碩士學位論文

Zero-term Rank Preservers Of Nonnegative Integer Matrices

濟州大學校 大學院

數 學 科

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2002年 12月

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이 論文을 理學 碩士學位 論文으로 提出함

2002年 12月

韓希政의 理學 碩士學位 論文을 認准함

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2002年 12月

Zero-term Rank Preservers Of Nonnegative Integer Matrices

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL
CHEJU NATIONAL UNIVERSITY

December 2002

CONTENTS

< **Abstract** >

ZERO-TERM RANK PRESERVERS OF NONNEGATIVE INTEGER MATRICES

There are many papers on the ranks of matrices and their preservers. They gave us the motivation to research on the zero-term rank of matrices and its preserver. Recently, Beasley, Song and Lee obtained characterizations of the linear operators that preserve zero-term rank of Boolean matrices in ([3]). The zero-term rank of a matrix over algebraic structures is the minimum number of lines (rows or columns) needed to cover all the zero entries of the given matrix. In this thesis, we extend their results to the matrices over nonnegative integers. Namely we characterize the linear operators that preserve the zero-term rank of the $m \times n$ matrices over nonnegative integers.

1. Introduction

A *semiring* consists of a set $\mathbb{S}$, and two binary operations on $\mathbb{S}$, addition(+) and multiplication($\cdot$), such that

(1) $(\mathbb{S}, +)$ is an Abelian monoid under addition (identity denoted by 0);

(2) $(\mathbb{S}, \cdot)$ is a monoid under multiplication (identity denoted by 1);

(3) multiplication distributes over addition ;

(4) $s0 = 0s = 0$ for all $s \in \mathbb{S}$; and

(5) $0 \neq 1$.

Usually $\mathbb{S}$ denotes both the semiring and the set. The set of nonnegative integers with usual addition and multiplication, binary Boolean algebra and fuzzy sets are important example of semirings in the combinatorial mathematics.

Here are some examples of semirings which occur in combinatorics. Let $\mathbb{B}$ be any Boolean algebra; then $(\mathbb{B}, \cup, \cap)$ is an semiring. Let $\mathcal{C}$ be any chain with lower bound 0 and upper bound 1; then ($\mathcal{C}$,max, min) is a semiring (a chain semiring). In particular, if $\mathbb{F}$ is the real unit interval $[0, 1]$, then $\mathbb{F}$ is a semiring with max for + and min for $\times$. This ($\mathbb{F}$, max, min) is called a fuzzy semiring. If $\mathbb{P}$ is a subring of the reals $\mathbb{R}$(under real addition and multiplication) and $\mathbb{P}^+$ denotes the nonnegative members of $\mathbb{P}$, then $\mathbb{P}^+$ is a semiring. In particular $\mathbb{Z}^+$, the nonnegative integers, is a semiring.

There is much literature on the study of those linear operators on matrices that leave certain properties or subsets invariant. Boolean matrices also have been the subject of research by many authors. Beasley and Pullman characterized those linear operators that preserve Boolean rank in ([1]) and term rank of matrices over semirings in ([3]). But there are few papers on the linear operators that preserve zero-term rank of the matrices. Recently Beasley, Song, and Lee obtained characterizations of the linear operators that preserve zero-term rank of Boolean matrices in ([4]).

In this thesis, we investigate the zero-term rank of nonnegative integers. We obtain characterizations of linear operators that preserve zero-term rank of the $m \times n$ matrices over $\mathbb{Z}^+$. This results extend the results in ([4]) over nonnegative integer matrices.

In Chapter 2, we introduce the definitions, notations, and well-known fact.

In Chapter 3, we study the rank-1-preserving operators over binary Boolean matrices.

In Chapter 4, we study the Boolean rank preservers and review the characterizations of the linear operators that preserve the rank of Boolean matrices which is studied in ([3]).

In Chapter 5, we give some characterizations of linear operators that preserve zero-term rank of the $m \times n$ matrices over $\mathbb{Z}^+$.

2. Preliminaries

We introduce some definitions and notations that we shall use in this thesis. Let $M_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ matrices with entries in $\mathbb{B} = \{0, 1\}$, the binary Boolean algebra. Arithmetic in $\mathbb{B}$ follows the usual rules except that 1+1=1. The usual definition for adding and multiplying matrices over fields are applied to Boolean matrices as well.

Definition 2.1 . Let A be a nonzero $m \times n$ Boolean matrix. If there is the least integer k for which there exist $m \times k$ and $k \times n$ Boolean matrices B and C with $A = BC$, then we call that A has *Boolean rank* k and denote $b(A) = k$.

Definition 2.2 . An $m \times n$ Boolean matrix A is called a *singular* if $A\mathbf{x} = 0$ for some nonzero $\mathbf{x}$ in $M_{m,1}(\mathbb{B})$. And A is called a *nonsingular* matrix if it is not singular.

Trivially A is nonsingular if and only if A has no zero column.

Definition 2.3 . An $n \times n$ Boolean matrix A is said to be *invertible* if there exists some $X \in M_{n,n}(\mathbb{B})$ such that $AX = XA = I_n$, where I_n is the $n \times n$ identity matrix.

It is well-known that the permutation matrices are the only invertible matrices in $M_{n,n}(\mathbb{B})$ and $A^{-1} = A^t$ when A is invertible. Moreover invertible matrices are all nonsingular.

Definition 2.4 . A *Boolean vector space* is any subset of $\mathbb{B}^m[= M_{m,1}(\mathbb{B})]$ containing $\mathbf{0}$ which is closed under addition.

Definition 2.5 . For any $\mathbf{x}, \mathbf{y} \in \mathbb{B}^m$, if $y_i = 0$ whenever $x_i = 0$, for all $1 \leq i \leq m$, then we say $\mathbf{x}$ *absorbs* $\mathbf{y}$, which is defined by $\mathbf{x} \geq \mathbf{y}$, .

Definition 2.6 . If $\mathbf{V}, \mathbf{W}$ are vector space with $\mathbf{V} \subseteq \mathbf{W}$, then $\mathbf{V}$ is called a *subspace* of $\mathbf{W}$.

Definition 2.7. Let $\mathbf{V}$ be a Boolean vector space. If S is a subset of $\mathbf{V}$, then

(1) The intersection of all subspaces of $\mathbf{V}$ containing S becomes a subspace of $\mathbf{V}$. This subspace is called the subspace *generated by* S and is denoted by $\langle S \rangle$.

(2) If $S = \{s_1, s_2, ..., s_p\}$, then $\langle S \rangle = \{\sum_{i=1}^{p} x_i s_i : x_i \in \mathbb{B}\}$ is the set of all *linear combinations* of the elements in S. In particular, $\langle \phi \rangle = \{\mathbf{0}\}$.

Definition 2.8. The *dimension* of $\mathbf{V}$, written as $\dim(\mathbf{V})$, is the minimum of the cardinalities of all subsets S of $\mathbf{V}$ generating $\mathbf{V}$. We call a generating set S of cardinality equal to $\dim(\mathbf{V})$ by a *basis* of $\mathbf{V}$.

And it is well known that every Boolean vector space $\mathbf{V}$ has only one basis([1]).

Definition 2.9. A subset $\mathbf{I}$ of $\mathbf{V}$ is called *independent* if none of its members is a linear combination of the others. And a subset $\mathbf{J}$ of $\mathbf{V}$ is called *dependent* if it is not independent.

Evidently every basis is independent. The following Lemma proves the uniqueness of the basis and establishes the fact that every independent set is the basis for the space it generates.

Lemma 2.1 . If S is an independent subset of the Boolean vector space $\mathbf{V}$, then S is contained in every subset of $\mathbf{V}$ generating $\langle S \rangle$.

Proof. We may assume that $S \neq \phi$. Suppose $\langle T \rangle = \langle S \rangle$ and $T \subseteq \mathbf{V}$. Let $\mathbf{c} \in S$; then $\mathbf{c}$ is a linear combination of members of T, each of which is a linear combination of members of S. But S is independent, so $\mathbf{c} \geq \mathbf{b} \geq \mathbf{c}$ for some $\mathbf{b} \in T$. Therefore $\mathbf{c} \in T$. Hence $S \subseteq T$. □

In contrast with vector spaces over fields, a Boolean vector space V may have several subspaces with the same dimension as V. For example, let $\mathbf{x} = [0, 1, 1], \mathbf{y} = [1, 1, 0]$ and $\mathbf{z} = [1, 1, 1]$. Let $V = \langle \mathbf{x}, \mathbf{y} \rangle$; then $\langle \mathbf{x}, \mathbf{z} \rangle$ and $\langle \mathbf{y}, \mathbf{z} \rangle$ are two-dimensional subspaces of $\mathbf{V}$, neither of which equals $\mathbf{V}$.

Even more disconcerting, $\mathbf{V}$ can have subspaces whose dimensions exceed $\dim(\mathbf{V})$. For example, let $\mathbf{V}$ be the subspace of $\mathbb{B}^4$ generated by the set S of six vectors $\mathbf{x}$ having exactly two entries equal to 0. Then $\dim(\mathbf{V})$=6 because S is independent, even though $\mathbf{V}$ is subspace of a 4-dimensional space.

As with vector spaces over a field, the intersection of two subspaces $\mathbf{U}, \mathbf{W}$ of a Boolean vector space is always a subspace, but union seldom is. However, if $\mathbf{W}$ absorbs $\mathbf{U}$ (that is, $\mathbf{w} \geq \mathbf{u}$ for all nonzero $\mathbf{w}$ in $\mathbf{W}$ and all $\mathbf{u}$ in $\mathbf{U}$), then it's easy to verify that $\mathbf{U} \cup \mathbf{W}$ is a Boolean vector space.

Lemma 2.2 . If $\mathbf{U}, \mathbf{W}$ are subspaces of the same Boolean vector space with $\mathbf{W} \geq \mathbf{U}$ and $\mathbf{U} \cap \mathbf{W} = \{0\}$, then

$$\dim(\mathbf{U} \cup \mathbf{W}) = \dim(\mathbf{U}) + \dim(\mathbf{W}).$$

Proof. Let $\mathcal{C}, \mathcal{D}$ be the bases of $\mathbf{U}$ and $\mathbf{W}$ respectively, and $\mathcal{B} = \mathcal{C} \cup \mathcal{D}$. Then $\mathcal{B}$ is independent and generates $\mathbf{U} \cup \mathbf{W}$, so $\mathcal{B}$ is a basis for $\mathbf{U} \cup \mathbf{W}$ by Lemma 2.1. □

Definition 2.10 . If $\mathbf{V}$,$\mathbf{W}$ are Boolean vector spaces, a mapping $T : \mathbf{V} \longrightarrow \mathbf{W}$ which preserves sums and 0 is said to be a (Boolean) *linear transformation*. If $\mathbf{V} = \mathbf{W}$, the word *operator* is used instead of "transformation".

Evidently, when T is linear, its behavior on the basis of $\mathbf{V}$ determines its behavior completely. As with transformations of vector spaces over fields, by ordering the bases of $\mathbf{V}$ and $\mathbf{W}$, we can represent T by an $m \times n$ matrix $[t_{ij}]$ in an analogous way. But the t_{ij}'s are not uniquely defined by Boolean T in general, so T may have several matrix representations for the same bases orderings.

A matrix $A \in M_{m,n}(\mathbb{B})$ determines a linear transformation T_A of $\mathbb{B}^n$ into $\mathbb{B}^m$ by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

for all $\mathbf{x} \in \mathbb{B}^n$

The image $T(\mathbf{V})$ of $\mathbf{V}$ in $\mathbf{W}$ is generated by the *image* $T(\mathcal{B})$ of the basis $\mathcal{B}$ of $\mathbf{V}$.

Lemma 2.3 . For every linear Boolean transformation T,

$$\dim(T(\mathbf{V})) \le \dim(\mathbf{V}).$$

Proof. Let $\dim(\mathbf{V}) = k$. Then there are k vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ in $\mathbf{V}$, which constitute a basis of $\mathbf{V}$. For any vector $\mathbf{w}$ in $T(\mathbf{V})$, there exist a vector $\mathbf{v}$ in $\mathbf{V}$ such that $T(\mathbf{v}) = \mathbf{w}$. Since $\mathbf{v} \in \mathbf{V}$, $\mathbf{v}$ is spanned by $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$. That is, there are $\alpha_i \in \mathbb{B}$ such that $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots, +\alpha_k\mathbf{v}_k$. Hence $\mathbf{w} = T(\mathbf{v}) = T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots, +\alpha_k\mathbf{v}_k) = \alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2) + \cdots, +\alpha_k T(\mathbf{v}_k)$, which shows that $\mathbf{w}$ is spanned by k vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), \cdots, T(\mathbf{v}_k)$ in $T(\mathbf{v})$. Therefore $\dim T(\mathbf{V}) \le k$. □

Lemma 2.4. If the Boolean linear transformation $T : \mathbf{V} \longrightarrow \mathbf{W}$ is injective, then dim($\mathbf{V}$)=dim($T(\mathbf{V})$) and T maps the basis of $\mathbf{V}$ onto the basis of $T(\mathbf{V})$.

Proof. In the proof of Lemma 2.3, we have shown that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \cdots, T(\mathbf{v}_k)\}$ spans $T(\mathbf{V})$. Since T is injective, $T(\mathbf{v}_1), T(\mathbf{v}_2), \cdots, T(\mathbf{v}_k)$ are linearly independent, and hence it is a basis for $T(\mathbf{V})$. Thus dim$T(\mathbf{V}) = k$ and T maps the basis $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ of $\mathbf{V}$ onto the basis $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \cdots, T(\mathbf{v}_k)\}$ of $T(\mathbf{V})$. □

Proposition 2.1. A transformation T : $\mathbf{V} \longrightarrow \mathbf{W}$ is *invertible* if and only if T is injective and T($\mathbf{V}$)=$\mathbf{W}$.

It is obvious that the inverse T^{-1} of a Boolean linear transformation T is also linear.

Lemma 2.5 . If T : $\mathbf{V} \longrightarrow \mathbf{W}$ is a surjective Boolean linear transformation, then T is invertible if and only if T preserves the dimension of every subspace of $\mathbf{V}$.

Proof. If T is not injective, then for some $\mathbf{x} \neq \mathbf{y}$, T reduces the dimension of $\langle \mathbf{x}, \mathbf{y} \rangle$. Conversely, if T is invertible, then the conclusion follows by Lemma 2.4. □

Corollary 2.5.1 If T is a Boolean linear operator on $\mathbf{V}$, then the following statements are equivalent:

(a) T is invertible ;

(b) T is injective ;

(c) T is a surjective ;

(d) T permutes the basis of $\mathbf{V}$;

(b) T preserves the dimension of every subspace of $\mathbf{V}$.

Proposition 2.2. Suppose that T is a linear operator on V. Then the following statement are equivalent :

(1) T_A is invertible;

(2) A is invertible;

(3) A is a permutation matrix.

Let's use the following notions:

(1) $\Delta_{m,n} = \{(i,j)|1 \leq i \leq m,\ 1 \leq j \leq n\}$.

(2) $E_{i,j}^{m,n}$ be the $m \times n$ matrix whose (i,j)th entry is 1 and whose other entries are all 0.

(3) $\mathcal{E}_{m,n}$=$\{E_{i,j}^{m,n} : (i,j) \in \Delta_{m,n}\}$.

Corollary 2.5.2. The linear operator T on $M(\mathbb{B})$ is invertible if and only if T permutes $\mathcal{E}$ if and only if T preserves the dimension of every subspace of $M(\mathbb{B})$.

We can describe any operator T on $M(\mathbb{B})$ by expressing $(T(X))_{ij}$ as a scalar-valued function of X for all $(i,j) \in \Delta$. The operator T will be linear if and only if each component function $t_{ij} : X \to (T(X))_{ij}$ is a linear transformation of $M(\mathbb{B})$ onto $\mathbb{B}$. Applying corollary 2.5.2, we see that the operator T on $M(\mathbb{B})$ is invertible if and only if there exist a permutation τ of Δ such that $T([x_{ij}]) = [x_{\tau(i,j)}]$ for all X in $M(\mathbb{B})$.

Proposition2.3. Let A be a Boolean matrix with rank 1 in $M_{m,n}(\mathbb{B})$.

(1) If $A = \mathbf{x}\mathbf{y}^t$ is a factorization of A, then the vectors $\mathbf{x}$ and $\mathbf{y}$ are uniquely determined by A. We call $\mathbf{x}$ and $\mathbf{y}$ the *left factor* and the *right factor* of A, respectively.

(2) There are exactly $(2^m - 1)(2^n - 1)$ rank-1 $m \times n$ Boolean matrices.

Definition 2.11. Let $A, B \in M_{m,n}(\mathbb{B})$. We define $A \leq B$ if $a_{ij} = 0$ whenever $b_{ij} = 0$. Equivalently $A \leq B$ if and only if $A + B = B$.

Definition 2.12. For any vector $\mathbf{x}$, let $|\mathbf{x}|$ be the number of nonzero entries in $\mathbf{x}$. If $A = \mathbf{a}\mathbf{b}^t$ is not zero, then we define *perimeter* of A as $|\mathbf{a}| + |\mathbf{b}|$, and we denote the perimeter of A by $p(A)$.

Lemma 2.5. If $A \leq B$ and $b(A) = b(B) = 1$, then $p(A) < p(B)$ unless $A = B$.

Definition 2.13 . A subspace of $M_{m,n}(\mathbb{B})$ whose nonzero members have Boolean rank 1 is defined by a *rank-1 space* .

Lemma 2.6 . If A, B, and $A + B$ are rank-1 matrices and neither $A \leq B$ nor $B \leq A$, then A, B, and $A + B$ have a common factor.

Proof. Let $A = \mathbf{a}\mathbf{x}^t$, $B = \mathbf{b}\mathbf{y}^t$, and $C = A + B = \mathbf{c}\mathbf{z}^t$ be the factorizations of A, B, and C,respectively. We have for all i, j

$$a_i\mathbf{x} + b_i\mathbf{y} = c_i\mathbf{z} \quad \text{and} \quad x_j\mathbf{a} + y_j\mathbf{b} = z_j\mathbf{c}.$$

If $\mathbf{a} \not\leq \mathbf{b}$ and $\mathbf{b} \not\leq \mathbf{a}$, then for some i, j, $\mathbf{x} = c_i\mathbf{z}$ and $\mathbf{y} = c_j\mathbf{z}$. But $\mathbf{x} \neq 0$ and $\mathbf{y} \neq 0$, so $\mathbf{x} = \mathbf{y} = \mathbf{z}$. Thus A, B, C have a common right factor. If $\mathbf{a} \leq \mathbf{b}$, then $\mathbf{x} \not\leq \mathbf{y}$ (as $A \not\leq B$). Therefore $\mathbf{a} = \mathbf{b}$ and A, B,and C have a common left factor. A parallel argument holds if $\mathbf{b} \leq \mathbf{a}$. $\square$

3. Linear operators that preserve Boolean rank 1

Definition 3.1. Suppose that T is a linear operator on M which is the set of the $m \times n$ matrices over semirings $\mathbb{S}$. Say that T is a

(1) (U, V)-*operator* if there exist invertible matrices U and V such that $T(A) = UAV$ for all A in M, or $m = n$ and $T(A) = UA^tV$ for all A in M.

(2) *rank preserver* if rank($T(A)$)=rank(A) for all A in M.

(3) *rank-1 preserver* if rank($T(A)$) $= 1$ whenever rank(A)=1 for all A in M.

Marcus, Moyls, and Westwick([5]) showed that if T is a linear operator on $M_{m,n}(\mathbb{F})$($\mathbb{F}$ algebraically closed field) and T maps rank-1 matrices to rank-1 matrices(*i.e.* T preserves rank-1 matrices), then (and only then) T is a (U, V)-operator. This result does not hold for the Boolean case. All of the contents of this chapter are due to Beasly and Pullman in ([1]). The following example shows that not all rank-1-preserving operators T are of the form $T(X) = UXV$ or $T(X) = UX^tV$ for some nonsingular U, V, contrary to the situation for algebraically closed fields. Since invertible Boolean matrices are nonsingular, it also shows that not all rank-1-preserving operators T are (U, V)-operators.

Example. Let

$$T\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = (b + e + c + f)\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 & d \\ 0 & 0 & 0 \end{bmatrix}$$

Here, T is a linear operator and $b(T(X)) = 1$ whenever $b(X) = 1$ (in fact whenever $X \neq 0$). If there existed nonsingular U and V such that $T(X) = UXV$ for all $X \in M_{23}(\mathbb{B})$, then for $j = 1, 2, 3$, we have $T(E_{1j}) = \mathbf{u}\mathbf{v}_j^t$, where $\mathbf{u}$ is the first column of U and $\mathbf{v}_j$ is the jth column of V. But

$$T(E_{11}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1, 0, 0] \text{ and } T(E_{12}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1, 1, 1],$$

and hence

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which is a contradiction. □

Suppose that U and V are nonsingular member of $M_{m,m}(\mathbb{B})$and $M_{n,n}(\mathbb{B})$ respectively, and T is the operator on $M_{m,n}(\mathbb{B})$ defined by $T(X) = UXV$ for all X. Clearly T is linear. Moreover $T(X)$ has rank 1 whenever X has rank 1. For, suppose X has rank 1, so that $X = \mathbf{a}\mathbf{b}^t$ where $\mathbf{a} \neq 0$, $\mathbf{b} \neq 0$. Then $T(X) = U\mathbf{a}\mathbf{b}^tV = (U\mathbf{a})(V^t\mathbf{b})^t$, and since U and V^t are nonsingular, neither $U\mathbf{a}$ nor $V^t\mathbf{b}$ is 0, so $T(X)$ has rank 1. It follows that all Boolean (U, V)-operators are rank-1 preservers.

Example. Suppose that C is a fixed rank-1 member of $M_{m,n}(\mathbb{B})$, and T is the operator defined by $T(X) = C$ if $X \neq 0$ and $T(0) = 0$. Then T preserves Boolean rank 1. But T is not a (U, V)-operator. □

This example shows that for each $k (1 \leq k \leq n)$ there is a linear operator T_k that preserves the Boolean rank of every rank-k $m \times n$ matrix but is not a (U, V)-operator when $k > 1$. [Just take C to be fixed rank-k matrix]

Beasley ([6]) showed that for most $k \leq n$, each operator on field-valued matrices preserves the rank of rank-k matrices if and only if it is a (U, V)-operator.

They were unable to find a condition necessary and sufficient for a Boolean operator to preserve the rank of all rank-1 matrices. They have, however, found two conditions, one necessary (but not sufficient), and the other sufficient(but not necessary), which are of some help in constructing example. These are described in the next few paragraphs.

Suppose that T is a linear operator on $M_{m,n}(\mathbb{B})$. Let $\mathcal{R}_i = \{T(E_{ik}) : 1 \leq k \leq n\}$ and $\mathcal{C}_j = \{T(E_{kj}) : 1 \leq k \leq m\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Lemma 3.1. T preserves the rank of all rank-1 matrices only if there exist rank-1 spaces $\mathbf{R}_i$ and $\mathbf{C}_j$ such that $\mathcal{R}_i \subseteq \mathbf{R}_i$ and $\mathcal{C}_j \subseteq \mathbf{C}_j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Proof. Suppose T preserves the rank of all Boolean rank-1 matrices. Then as $\{E_{ij} : 1 \leq j \leq n\}$ is in the rank-1 space $\mathbf{V}_i$ of all $A \in M_{m,n}(\mathbb{B})$ whose nonzero entries all lie in its ith row, it follows that $\mathcal{R}_i \subseteq T(\mathbf{V}_i)$. But $T(\mathbf{V}_i)$ is also a rank-1 space. Therefore, $T(\mathbf{V}_i)$ will serve for $\mathbf{R}_i$ of the conclusion. Similarly for $\mathbf{C}_j$. □

Lemma 3.2. T preserves the rank of all rank-1 matrices if there is a rank-1 spaces $\mathbf{V}$ such that

(i) $\bigcup_{i=1}^{m} \mathcal{R}_i \subseteq \mathbf{V}$ or

(ii) $\bigcup_{j=1}^{n} \mathcal{C}_j \subseteq \mathbf{V}$.

Proof. (i) : If $b(X) = 1$, then $T(X) = \sum_{j=1}^{m} \sum_{j=1}^{n} x_i y_j T(E_{ij}) = \sum_{i=1}^{m} x_i [\sum_{j=1}^{n} y_j T(E_{ij})] = \sum_{i=1}^{m} x_i M_i$, where M_i is in $\langle \mathcal{R}_i \rangle$. Therefore $T(X)$ is a sum of members of **V** and hence has rank 1. The proof of (ii) is similar. □

The identity operator I on $M_{n,n}(\mathbb{B})$ provides an example of a rank-1-preserving operator for which neither (i) nor (ii) of Lemma 3.2 holds. Thus those conditions are sufficient, but not necessary.

If we add the hypothesis that T preserves the dimension of any rank-1 space (unlike matrices over fields, for which this is always true of rank-1 preservers), then the conclusion is much more restrictive, as we shall see in Theorem 3.1.

Lemma 3.3. If T is a linear operator on $M_{m,n}(\mathbb{B})$ that preserves the dimension of all rank-1 spaces, then the restriction of T to the rank-1 matrices is injective or T reduces the rank of some rank-2 matrix to 1.

Proof. Let $M^1 = \{A \in M_{m,n}(\mathbb{B}) : b(A) = 1\}$ and $\mathbf{W} \equiv \{0\} \cup \{X \in M^1 : T(X) = T(B)\}$ for each $B \in M^1$. If **W** is a rank-1 space, then $\dim(\mathbf{W}) = \dim(T(\mathbf{W})) = 1$, so $\mathbf{W} = \langle B \rangle$. Thus T is injective. Otherwise there are X, Y in **W** such that $b(X + Y) = 2$. □

Corollary 3.3. If T is a linear operator on $M_{m,n}(\mathbb{B})$ that

(i) preserves the ranks of all rank-1 and rank-2 matrices and

(ii) preserves the dimension of all rank-1 spaces,

then

(a) T is invertible and

(b) T^{-1} satisfies (i) and (ii).

Proof. Part (a) : Let $\mathcal{E}$ be the basis of $M_{m,n}(\mathbb{B})$, as in chapter 2. According to Corollary 2.5.1(d), T is invertible if it permutes $\mathcal{E}$. Lemma 3.3 implies that T permutes M^1. But $M^1 \supseteq \mathcal{E}$, so it suffices to show that $T(\mathcal{E}) \supseteq \mathcal{E}$. Let $E \in \mathcal{E}$; then $E = T(C)$ for some $C \in M^1$. Since $C \neq 0$, we have $C \geq F$ for some F in the basis $\mathcal{E}$. Therefore $E \geq T(F)$. Then $E = T(F)$ by Lemma 2.5, completing the proof of part (a). Part (b) follows directly. □

Lemma 3.4. Let T is an invertible linear operator on $M_{m,n}(\mathbb{B})$ that preserves the rank of every rank-1 matrix and τ is the permutation of Δ representing T. Then there exist permutation α, β of $\{1, 2, ..., m\}$ and $\{1, 2, ..., n\}$ respectively such that

(a) $\tau(i, j) = (\alpha(i), \beta(j))$ *for all* $(i, j) \in \Delta$ or

(b) $m = n$ *and* $\tau(i, j) = (\beta(j), \alpha(i))$ *for all* $(i, j) \in \Delta$.

Lemma 3.5. If τ satisfies the conclusion of Lemma 3.4, then T is a (U, V)-operator.

Theorem 3.1. If T is a linear operator on $M_{m,n}(\mathbb{B})$, then the following statements are equivalent:

(a) T is invertible and preserves the rank of all rank-1 matrices;

(b) T preserves the ranks of all rank-1 matrices and rank-2 matrices and preserves the dimension of all rank-1 spaces;

(c) T is a (U, V)-operator.

Proof. Lemmas 3.4 and 3.5 show that (a) implies (c). Statement (b) implies (a) by corollary 3.3. So it sufficies to show that (c) implies (b). Any operator T that satisfies (c) is invertible; in fact $T^{-1}(A) = U^{-1}AV^{-1}$ or $T^{-1}(A) = U^{-1}A^tV^{-1}$. Such operators are clearly rank-1 preservers. The rest is implied by Lemma 2.4. □

4. Linear operators that preserve the Boolean rank

In this chapter, we reviewed the linear operators that preserve Boolean rank.

Lemma 4.1. For A, B are in $M_{m,n}(\mathbb{B})(m > 1)$, $A \neq B$, if $p(A) \geq p(B)$, and $b(A) = b(B) = 1$, then there exists C in $M_{m,n}(\mathbb{B})$ such that $b(A+C) = 1$ and $b(B+C) = 2$.

Proof. If $b(A+B) = 2$, then the conclusion is obtained by letting $C = A$. So we may assume that $b(A+B) = 1$. we define E_{pq} as in Chapter 2.

Factoring A, B, and E_{pq}, we have $A = \mathbf{a}\mathbf{x}^t$, $B = \mathbf{b}\mathbf{y}^t$, and $E_{pq} = \mathbf{e}_p\mathbf{f}_q^t$. By our hypotheses and Lemma 2.5, $A \not\leq B$. Therefore, Lemma 2.6 implies that there are three cases; (i) $\mathbf{a} = \mathbf{b}$ and $\mathbf{x} \neq \mathbf{y}$, or (ii) $\mathbf{x} = \mathbf{y}$ and $\mathbf{a} \neq \mathbf{b}$, or (iii) $\mathbf{b} \leq \mathbf{a}$, $\mathbf{b} \neq \mathbf{a}$, $\mathbf{y} \leq \mathbf{x}$, and $\mathbf{y} \neq \mathbf{x}$. In any case, there exist k, l such that $b_k = y_l = 1$, because $B \neq 0$.

Case(i): We have $\mathbf{x} \not\leq \mathbf{y}$ because $\mathbf{a} = \mathbf{b}$ and $A \not\leq B$. So we can select $j \leq n$ so that $x_j = 1$ and $y_j = 0$. Since $m > 1$, we can choose $i \leq m$ so that $i \neq k$. Now $b(E_{ij} + E_{kl}) = 2$, because $k \neq i$ and $l \neq j$. Let $C = (\mathbf{a} + \mathbf{e}_i)\mathbf{x}^t$. Then $B + C \geq E_{ij} + E_{kl}$. Thus $b(B+C) = 2$. On the other hand, $A \leq C$, so $b(A+C) = b(C) = 1$.

The case (ii),(iii) are proved similarly. □

Lemma 4.2. If T is a linear operator on $M_{m,n}(\mathbb{B})$ with $m > 1$, and T is not invertible but preserves the rank of rank-1 matrices, then T decreases the rank of some rank-2 matrix.

Proof. By the proof of Corollary 3.3, T is not injective on $M^1 = \{A \in M_{m,n}(\mathbb{B}) : b(A) = 1\}$ so $T(X) = T(Y)$ for some X, Y in M^1 with $X \neq Y$. Without loss of generality we may suppose that $p(X) \geq p(Y)$. By Lemma 4.1, there is some matrix D such that $b(X + D) = 2$ while $b(Y + D) = 1$. However, $T(X + D) = T(X) + T(D) = T(Y + D)$. □

Theorem 4.1. Suppose that T is a linear operator on $M_{m,n}(\mathbb{B})$ with $m > 1$. Then T is a rank preserver if and only if T is a (U, V)-operator.

Proof. Theorem 3.1 and Lemma 4.2 prove the necessity of the condition given for rank preservation. To prove the sufficiency, we note that $b(A)$ is the least integer k for which k rank-1 matrices whose sum is A exist. Therefore $b(L(A)) \leq b(A)$ whenever L is a linear rank-1 preserver. Now each (U, V)-operator and its inverse are rank-1 preservers, so such operators preserves all ranks. □

Theorem 4.2. Suppose that T is a linear operator on $M_{m,n}(\mathbb{B})$. Then T is a rank preserver if and only if T preserves the ranks of all rank-1 and rank-2 matrices.

Proof. We may assume that $m > 1$. If T preserves ranks 1 and 2, then T is invertible(by Lemma 4.2) and hence a rank-preserver by Theorem 3.1 and 4.1. □

5. Zero-term rank preserver of matrices over nonnegative integer

In this chapter, we obtain the properties of zero-term rank of matrices over nonnegative integers and also have the characterizations of the linear operators that preserve the zero-term rank of the matrices. We extend the results over Boolean matrices of Beasley, Song and Lee([3]) to matrices over nonnegative integer.

We let $M_{m,n}(\mathbb{Z}^+)$ denote the set of all $m \times n$ matrices with entries in $\mathbb{Z}^+ = \{0, 1, 2, 3...\}$, the nonnegative integers.

Definition 5.1.

(1) Let E_{ij} be the $m \times n$ matrix, whose (i, j)th entry is 1 and whose other entries are all zero, which is called a *cell*.

(2) Let J denote the $m \times n$ matrix all of whose entries are 1,
$\Delta = \{E_{ij} \mid 1 \le i \le m,\ 1 \le j \le n\}$ denote the set of cells, and
$\mathcal{E} = \{(i, j) \mid 1 \le i \le m,\ 1 \le j \le n\}$ denote the set of indices.

Definition 5.2. The *zero-term rank* of a matrix X, $z(X)$, is the minimum number of lines (rows or columns) needed to cover all zero entries of X.

Definition 5.3. The *term rank* of X, $t(X)$, is the minimum number of lines (rows or columns) needed to cover all the nonzero entries of X.

Definition 5.4. For any $A, B \in M_{m,n}(\mathbb{Z}^+)$, we say A *dominates* B (written $A \ge B$ or $B \le A$) if $a_{ij} \ge b_{ij}$ for all i, j.

Then we obtain the following Lemma for the zero-term rank.

Lemma 5.1 . For any $A, B \in M_{m,n}(\mathbb{Z}^+)$, $A \geq B$ implies that $z(A) \leq z(B)$.

Proof . If $z(B) = k$, then there are k lines which cover all zero entries in B. Since $A \geq B$, this k lines can also cover all zero entries in A. Hence $z(A) \leq k = z(B)$. □

Definition 5.5. A function T mapping $M_{m,n}(\mathbb{Z}^+)$ into itself is called a *linear operator* if T satisfies $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all α, $\beta \in \mathbb{Z}^+$ and for $A, B \in M_{m,n}(\mathbb{Z}^+)$.

From now on we will assume that $2 \leq m \leq n$ for all $m \times n$ matrices, and a mapping T will denote a linear operator on $M_{m,n}(\mathbb{Z}^+)$.

Definition 5.6. If $z(T(X)) = k$ whenever $z(X) = k$, we say T *preserves zero-term rank k.* If T preserves zero-term rank k for every $k \leq min\{m, n\}$, then we say T *preserves zero-term rank.*

Definition 5.7. If $t(T(X)) = k$ whenever $t(X) = k$, we say T *preserves term rank k.* If T preserves term rank k for every $k \leq min\{m, n\}$, then we say T *preserves term rank.*

Consider the semiring $\mathbb{Z}^+$. Which linear operators on $M_{m,n}(\mathbb{Z}^+)$ preserve zero-term rank? The operations of (1) permuting rows, (2) permuting columns, and (3)(if $m = n$) transposing the matrices in $M_{m,n}(\mathbb{Z}^+)$ are all linear operators that preserve zero-term rank of the matrices on $M_{m,n}(\mathbb{Z}^+)$.

If we take a fixed $m \times n$ matrix B in $M_{m,n}(\mathbb{Z}^+)$, then its *Schur product* is defined by $B \circ X = [b_{ij}x_{ij}]$ for all $X \in M_{m,n}(\mathbb{Z}^+)$.

Proposition 5.1. Suppose that T is an operator on $M_{m,n}(\mathbb{Z}^+)$ such that $T(X) = B \circ X$, where B is fixed in $M_{m,n}(\mathbb{Z}^+)$. Then T is linear.

Proof. For all $\alpha, \beta \in \mathbb{Z}^+$, A, $B \in M_{m,n}(\mathbb{Z}^+)$,

$$\begin{aligned} T(\alpha X + \beta Y) &= B \circ (\alpha X + \beta Y) = B \circ (\alpha X) + B \circ (\beta Y) \\ &= \alpha(B \circ X) + \beta(B \circ Y) = \alpha T(X) + \beta T(Y). \end{aligned}$$ □

Proposition 5.2. Suppose that T is a linear operator on $M_{m,n}(\mathbb{Z}^+)$ such that $T(X) = B \circ X$, where B is fixed in $M_{m,n}(\mathbb{Z}^+)$, none of entries is zero in $\mathbb{Z}^+$. Then T preserves zero-term rank.

Proof. It follows the definition of *Schur product.* □

Definition 5.8. Let P and Q be $m \times m$ and $n \times n$ permutation matrices and B is an $m \times n$ matrix over $\mathbb{Z}^+$ with none of whose entries is zero. Then T is a (P, Q, B)*-operator* if

(1) $T(X) = P(B \circ X)Q$ for all X in $M_{m,n}(\mathbb{Z}^+)$ or

(2) $m = n$, and $T(X) = P(B \circ X^t)Q$ for all X in $M_{m,n}(\mathbb{Z}^+)$.

Definition 5.9. Let $\mathcal{E} = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$. That is, $\mathcal{E}$ is the set of indices. Define $T' : \mathcal{E} \to \mathcal{E}$ by $T'(i, j) = (u, v)$ whenever $T(E_{ij}) = b_{ij}E_{uv}$.

Now we have some Lemmas which are need to obtain the main Theorem.

Lemma 5.2. Suppose that T preserves zero-term rank 1 and $T(J) \geq J$. Then T maps a cell onto a cell with a scalar multiple and hence T' is a bijection on $\mathcal{E}$.

Proof. If $T(E_{ij}) = 0$ for some $E_{ij} \in \Delta$, then we can choose $mn-1$ cells $E_1, E_2, ..., E_{mn-1}$ which are different from E_{ij}. Thus we have

$$\begin{aligned} J \leq T(J) &= T(E_{ij} + \sum_{k=1}^{mn-1} E_k) \\ &= T(E_{ij}) + T(\sum_{k=1}^{mn-1} E_k) \\ &= 0 + T(\sum_{k=1}^{mn-1} E_k) \\ &= T(\sum_{k=1}^{mn-1} E_k). \end{aligned}$$

But $z(J) = 0$ and $z(\sum_{k=1}^{mn-1} E_k) = 1$. Since T preserves zero-term rank 1, we have $z(T(\sum_{k=1}^{mn-1} E_k)) = 1$. Since $J \leq T(J) = T(\sum_{k=1}^{mn-1} E_k)$, we have $0 = z(J) \geq z(T(\sum_{k=1}^{mn-1} E_k)) = 1$ by Lemma 5.1. Then we have a contradiction. Therefore $T(E_{ij})$ dominates at least one cell with a scalar multiple.

For some cell $E_{ij} \in \Delta$, suppose $T(E_{ij})$ dominates two cells, that is, $T(E_{ij}) \geq b_{ij}E_{kl} + b'_{ij}E_{uv}$. For each cell E_{rs} except for both E_{kl} and E_{uv}, we can choose one cell E_h such that $T(E_h)$ dominates E_{rs} because $T(J) \geq J$. Since the number of cells except for both E_{kl} and E_{uv} is mn-2, there exist at most mn-1 cells $E_1, E_2, ..., E_{mn-1}$ containing E_{ij} such that $T(\sum_{h=1}^{mn-1} E_h)) \geq J$. Since T preserves zero-term rank 1, we have $z(T(\sum_{h=1}^{mn-1} E_h)) = z(\sum_{h=1}^{mn-1} E_h) =$

1. But $1 = z(\sum_{h=1}^{mn-1} E_h) \leq z(J) = 0$ by Lemma 5.1. Thus we have a contradiction. Hence $T(E_{ij})$ dominates only one cell with a scalar multiple. That is, T maps a cell into a cell with a scalar multiple.

Now, we show that T' is a bijection on $\mathcal{E}$. If $T'(i,j) = T'(k,l) = (u,v)$ for some distinct pairs $(i,j),(k,l)$, then we have $T(E_{ij}) = a_{ij}E_{uv}$ and $T(E_{kl}) = b_{kl}E_{uv}$. Thus we have

$$\begin{aligned} J \leq T(J) &= T(J-(E_{ij}+E_{kl})+(E_{ij}+E_{kl})) \\ &= T(J-(E_{ij}+E_{kl}))+T(E_{ij})+T(E_{kl}) \\ &= T(J-(E_{ij}+E_{kl}))+a_{ij}E_{uv}+b_{kl}E_{uv} \\ &= T(J-(E_{ij}+E_{kl}))+(a_{ij}+b_{kl})E_{uv} \end{aligned}$$

But we have

$$\begin{aligned} & z(T(J-(E_{ij}+E_{kl}))+(a_{ij}+b_{kl})E_{uv}) \\ = \; & z(T(J-(E_{ij}+E_{kl}))+a_{ij}E_{uv}) \\ = \; & z(T(J-(E_{ij}+E_{kl}))+T(E_{ij})) \\ = \; & z(T(J-(E_{ij}+E_{kl})+E_{ij})) \\ = \; & z(T(J-E_{kl})). \end{aligned}$$

Since T preserves zero-term rank 1 and $z(J-E_{kl}) = 1, z(T(J-E_{kl}) = 1$. Since $J \leq T(J-(E_{ij}+E_{kl}))+(a_{ij}+b_{kl})E_{uv}$ we have $0 = z(J) \geq z(T(J-(E_{ij}+E_{kl}))+(a_{ij}+b_{kl})E_{uv}) = z(T(J-E_{kl}) = 1$. This is a contradiction. Therefore T' is an injection on $\mathcal{E}$ and hence a bijection on $\mathcal{E}$. □

Lemma 5.3. If T preserves zero-term rank 1 and $T(J) \geq J$, then T preserves term rank 1.

Proof. Suppose that T does not preserve term rank 1. Then there exist some cells E_{ij} and E_{il} on the same row(or column) such that $T(E_{ij}+E_{il}) =$

$T(E_{ij})+T(E_{il}) = b_{ij}E_{pq}+b_{il}E_{rs}$ with $p \neq r$ and $q \neq s$, where $T'(i,j) = (p,q)$ and $T'(i,l) = (r,s)$. Since T preserves zero-term rank 1 and $T(J) \geq J$, we have that T' is a bijection on $\mathcal{E}$ by Lemma 5.2. Thus we have $T(J) = B = (b_{uv})_{m\times n}$, for some $B \in M_{m,n}(\mathbb{Z}^+)$ with $b_{uv} \geq 1$. Since T preserve zero-term rank 1 and $z(J - E_{ij} - E_{il}) = 1$ we have $z(T(J - E_{ij} - E_{il})) = 1$. But $T(J - E_{ij} - E_{il})$ has zeros in the (p,q) and (r,s) positions because $T(E_{ij} + E_{il}) = b_{ij}E_{pq} + b_{il}E_{rs}$. Then $z(T(J - E_{ij} - E_{il})) = 2$. This is impossible. Hence T preserves term rank 1. □

Lemma 5.4. If T preserves zero-term rank 1 and $T(J) \geq J$, then T maps a row of a matrix onto a row with a scalar multiple(or column if m=n).

Proof. Suppose that T does not map a row into a row with a scalar multiple(or column if m=n). Then T does not preserve term rank 1. This contradicts to Lemma 5.3. Hence T maps a row into a row with a scalar multiple(or column if m=n). Lemma 5.2 implies that T' is a bijection on $\mathcal{E}$. Then the bijectivity of T' implies that T maps a row onto a row with a scalar multiple(or may be a column if m=n). □

Lemma 5.5. For the case m=n, suppose T preserves zero-term rank 1 and $T(J) \geq J$. If T maps a row onto a row(or column) with a scalar multiple, then all rows of a matrix must be mapped some rows(or columns, respectively) with scalar multiple.

Proof. Lemma 5.2 implies that T' is a bijection on $\mathcal{E}$. Let $R_i = \sum_{j=1}^{n} E_{ij}$, $C^{(j)} = \sum_{i=1}^{n} E_{ij}$, for $i, j = 1, 2, ..., n$. Suppose T maps a row, say R_1 , onto an ith row R_i with a scalar multiple B_i and another row, say R_2 , onto a jth column $C^{(j)}$ with a scalar multiple $B^{(j)}$. That is, $T(R_1) = B_i \circ R_i$ and

$T(R_2) = B^{(j)} \circ C^{(j)}$. Then $R_1 + R_2$ has $2n$ cells but $B_i \circ R_i + B^{(j)} \circ C^{(j)}$ has $2n-1$ cells. This contradicts to the bijectivity of T' on $\mathcal{E}$. Hence all row must be mapped some rows(or columns, respectively) with a scalar multiple. □

Thus we have the following characterization theorem for zero-term rank preserver on $M_{m,n}(\mathbb{Z}^+)$.

Theorem 5.1. Suppose T is a linear operator on $M_{m,n}(\mathbb{Z}^+)$. Then the following statements are equivalent:

(i) T is a (P, Q, B)-operator ;

(ii) T preserves zero-term rank ;

(iii) T preserves zero-term rank 1 and $T(J) \geq J$.

Proof. (i)$\Longrightarrow$ (ii): Suppose T is a (P, Q, B)-operator and $X \in M_{m,n}(\mathbb{Z}^+)$. Then $T(X) = P(B \circ X)Q$(or $m = n$, and $T(X) = P(B \circ X^t)Q$), where P and Q are $m \times m$ and $n \times n$ permutation matrices and B is an $m \times n$ matrix over $\mathbb{Z}^+$, none of whose entries is zero. Hence $z(T(X)) = z(P(B \circ X)Q) = z(X)$ or $z(T(X)) = z(P(B \circ X^t)Q) = z(X)$. Since X is arbitrary, T preserves zero-term rank.
(ii)$\Longrightarrow$ (iii): clearly.
(iii)$\Longrightarrow$ (i): Suppose T preserves zero-term rank 1 and $T(J) \geq J$. Lemmas 5.4 and 5.5 imply that T maps all rows of a matrix onto rows(or columns if m=n) with a scalar multiple. Thus T is of the form $T(X) = P(B \circ X)Q$ or $T(X) = P(B \circ X^t)Q$, where P and Q are permutation matrices and B is a fixed $m \times n$ matrix over $\mathbb{Z}^+$, none of whose entries is zero. Hence T is a (P, Q, B)-operator. □

Lemma 5.6 . For A, B in $M_{m,n}(\mathbb{Z}^+)$, $A \geq B$ implies $T(A) \geq T(B)$.

Proof. By definition of $A \geq B$, we have $a_{ij} \geq b_{ij}$ for all i, j.
Using $A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij}$ and $B = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} E_{ij}$, we have

$$\begin{aligned} T(A) &= T(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij}) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} T(E_{ij}) \\ &\geq \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} T(E_{ij}) \\ &= T(\sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} E_{ij}) = T(B). \end{aligned}$$

□

Definition 5.10. We say that a linear operator T *strongly preserves zero-term rank* k provided that $z(T(A)) = k$ if and only if $z(A) = k$. And a linear operator T *strongly preserves term rank* k provided that $t(T(A)) = k$ if and only if $t(A) = k$.

Lemma 5.7. If T strongly preserves zero-term rank 1, then we have $T(J) \geq J$.

Proof. Since T strongly preserves zero-term rank 1 and $z(J) \neq 1$. we have $z(T(J)) = 0$ or $z(T(J)) \geq 2$. Suppose $z(T(J)) \geq 2$. Since $J \geq J - E_{ij}$ for any cell E_{ij} in $\mathcal{E}$, $T(J) \geq T(J - E_{ij})$ by Lemma 5.6. But Lemma 5.1 implies

$z(T(J)) \leq z(T(J - E_{ij}))$. Since T strongly preserves zero-term rank 1 and $z(J - E_{ij}) = 1$, we have $z(T(J - E_{ij})) = 1$. This is a contradiction because $z(T(J)) \geq 2$ and $z(T(J)) \leq z(T(J - E_{ij})) = 1$. Thus $z(T(J)) = 0$ and hence $T(J) \geq J$. □

Theorem 5.2. Suppose that T is a linear operator on $M_{m,n}(\mathbb{Z}^+)$. Then T preserves zero-term rank if and only if it strongly preserves zero-term rank 1.

Proof. Suppose that T strongly preserves zero-term rank 1. Then Lemma 5.7 implies that $T(J) \geq J$. By Theorem 5.1, T preserves zero term rank .

Conversely, suppose T preserves zero-term rank. If $z(T(X)) = 1$ and $z(X) \neq 1$, then $z(X) = 0$ or $z(X) \geq 2$. If $z(X) = 0$, then $z(T(X)) = 0$ by assumption. If $z(X) \geq 2$, then $z(T(X)) \geq 2$ by assumption. Those contradict to $z(T(X)) = 1$. Hence T strongly preserves zero-term rank 1. □

Thus we have characterized the linear operators that preserve the zero-term rank on $M_{m,n}(\mathbb{Z}^+)$, which extend the results on Boolean case in ([3])

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<국문 초록>

음이 아닌 정수 행렬들의 영인자 계수를 보존하는 선형 연산자들

본 논문에서는 부울대수 행렬의 영인자 계수를 보존하는 선형 연산자들의 특성에 관한 기존의 논문결과가 음이 아닌 정수 행렬의 경우에도 적용할 수 있는가를 고찰하여 이 행렬 위에서의 영인자 계수를 보존하는 선형 연산자의 특성을 밝혔다.

그 결과, $m \times n$ 의 음이 아닌 정수 행렬 위에서의 선형연산자 T가 영인자 계수를 보존하기 위한 필요충분조건은 T가 (P, Q, B)-연산자이고, T가 영인자 계수 1과 0을 보존하고, T가 강하게 영인자 계수 1을 보존하는 것이다.

감사의 글

먼저 지금까지 인도하시고 도와주신 하나님께 감사와 영광을 돌려 드립니다. 너무나도 훌쩍 지나버린 2년이라는 시간이 많은 후회와 아쉬움으로 남지만 미약하나마 이렇게 석사학위 논문을 내 놓을 수 있게 되어서 대단히 기쁘게 생각합니다. 여러 가지로 많은 어려움이 있었지만 많은 분들의 도움으로 무사히 마칠 수 있었습니다.

우선 본 논문이 완성되기까지 연구에 바쁘신 가운데도 부족한 저에게 항상 깊은 관심과 배려로 많은 가르침을 주시고 지도해주신 송석준 교수님께 깊은 감사를 드립니다. 그리고 대학원 4학기 동안 훌륭한 강의를 해주시고 여러 가지로 도움을 주신 양영오, 방은숙, 윤용식, 고윤희 교수님들께도 감사를 드리며, 논문편집과정에서 아낌없이 시간을 내어 섬세한 검토와 조언을 해주신 정승달 교수님께도 감사의 마음을 드립니다. 대학원 생활뿐만 아니라 논문이 완성되기까지 많은 관심과 충고로 저를 이끌어 주신 강경태 선생님의 고마움도 잊지 않겠습니다. 그리고 짧은 기간이나마 강의를 함께 들었던 박사 과정의 선생님들에게도 감사의 마음을 전하고 싶고, 4학기 동안 기쁠 때 같이 웃고 힘들 때 서로 의지하며 함께 생활해 온 동기 김춘심에게도 고마움과 사랑의 마음을 전합니다. 무엇보다도 저를 아낌없이 지원해 주고 변함없이 사랑해주신 저의 소중한 부모님과 가족들에게 깊은 감사와 사랑의 마음을 전합니다. 앞으로도 이 모든 분들과 주위에서 격려하고 용기를 주신 모든 분들의 사랑과 기대에 어긋남이 없도록 열심히 생활하겠습니다.

2002년 12월