corollary 3.14. Let $T_n \to T$ and $\sigma_w(T_n) = \sigma_e(T_n)$ for all n. Then $\sigma_w(T) = \sigma_e(T)$ if one of the following cases holds.

(1) $T_nT = TT_n$ for all n.

(2) $\sigma(T)$ is totally disconnected.

We recall that two operators A and B are *similar* if there is an invertible operator P such that $P^{-1}AP = B$.

Theorem 3.15. Let $S \in \mathcal{L}(X)$ be similar to $T \in \mathcal{L}(X)$. Then

(1) $\sigma(T) = \sigma(S)$

(2) $\sigma_p(T) = \sigma_p(S)$

(3) $\sigma_{com}(T) = \sigma_{com}(S)$

(4) $\sigma_e(T) = \sigma_e(S)$

(5) $\sigma_w(T) = \sigma_w(S)$

(6) $\sigma_b(T) = \sigma_b(S)$

Proof. Since S and T are similar, there is an invertible operator U such that $U^{-1}TU = S$ and $T = USU^{-1}$.

(1) If $\lambda \notin \sigma(T)$, then $T - \lambda$ is invertible. Thus $S - \lambda = U^{-1}TU - \lambda = U^{-1}(T - \lambda)U$ is invertible and so $\lambda \notin \sigma(S)$. Hence $\sigma(S) \subseteq \sigma(T)$.

Similarly, we have $\sigma(T) \subseteq \sigma(S)$. Therefore $\sigma(T) = \sigma(S)$.

(2) If $\lambda \in \sigma_p(T)$, then $Tx = \lambda x$ for any nonzero vector $x \in X$. Then $S(U^{-1}x) = U^{-1}TU(U^{-1}x) = U^{-1}(Tx) = U^{-1}(\lambda x) = \lambda(U^{-1}x)$. Thus $Sy = \lambda y$ for any nonzero vector $y \in X$, i.e., $\lambda \in \sigma_p(S)$. Hence $\sigma_p(T) \subseteq \sigma_p(S)$.

Similarly, we have $\sigma_p(S) \subseteq \sigma_p(T)$. Therefore $\sigma_p(T) = \sigma_p(S)$.

(3) If $\lambda \in \sigma_{com}(T)$, then $\overline{R(T-\lambda)} \neq X$. Then there is $x \in X$ such that $x \notin \overline{R(T-\lambda)}$. And $R(S-\lambda) = R(U^{-1}TU-\lambda) = R(U^{-1}(T-\lambda)U) \subseteq R(T-\lambda)$. Then $x \notin \overline{R(S-\lambda)}$. Thus $\overline{R(S-\lambda)} \neq X$. So $\lambda \in \sigma_{com}(S)$. Hence $\sigma_{com}(T) \subseteq \sigma_{com}(S)$.

Similarly, we have $\sigma_{com}(S) \subseteq \sigma_{com}(T)$. Therefore $\sigma_{com}(S) = \sigma_{com}(T)$.

(4) If $\lambda \notin \sigma_e(T)$, then $T - \lambda \in \Phi(X)$ and so $\pi(T) - \lambda$ is invertible. Then $\pi(S-\lambda) = \pi(U^{-1}TU - \lambda) = \pi(U^{-1}(T-\lambda)U) = \pi(U^{-1})\{\pi(T)-\lambda\}\pi(U)$ is invertible. Thus $S - \lambda \in \Phi(X)$ and so $\lambda \notin \sigma_e(S)$. Hence $\sigma_e(S) \subseteq \sigma_e(T)$.

Similarly, we have $\sigma_e(T) \subseteq \sigma_e(S)$. Therefore $\sigma_e(T) = \sigma_e(S)$.

(5) If $\lambda \notin \sigma_w(T)$, then $T - \lambda \in \Phi_0(X)$. There is an invertible operator A and a compact operator B such that $T - \lambda = A + B$ by Corollary 2.20. Then $S - \lambda = U^{-1}TU - \lambda = U^{-1}(T-\lambda)U = U^{-1}(A+B)U = U^{-1}AU + U^{-1}BU$. Here $U^{-1}AU$ is invertible and $U^{-1}BU$ is compact. Then $S - \lambda \in \Phi_0(X)$. Thus $\lambda \notin \sigma_w(S)$. Hence $\sigma_w(S) \subseteq \sigma_w(T)$.

Similarly, we have $\sigma_w(T) \subseteq \sigma_w(S)$. Therefore $\sigma_w(T) = \sigma_w(S)$.

(6) By (1) and (4), $\sigma_e(T) = \sigma_e(S)$ and $\sigma(T) = \sigma(S)$. Then $\sigma_b(T) = \sigma_e(T) \cup \text{acc}\,\sigma(T) = \sigma_e(S) \cup \text{acc}\,\sigma(S) = \sigma_b(S)$. □

From the above theorem, several spectra are invariant under similarity. Moreover, the index of Fredholm is also invariant under similarity. For if there is an invertible operator U such that $T = USU^{-1}$ then $i(T) = i(USU^{-1}) =$

$i(U)+i(S)+i(U^{-1})=i(S)$ since $i(TS)=i(T)+i(S)$ and the index of invertible is zero.

Theorem 3.16. ([2])(the spectral mapping theorem) If $T \in \mathcal{L}(X)$ and p is any polynomial, then $\sigma(p(T)) = p(\sigma(T))$.

Proof. Let $\lambda \in \sigma(T)$. Then we show that $p(\lambda) \in \sigma(p(T))$, i.e., $p(T) - p(\lambda)$ is not invertible. We may write $p(T)-p(\lambda) = (T-\lambda)q(T)$ where q is a polynomial. If $p(\lambda) \notin \sigma(p(T))$, then $p(T) - p(\lambda)$ is invertible. Then

$$I = (p(T) - p(\lambda))^{-1}(T-\lambda)q(T).$$

Thus $T-\lambda$ is invertible. This is a contradiction to the fact that $\lambda \in \sigma(T)$. Hence $p(\lambda) \in \sigma(p(T))$, i.e., $p(\sigma(T)) \subseteq \sigma(p(T))$.

Conversely, if $\mu \notin p(\sigma(T))$, then $p(\sigma(T)) - \mu \neq 0$. Then

$$q(T) = (p(T) - \mu)^{-1}$$

is polynomial and so $q(T)\{p(T) - \mu\} = I$, i.e., $p(T) - \mu$ is invertible. Thus $\mu \notin \sigma(p(T))$. Hence $\sigma(p(T)) \subseteq p(\sigma(T))$. □

Theorem 3.17. For any operator T and for all polynomial p, $\sigma_w(p(T))$ is a subset of $p(\sigma_w(T))$, i.e., $\sigma_w(p(T)) \subset p(\sigma_w(T))$.

Proof. Let $\mu \notin p(\sigma_w(T))$ and $p(\lambda) - \mu = a(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$. Then $p(T) - \mu I = a(T-\lambda_1)(T-\lambda_2)\cdots(T-\lambda_n)$ and $p(\lambda_j) - \mu = 0$ for $j = 1, 2, \cdots, n$. Thus $\mu = p(\lambda_j) \notin p(\sigma_w(T))$ and so $\lambda_j \notin \sigma_w(T)$. Thus $T - \lambda_j \in \Phi_0(X)$ for $j = 1, 2, \cdots, n$. By Theorem 2.8, $(T-\lambda_1)(T-\lambda_2)\cdots(T-\lambda_n) \in \Phi_0(X)$ and so $p(T) - \mu I \in \Phi_0(T)$. Thus $\mu \notin \sigma_w(p(T))$. Hence $\sigma_w(p(T)) \subset p(\sigma_w(T))$. □

Remark 3.2. We give the following example of an operator T such that $\sigma_w(p(T)) \neq p(\sigma_w(T))$.

For example, let H be a Hilbert space, let $T = U \oplus (U^* + 2I)$ where U is the unilateral shift operator and let $p(\lambda) = \lambda(\lambda - 2)$. Then $R(U) = \{(0, x_1, x_1, \cdots) : (x_1, x_2, x_3, \cdots) \in l^2\}$ and so $R(U)$ is closed. Then $N(U) = \{0\}$ and $X/R(U) = \{(x, 0, 0, \cdots) : x \in \mathbb{C}\}$. Thus $i(U) = 0 - 1 = -1$. For all $x = \{x_n\}, y = \{y_n\}$ in H,

$$\begin{aligned} < Ux, y > &= < (0, x_1, x_1, \cdots),\ (y_1, y_2, y_3, \cdots) >= x_1\overline{y_2} + x_2\overline{y_3} + \cdots \\ &= < (x_1, x_2, x_3, \cdots), (y_2, y_3, y_4, \cdots) >=< x, U^*y > . \end{aligned}$$

Thus $U^*(y_1, y_2, \cdots) = (y_2, y_3, \cdots)$. Also $R(U^*) = l^2,\ N(U^*) = \{(x, 0, 0, \cdots) : x \in \mathbb{C}\}$ and $X/R(U^*) = \{0\}$. Thus $i(U^*) = 1 - 0 = 1$. Since $-2 \notin \sigma(U^*)$ and $2 \notin \sigma(U)$, $U^* + 2I$ and $U - 2I$ are invertible and so $i(U^* + 2I) = 0 = i(U - 2I)$. Since $p(\lambda) = \lambda(\lambda - 2)$,

$$\begin{aligned} p(T) &= T(T - 2I) = [U \oplus (U^* + 2I)][U \oplus (U^* + 2I) - 2I] \\ &= [U \oplus (U^* + 2I)][(U - 2I) \oplus U^*]. \end{aligned}$$

Note that $i(T \oplus S) = i(T) + i(S)$ and $i(TS) = i(T) + i(S)$. Thus $i(T) = i(U \oplus (U^* + 2I)) = -1 + 0 = -1$, $i(T - 2I) = i((U - 2I) \oplus U^*) = 0 + 1 = 1$. Then $i(p(T)) = i(T(T - 2I)) = i(T) + i(T - 2I) = -1 + 1 = 0$. Thus $p(T) \in \Phi_0(X)$ and so $0 \notin \sigma_w(p(T))$. Since $i(T) = -1$, $T \notin \Phi_0(X)$. Then $0 \in \sigma_w(T)$. Hence $0 \in p(\sigma_w(T))$.

Theorem 3.18. If T is a normal operator. Then $\sigma_w(f(T)) = f(\sigma_w(T))$ for every continuous complex-valued function f on $\sigma(T)$.

Proof. If T is normal, then $\widehat{T}$ is also normal in $\mathcal{L}(X)/\mathcal{K}(X)$. Because $\widehat{T}\widehat{T}^* = \pi(T)\pi(T)^* = \pi(T)\pi(T^*) = \pi(TT^*) = \pi(T^*T) = \pi(T^*)\pi(T) = \pi(T)^*\pi(T) = \widehat{T}^*\widehat{T}$. By the standard C^*-algebra theory, $f(\widehat{T}) = \widehat{f(T)}$. Since T is normal, $f(T)$ is normal. Note that if S is normal, then $\sigma_w(S) = \sigma_e(S)$. We have $\sigma_w(S) = \sigma_e(S) = \sigma(\pi(S)) = \sigma(\widehat{S})$, i.e., $\sigma_w(S) = \sigma(\widehat{S})$. Hence $\sigma_w(f(T)) = \sigma(\widehat{f(T)}) = \sigma(f(\widehat{T})) = f(\sigma(\widehat{T})) = f(\sigma_w(T))$. □

Theorem 3.19. Let $T \in \mathcal{L}(X)$ be any operator. Then for any polynomial p, $p(\sigma_b(T)) = \sigma_b(p(T))$.

Proof. Let $\mu \in \sigma_b(p(T))$.

Case I. μ is not an isolated point of $\sigma(p(T)) = p(\sigma(T))$. Then there is a sequence $\{\lambda_n\}$ in $\sigma(T)$ such that $\mu = \lim p(\lambda_n)$. Since $\sigma(T)$ is a compact subset of $\mathbb{C}$, $\{\lambda_n\}$ has a convergent subsequence, say $\{\lambda_{n_k}\}$. Let $\lim \lambda_{n_k} = \lambda$. Then $\lambda \in \sigma_b(T)$. Since $p(\lambda) = p(\lim \lambda_{n_k}) = \lim p(\lambda_{n_k}) = \mu$, $p(\lambda) \in p(\sigma_b(T))$. Thus $\mu \in p(\sigma_b(T))$.

Case II. μ is an isolated point of $\sigma(p(T)) = p(\sigma(T))$. Then $\mu \in \sigma_e(p(T))$ by $\sigma_b(T) = \sigma_e(T) \cup \mathrm{acc}\sigma(T)$, i.e., $p(T) - \mu = (T - \lambda_1)(T - \lambda_2) \cdots (T - \lambda_n) \notin \Phi(X)$. Then $T - \lambda_k \notin \Phi(X)$ for some k. Thus $\lambda_k \in \sigma_e(T)$ and so $\mu = p(\lambda_k) \in p(\sigma_e(T)) \subset p(\sigma_b(T))$. Hence $\sigma_b(p(T)) \subset p(\sigma_b(T))$. Let $\lambda \in \sigma_b(T)$. If λ is not an isolated point of $\sigma(T)$, then $p(\lambda)$ is also not an isolated point of $\sigma(p(T))$. Thus $p(\lambda) \in \sigma_b(p(T))$. If λ is an isolated point of $\sigma(T)$, then $\lambda \in \sigma_e(T)$. Thus $T - \lambda \notin \Phi(X)$ and so $p(T) - p(\lambda I) = (T - \lambda_1)(T - \lambda_2) \cdots (T - \lambda_n) \notin \Phi(X)$. Note that if $TS = ST$ and $T \notin \Phi(X)$, then $TS \notin \Phi(X)$. Hence $p(\lambda) \in \sigma_e(p(T)) \subset \sigma_b(p(T))$. □

4 Weyl's theorem, Browder's theorem, a-Weyl's theorem and a-Browder's theorem

Definition 4.1. Let $T \in \mathcal{L}(X)$ be any operator.

(1) If $\pi_{00}(T) = \sigma(T) \backslash \sigma_w(T)$, then we say that *Weyl's theorem holds for T* where $\pi_{00}(T)$ denotes the isolated points of $\sigma(T)$ that are eigenvalue of finite multiplicity.

(2) If $p_{00}(T) = \sigma(T) \backslash \sigma_w(T)$ then we say that *Browder's theorem holds for T* where $p_{00}(T) = \sigma(T) \backslash \sigma_b(T)$ is the set of Riesz points of T.

Theorem 4.2. ([12]) Let $T \in \mathcal{L}(X)$ be any operator. Then for any polynomial p, we have $\sigma(p(T)) \backslash \pi_{00}(p(T)) \subset p(\sigma(T) \backslash \pi_{00}(T))$.

Proof. Let $\lambda \in \sigma(p(T)) \backslash \pi_{00}(p(T)) = p(\sigma(T)) \backslash \pi_{00}(p(T))$.

Case I. λ is not an isolated point of $p(\sigma(T))$. Then there is a sequence $\{\lambda_n\}$ in $p(\sigma(T))$ such that $\lambda_n \to \lambda$, and so there is a sequence $\{\mu_n\}$ in $\sigma(T)$ such that $p(\mu_n) = \lambda_n \to \lambda$, i.e., $\lim p(\mu_n) = \lambda$. Then $\{p(\mu_n)\}$ is bounded and so $\{\mu_n\}$ is bounded. Thus $\{\mu_n\}$ has a convergent subsequence, say $\{\mu_n\}$. Let $\lim \mu_n = \mu_0$. Then $p(\mu_0) = p(\lim \mu_n) = \lim p(\mu_n) = \lambda$. Since $\sigma(T)$ is closed, $\mu_0 \in \sigma(T)$. Since $\mu_0 \in \sigma(T) \backslash \pi_{00}(T)$, $\lambda \in p(\sigma(T) \backslash \pi_{00}(T))$.

Case II. λ is an isolated point of $\sigma(p(T))$. Since $\lambda \notin \pi_{00}(p(T))$, either λ is not an eigenvalue of $p(T)$ or it is an eigenvalue of infinite multiplicity. Let $p(T) - \lambda I = a_0(T - \mu_1 I)(T - \mu_2 I) \cdots (T - \mu_n I)$. If λ is not an eigenvalue of $p(T)$, then none of $\mu_1, \mu_2, \cdots, \mu_n$ can be an eigenvalue of T. If $\mu_i \notin \sigma(T)$ for all i, then $T - \mu_i$ is invertible and so $p(T) - \lambda I$ is invertible. This is contradiction to the fact that $\lambda \in \sigma(p(T))$. Thus for some k, $\mu_k \in \sigma(T)$, μ_k is not an eigenvalue of T and $\mu_k \in \sigma(T)\backslash\pi_{00}(T)$. Since $p(\mu_k) - \lambda = 0$, $\lambda = p(\mu_k) \in p(\sigma(T)\backslash\pi_{00}(T))$. If λ is an eigenvalue of infinite multiplicity, then $N(p(T) - \lambda) = \bigcup_{k=1}^{\infty} N(T - \mu_k)$. Since $\dim N(p(T) - \lambda) = \infty$, $\dim N(T - \mu_k) = \infty$ for some k. Thus μ_k is an eigenvalue of T with infinite multiplicity. Hence $\mu_k \in \sigma(T)\backslash\pi_{00}(T)$ and $\lambda = p(\mu_k) \in p(\sigma(T)\backslash\pi_{00}(T))$. □

Recall that an operator $T \in \mathcal{L}(X)$ is said to be an *isoloid* if isolated points of $\sigma(T)$ are eigenvalues of T.

Theorem 4.3. ([12]) If T is isoloid, then $\sigma(p(T))\backslash\pi_{00}(p(T)) = p(\sigma(T)\backslash\pi_{00}(T))$ for any polynomial p.

Proof. We show that $p(\sigma(T)\backslash\pi_{00}(T)) \subset \sigma(p(T))\backslash\pi_{00}(p(T))$.

Let $\lambda \in p(\sigma(T)\backslash\pi_{00}(T))$. Then there is $\mu \in \sigma(T)\backslash\pi_{00}(T)$ such that $\lambda = p(\mu)$ and $\lambda \in \sigma(p(T))$. Suppose that $\lambda \in \pi_{00}(p(T))$, i.e., λ is an isolated point of $\sigma(p(T))$ and an eigenvalue of $p(T)$ of infinite multiplicity. Let $p(T) - \lambda I = a_0(T - \mu_1 I)(T - \mu_2 I) \cdots (T - \mu_n I)$. Then $\mu = \mu_k$ for some k. Since $\lambda = p(\mu)$ and $\mu \in \sigma(T)\backslash\pi_{00}(T)$, $\lambda = p(\mu_k)$ where $\mu_k \in \sigma(T)\backslash\pi_{00}(T)$. Thus μ_k is an isolated point of $\sigma(T)$. Hence μ_k is an eigenvalue since T is isoloid. Since $N(T - \mu_k) \subset N(p(T) - \lambda)$ and $\dim N(p(T) - \lambda) < \infty$, $\dim N(T - \mu_k) < \infty$. Thus $\mu_k \in \pi_{00}(T)$. This is contradiction to the fact that $\lambda = p(\mu_k) \in p(\sigma(T)\backslash\pi_{00}(T))$. Hence $\lambda = p(\mu_k) \notin \pi_{00}(p(T))$ and $\lambda \in \sigma(p(T))\backslash\pi_{00}(p(T))$. □

Theorem 4.4. ([12]) Let T be an isoloid operator and let Weyl's theorem holds for T. Then for any polynomial p, Weyl's theorem holds for $p(T)$ if and only if $p(\sigma_w(T)) = \sigma_w(p(T))$.

Proof. Suppose that Weyl's theorem holds for $p(T)$. Then by hypothesis $\sigma_w(T) = \sigma(T)\backslash\pi_{00}(T)$ and $\sigma_w(p(T)) = \sigma(p(T))\backslash\pi_{00}(p(T))$. Thus $\sigma_w(p(T)) = \sigma(p(T))\backslash\pi_{00}(p(T)) = p(\sigma(T)\backslash\pi_{00}(T)) = p(\sigma_w(T))$ by Theorem 4.3. Hence $\sigma_w(p(T)) = p(\sigma_w(T))$.

Conversely, if Weyl's theorem holds for T, then $\sigma_w(T) = \sigma(T)\backslash\pi_{00}(T)$. Suppose that $p(\sigma_w(T)) = \sigma_w(p(T))$ for any polynomial. Then $\sigma_w(p(T)) = p(\sigma_w(T))$ $= p(\sigma(T)\backslash\pi_{00}(T)) = \sigma(p(T))\backslash\pi_{00}(p(T))$ by Theorem 4.3. Thus $\sigma_w(p(T)) =$ $\sigma(p(T))\backslash\pi_{00}(p(T))$. Hence Weyl's theorem holds for $p(T)$. □

Theorem 4.5. Let $T \in \mathcal{L}(X)$ be such that for any polynomial p then $p(\sigma_w(T)) = \sigma_w(p(T))$. Then if f is a holomorphic function defined in a neighborhood of $\sigma(T)$, then $f(\sigma_w(T)) = \sigma_w(f(T))$.

Proof. By Runge's theorem, let p_n be a sequence of polynomial converging uniformly in a neighborhood of $\sigma(T)$ to f so that $p_n(T) \to f(T)$, i.e., $\lim_{n\to\infty} p_n(T) = f(T)$. By Theorem 3.12,

$$\sigma_w(f(T)) = \sigma_w(\lim p_n(T)) = \lim \sigma_w(p_n(T)) = \lim p_n(\sigma_w(T)) = f(\sigma_w(T)).$$

The proof is complete. □

Theorem 4.6. Let $T \in \mathcal{L}(X)$ be an operator. If f is a holomorphic function defined in a neighborhood of $\sigma(T)$, then $f(\sigma_b(T)) = \sigma_b(f(T))$.

Proof. By Theorem 3.19, $p(\sigma_b(T)) = \sigma_b(p(T))$ for any polynomial p. By Runge's theorem, let p_n be a sequence of polynomial converging uniformly in a neighborhood of $\sigma(T)$ to f so that $p_n(T) \to f(T)$. Then

$$\sigma_b(f(T)) = \sigma_b(\lim p_n(T)) = \lim \sigma_b(p_n(T)) = \lim p_n(\sigma_b(T)) = f(\sigma_b(T)).$$

The proof is complete. □

Definition 4.7. Let $\Phi_+^-(X) = \{T \in \mathcal{L}(X) : T \in \Phi_+(X) \text{ and } i(T) \leq 0\}$. Then $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+^-(X)\} = \cap\{\sigma_{ap}(T + K) : K \in \mathcal{K}(X)\}$ is called the *essential approximate point spectrum* and $\pi_{00}^a(T) = \{\lambda \in \text{iso}\sigma_{ap}(T) : 0 < \alpha(T - \lambda) < \infty\}$ is called the set of eigenvalues of finite multiplicity which are isolated in $\sigma_{ap}(T)$. Let $\sigma_{ab}(T) = \cap\{\sigma_{ap}(T + K) : TK = KT \text{ and } K \in \mathcal{K}(X)\}$ be the *Browder essential approximate point spectrum.*

Clearly $\sigma_{ea}(T) \subseteq \sigma_{ab}(T)$ by the definition. In fact, $\sigma_{ab}(T) = \sigma_{ea}(T) \cup \text{acc}\sigma_{ap}(T)$([14]).

Definition 4.8. If $\sigma_{ea}(T) = \sigma_{ap}(T) \backslash \pi_{00}^a(T)$ then we say that *a-Weyl's theorem holds for* $T \in \mathcal{L}(X)$. If $\sigma_{ea}(T) = \sigma_{ab}(T)$ then we say that *a-Browder's theorem holds for* $T \in \mathcal{L}(X)$.

Theorem 4.9.

(1) a-Weyl's theorem $\Longrightarrow$ Weyl's theorem $\Longrightarrow$ Browder's theorem.

(2) a-Weyl's theorem $\Longrightarrow$ a-Browder's theorem $\Longrightarrow$ Browder's theorem.

Proof. (1)(i) If T does not hold Weyl's theorem, then $\pi_{00}(T) \subsetneq \sigma(T)\backslash\sigma_w(T)$ and then there exists $\lambda \in \sigma(T)$ such that $\lambda \notin (\pi_{00}(T) \cup \sigma_w(T))$. Thus $\lambda \notin \sigma_w(T)$ and $\lambda \notin \pi_{00}(T)$. We have $\lambda \notin \sigma_{ea}(T)$ and $\lambda \notin \pi_{00}^a(T)$ since $\sigma_{ea}(T) \subseteq \sigma_w(T)$ and $\pi_{00}^a(T) \subseteq \pi_{00}(T)$. Since $\lambda \notin \pi_{00}^a(T)$, $\lambda \in \text{acc}\sigma_{ap}(T)$. By Theorem 3.2(3) $\sigma_{ap}(T)$ is closed, $\lambda \in \sigma_{ap}(T)$. This is a contradiction to the fact that $\sigma_{ap}(T)\backslash\sigma_{ea}(T) = \pi_{00}^a(T)$. Thus if a-Weyl's theorem holds for T, then T obeys Weyl's theorem.

(ii) If T does not hold Browder's theorem, then $p_{00}(T) = \sigma(T)\backslash\sigma_b(T) \neq \sigma(T)\backslash\sigma_w(T)$, i.e., $\sigma_b(T) \neq \sigma_w(T)$. We can take $\lambda \in \sigma_b(T)\backslash\sigma_w(T)$. Since $\sigma_b(T) = \sigma_e(T) \cup \text{acc}\sigma(T)$, $\lambda \in \text{acc}\sigma(T)$. But since T obeys Weyl's theorem and $\lambda \notin \sigma_w(T)$. Then $\lambda \in \pi_{00}(T)$. This is contradiction. Thus if Weyl's theorem holds for T, then T obeys Browder's theorem.

(2)(i) If T holds a-Weyl's theorem, then $\sigma_{ea}(T) = \sigma_{ap}(T)\backslash\ \pi_{00}^a(T)$. Since $\text{acc}\ \sigma_{ap}(T) \cap \pi_{00}^a(T) = \phi$, $\text{acc}\sigma_{ap}(T) \subseteq \sigma_{ea}(T)$. Then $\sigma_{ab}(T) = \sigma_{ea}(T) \cup$ acc $\sigma_{ap}(T)$ $= \sigma_{ea}(T)$. Hence if a-Weyl's theorem holds for T, then T obeys a-Browder's theorem.

(ii) If T does not hold Browder's theorem, then $p_{00}(T) = \sigma(T)\backslash\sigma_b(T) \neq \sigma(T)\backslash\sigma_w(T)$ and then $\sigma_b(T) \neq \sigma_w(T)$, i.e., there exists $\lambda \in \mathbb{C}$ such that $\lambda \in \sigma_b(T)\backslash\sigma_w(T)$. Then $T - \lambda \in \Phi_0(X)$ and $a(T - \lambda) = \infty$, it follows from ([14]) that $\lambda \in \sigma_{ab}(T)$. Since a-Browder's theorem holds for T, $\sigma_{ea}(T) = \sigma_{ab}(T)$. Then $\lambda \in \sigma_{ea}(T) \subset \sigma_{ab}(T)$. This is contradiction to $\lambda \notin \sigma_w(T)$. Thus if a-Browder's theorem holds for T, then T obeys Browder's theorem. □

Theorem 4.10. ([9]) If Browder's theorem holds for $T \in \mathcal{L}(X)$ and if p is a polynomial. Then Browder's theorem holds for $p(T)$ if and only if $p(\sigma_w(T)) \subseteq$

$\sigma_w(p(T))$.

Proof. If Browder's theorem holds for $p(T)$, then $\sigma_b(p(T)) = \sigma_w(p(T))$, i.e., $\sigma_b(p(T)) \subseteq \sigma_w(p(T))$. Since Browder's theorem holds for T, $\sigma_b(T) = \sigma_w(T)$. Thus by Theorem 3.19, $p(\sigma_w(T)) \subseteq p(\sigma_b(T)) = \sigma_b(p(T)) \subseteq \sigma_w(p(T))$.

Conversely, if $p(\sigma_w(T)) \subseteq \sigma_w(p(T))$ then by hypothesis and Theorem 3.19, $\sigma_b(p(T)) = p(\sigma_b(T)) \subseteq p(\sigma_w(T)) \subseteq \sigma_w(p(T))$. Since $\sigma_w(p(T)) \subset \sigma_b(p(T))$, $\sigma_b(p(T)) = \sigma_w(p(T))$. □

From Theorem 3.17 and Theorem 4.10, if Browder's theorem holds for $T \in \mathcal{L}(X)$ and if p is a polynomial. Then Browder's theorem holds for $p(T)$ if and only if $p(\sigma_w(T)) = \sigma_w(p(T))$.

Theorem 4.11. ([9]) Browder's theorem holds for T if and only if $\mathrm{acc}\sigma(T) \subseteq \sigma_w(T)$.

Proof. Suppose that $\mathrm{acc}\sigma(T) \subseteq \sigma_w(T)$, then $\sigma(T)\backslash\sigma_w(T) \subseteq \mathrm{iso}\sigma(T)$ and then $\sigma(T)\backslash\sigma_w(T) \subseteq \mathrm{iso}\sigma(T)\backslash\sigma_e(T) = p_{00}(T)$. Since $p_{00}(T) = \sigma(T)\backslash\sigma_b(T) \subseteq \sigma(T)\backslash\sigma_w(T)$, $p_{00}(T) = \sigma(T)\backslash\sigma_w(T)$. Hence Browder's theorem holds for T.

Conversely, if Browder's theorem holds for T, then $\sigma_b(T) = \sigma_w(T)$. Since $\sigma_b(T) = \sigma_e(T) \cup \mathrm{acc}\sigma(T)$, $\mathrm{acc}\sigma(T) \subset \sigma_b(T) = \sigma_w(T)$. □

Theorem 4.12. Necessary and sufficient for Weyl's theorem is Browder's theorem together with either of the following:

(1) $\sigma_w(T) \cap \pi_{00}(T) = \phi$;

(2) $\pi_{00}(T) \subseteq p_{00}(T)$.

Proof. Notice that (2) always implies (1). First we show that Browder's theorem together with (1) implies Weyl's theorem. Since $p_{00}(T) = \sigma(T)\setminus\sigma_b(T) = \text{iso}\sigma(T)\setminus\sigma_e(T) \subseteq \pi_{00}(T)$ and Browder's theorem holds for T, $\sigma(T)\setminus\sigma_w(T) \subseteq \pi_{00}(T)$. By (1), $\pi_{00}(T) \subseteq \sigma(T)\setminus\sigma_w(T)$. Then $\sigma(T)\setminus\sigma_w(T) = \pi_{00}(T)$, i.e., Weyl's theorem holds for T. Second we show that Weyl's theorem implies (2). If $\lambda \in \pi_{00}(T)$, then $\lambda \in \text{iso}\sigma(T)$ and $\lambda \notin \sigma_w(T)$ by Weyl's theorem. Then $\lambda \in \text{iso}\sigma(T)\setminus\sigma_w(T) \subseteq \text{iso}\sigma(T)\setminus\sigma_e(T) = p_{00}(T)$. □

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<국문 초록>

바나하 공간위에서 선형작용소의 바일과 브라우더 스펙트럼에 관한 연구

본 논문에서는 무한차원의 바나하(Banach) 공간에서 유계 선형 작용소인 바일(Weyl)작용소, 브라우더(Browder)작용소와 그들의 스펙트럼들에 대한 여러 가지 성질을 연구하고 바일(a-바일) 정리와 브라우더(a-브라우더) 정리의 체계적인 관계를 조사하였으며, 주요 연구 결과는 다음과 같다.

(1) 바일과 브라우더 작용소의 집합은 열린집합이고 컴펙트 섭동(compact perturbation)하에서 안정적이다. 또한 바일 작용소이기위한 동치조건과 브라우더 작용소이기위한 동치조건도 제시했다.

(2) 진성, 바일, 브라우더 스펙트럼은 위로 반연속(upper semi-continuous) 함수이고 그들의 스펙트럼 반경도 위로 반연속함수이다. 또한 이 스펙트럼들은 닮음에 대하여 불변하다.

(3) 유계 선형 작용소들의 브라우더 스펙트럼에 대해 스펙트럼 사상정리가 만족한다. 또한 이 결과를 다음과 같이 확장했다. 만약 T가 선형 작용소이고 f가 T의 스펙트럼근방에서 정의된 정칙함수(holomorphic function) 일 때 $f(\sigma_b(T)) = \sigma_b(f(T))$이다.

(4) 유계 선형 작용소에 대해 a-바일 정리가 성립하면 바일 정리가 성립하고, 바일 정리가 성립하면 브라우더 정리가 성립한다. 또한 유계 선형 작용소에 대해 a-바일 정리가 성립하면 a-브라우더 정리가 성립하며, a-브라우더 정리가 성립하면 브라우더 정리가 성립한다.

감사의 글

이 논문이 나오기까지 많은 분들의 격려와 애정 어린 관심에 대한 고마움을 잊을 수 없습니다. 짧으나마 이 지면을 통해 깊은 감사의 마음을 전하고 싶습니다.

우선 무척 바쁘신 가운데 없는 시간을 쪼개어가며 부족한 저를 위해 세심한 지도와 격려로 여기까지 이끌어주신 양영오 교수님께 깊은 감사를 드립니다. 또한 저의 건강과 빠른 쾌유를 위해 기도해주신 송석준, 방은숙 교수님께도 감사드리고 대학원 2년 동안 더 많은 지식과 지혜를 쌓을 수 있도록 도와주신 정승달, 윤용식, 유상욱 교수님께도 감사드립니다. 그리고 논문을 쓰는 동안 조언을 주신 강경태 선생님께도 감사의 말을 전합니다. 그리고 언제나 나의 힘이 되어 곁에서 지켜봐주는 친구들과 선배님들 후배들에게도 고마움을 전하고 싶고 힘든 시기가 닥칠 때마다 격려와 사랑으로 나를 있게 해주신 부모님과 가족들께도 사랑의 말을 전하고 싶습니다. 앞으로도 이 모든 사람들의 기대에 어긋나지 않는 사람이 되도록 노력하겠습니다.

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