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Pricing Stock Option When there is no Riskless Asset : Discrete Time Valuation with Dothan's Method

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Introduction

Since Black and Scholes (1973) derived the original option pricing model, many researchers have derived alternative formulas or have extended the model by making different assumptions about the various determinants of the valuation model. Cox, Ross, and Rubinstein (CRR) (1979) developed a simple but a powerful option valuation model by using lattice binomial model in discrete time period and derived same formula as B-S model as a limiting case of discrete time model. However, these models used the assumption that the risk-free interest rate is constant all over the option's life.

Merton (1973) was the first to generalize the B-S option valuation model with the assumption of stochastic interest rate. Since Vasicek (1977) derived a general form of the term structure of interest rate, Jamshidian (1989), Rabinovitch (1989) derived separately a closed-form solution for

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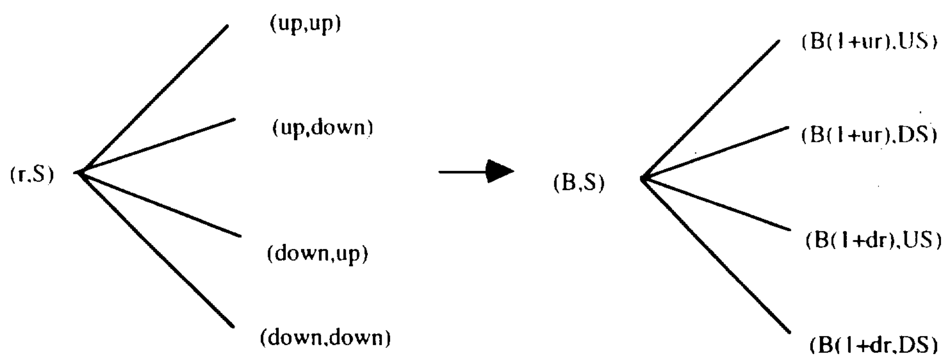
bond options when the default-free rate is stochastic. Also, Heath, Jarrow, and Morton (1990 & 1992) and Milne and Turnbull (1990) developed the similar model when the term structure of the interest rate is stochastic. They showed that there exists an equivalent martingale measure such that relative prices form a martingale when they are normalized by selected accumulation factor in the continuous time framework. In fact, they adopted no arbitrage pricing technique by choosing appropriate normalization factor and use risk-neutral valuation approach.

when the risk-neutral valuation approach is used, expected payoffs are discounted with respect to the martingale probability, which is not real. Instead, Dothan used the real probability in deriving B-S formula by using the change of the martingale technique. That is, according to classical risk neutral valuation method, the discounted prices process at the risk-free rate form a martingale so that no arbitrage opportunities are allowed and the discounted expected payoffs from any derivative securities with respect to this martingale measure are the values of the securities. Risk-neutrality assumption is merely an artificial device for obtaining the values of the derivative securities. Therefore, in order to use the nature's probabilities, some adjustments are necessary to price the derivative securities. After constructing the payoffs under nature's probability, Dothan adjust the payoffs by multiplying the likelihood ratios and finally gets the price of the derivative securities by discounting at the risk-less rate. This method provides same answer as the B-S valuation model but give us more intuition about the real world situation. In this paper, Dothan's method is used to value the option with the stochastic interest rate. By assuming very simple situation in one period model, I present a different valuation model from risk-neutral valuation formula. Although this idea is not original, by applying in a simple situation, we can get some interesting result and it can be a basis to understand the complicated continuous time model.

Simple One Period Model With no Riskless Asset

CRR lattice binomial model is modified by considering that default-free interest rate is no longer constant. Instead, it is assumed that the interest rate process forms a binomial model with the equal probability of upward and downward jump but is independent of stock price movement.



where B(1+ur) = value of bond after an up jump of default-free interest

US = stock value after an up jump

B(1+dr) = value of bond after a down jump of default-free interest rate

DS = stock value after a down jump

That is, the interest rate process is lognormally distributed in a limiting case so that riskless asset does not exist. To investigate the no arbitrage condition in this setting, I consider simple one period model. It is simple but give us more intuitive way to understand the relation between the existence of martingale measure and no arbitrage condition.

Consider two assets (stock and risky bond) with initial prices S and B. Over one period time interval, there are four situations with interest rate up or down and with stock price up or down. In this case, the market is not complete because there are only two assets whereas four independently different situations exist. Thus, even if no arbitrage conditions are met, martingale measure is not unique. In order for the above price system not to permit arbitrary strategies, there should exist martingale measures that make the risk-neutral valuation possible. If equilibrium martingale measures exist, then the initial cost of any attainable consumption is the expected value of that terminal consumption relative to any price measure, discounted at the default-free interest rate. However, because default-free interest rate is not constant, by making numeraire, we can pretend to have constant riskless rate. This is a same argument as continuous case in that relative price to riskless money market account form a martingale if the market is arbitrage free.

(That is, $\frac{S(s)}{\exp\int_0^s r(s)ds}$) forms a martingale.)

Therefore,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{US}{1+ur} & \frac{DS}{1+ur} & \frac{US}{1+dr} & \frac{DS}{1+dr} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} 1 \\ S \end{bmatrix} \text{ ----- (1)} \quad \text{and} \quad \sum_{i=1}^{4} Q_i = 1 \text{ -------- (2)}$$

Thus, equation (1) becomes

$$\frac{1}{1+ur}(UQ_1 + DQ_2) + \frac{1}{1+dr}(UQ_3 + DQ_4) = 1 \quad \text{======>} \quad E_Q\left(\frac{S_1/S_0}{1+r(s)}\right) = 1 \text{ --------- (3)}$$

where r(s) = ur or dr.

Equation (3) represents that expected rate of return during the period should equal to one if it is discounted at the interest rate of each situation.

Simple application of Dothan's Method when default-free interest rate is stochastic

In this section, time horizon is extended to T discrete multiperiod with the same setting as above economy. At each node, there are four possible states with equal probabilities 1/4. Thus, at time t, there are 4^t possible states. The notations used in this section as follows:

f_t = information structure upto time t

r_t = riskless rate during the time interval [t-1,t]

S_t = stock price at time t

B_t = risky bond price at time t ($= B_0 \prod_{i=1}^{t} (1+r_t)$)

T = time to maturity

a = exercise price of the option

w = state

E = expectation operator

Q = martingale measure that makes no arbitrage strategies

P = nature's probability of states

By assumption, the stock price represents all the information available to the traders. For $1 \le t \le T$, define

$$\mu_t = E_p\left(\frac{S_t}{S_{t-1}} \,\Omega\, f_{t-1}\right) - 1$$

$$w_t = w_{t-1} + \frac{S_t}{S_{t-1}} - E_p\left(\frac{S_t}{S_{t-1}} \,\Omega\, f_{t-1}\right)$$

where w_0 is an arbitrary constant.

That is, the stock price process, S_t is decomposed into a predictable process part $\{\mu_t\}$ and a martingale part $\{w_t\}$ using Doob Decomposition. The predictable process $\{\mu_t\}$ represents the conditional expected net rate of return of the stock during the time interval [t-1,t] , and the martingale process $\{w_t\}$ represents the innovation component of the rate of return of the stock during the time interval [t-1,t] . Then, the realized rate of return of the stock during [t-1,t] is

$$\frac{S_t}{S_{t-1}} = 1 + \mu_t + w_t - w_{t-1}$$

Let $\{z_t\}$ be the likelihood ratio for $\frac{Q}{P}$ such that $z_t = E_P(\frac{Q}{P}\Omega f_t)$ and define

$$v_t = w_t - \int_0^t \frac{d\langle w\, z\rangle_s}{z_{s-}}$$

, where $\langle w\, z\rangle$ is the predictable quadratic covariation process.

By applying Girsanov's Theorem, Dothan prices the derivative securities when default-free rate is constant. That is, the likelihood ratio process is used to change the risk-neutral martingale measure to the real probability measure. The probability measure Q is the same as Arrow-Debreu securities only if the process $\{\frac{S_t}{\prod_{s=1}^{t}(1+r_t)}\}$ is a Q-martingale.

Thus, the necessary and sufficient condition for this is that for all $1 \le t \le T$,

$$E_Q\left(\frac{1+\mu_t+v_t-v_{t-1}+\frac{\langle w\,z\rangle_t-\langle w\,z\rangle_{t-1}}{z_{t-1}}}{1+r_t}\,\Omega f_{t-1}\right)=1 \quad \text{----- (4)}$$

This is true because $\frac{S_t}{(1+r_t)^t} = S_0 \prod_{s=1}^{t} \frac{1+\mu_t+v_t-v_{t-1}+\frac{\langle w\,z\rangle_t-\langle w\,z\rangle_{t-1}}{z_{t-1}}}{1+r_t}$

Equation (4) has exactly same meaning as the equation (3).

But because an adapted process $\{S_t\}$ is a Q-martingale if and only if the process $\{z_tS_t\}$ is a P-martingale, we can use P-martingale probabilities to price the options.

Also, z_t has a representation,

$$z_t = E_P(\frac{Q}{P}\Omega f_t) = \prod_{s=1}^{t}\left[1 - \frac{\mu_s - r}{s_s^2}(w_s - w_{s-1})\right],$$

and, the martingale measure Q exists and is given by

$$Q(w) = P(w)\, z_T(w) = P(w) \prod_{t=1}^{T} [\, 1 - \frac{\mu_t - r}{s_t^2} (w_t - w_{t-1}) \,]. \quad \text{---------------- (5)}$$

Because the process $\{z_t \frac{S_t}{(1+r)^t}\}$ is a P-martingale, the likelihood ratio process z_t is also the risk adjustment process for this market. Also, the discounted stock price is a martingale relative to Q. implying,

$$E_Q (w_t - w_{t-1}\, \Omega\, f_{t-1}) = -(\mu_t - r_t)$$

That is, relative to Q, the conditional expected innovation of the rate of return on the stock equals the negative value of its risk premium. Therefore, the stock price has a representation as the stochastic exponential

$$S_t = S_0 \prod_{i=1}^{t} [\, 1 + \frac{1}{1 + r_t} (v_t - v_{t-1}) \,] \quad \text{-------------------------- (6)}$$

By combining equation (5) and (6), we can finally deliver a formula for the initial price of an option on the stock. A call option on the stock with exercise a has the initial value

$$C_0 = \frac{E_P [\max(0, S_t - a)\, z_T]}{\prod_{i=1}^{T} (1 + r_t)} = \frac{E_P [\max(0, S_t - a) \prod_{i=1}^{t} (1 - \frac{\mu - r_t}{s^2})(w_t - w_{t-1})\,]}{\prod_{i=1}^{t} (1 + r_t)} \quad \text{-------- (7)}$$

Numerical Example

In this section, I present a simple numerical example of the option pricing with stochastic interest rate. For simplicity, I assume that stochastic interest rate can be represented by simple one period lattice framework . Because each node has four different states, as I increase the subperiod , the lattice increases exponentially. For example, if we want divide the whole period into four subperiods, we need to calculate 256($= 4^4$) lattices to approximate. Thus, by trying to develop one period model, even though it is very simple, we can get intuition and easily extend to multi-period model. Two methods are presented. One is to employ Q-martingale probability for risk-neutral valuation and the other is to use nature's probability by applying the change of the martingale measure.

In this example, I assume that

$S = 50$	$s = 0.1$	$\mu = 0.1$ (drift term of stock price)
$B = 1$	$r = 8\%$	
$u = 1.1$	$d = 1/1.1 = 0.909...$	
$T = 1$	$a = 50$ (exercise price of a call option on stock)	

According to CRR method,

$$U = \exp[\, s\sqrt{h}\,] = \exp(0.1) = 1.1052$$

$$D = \exp[\, -s\sqrt{h}\,] = \exp(-0.1) = 0.9048$$

Thus, the payoffs of bond and stock are like right part of the tree in the previous page. By applying equation (2) and (3), we can get Q-martingale measures. But, in this case, the market is not complete because there are four independent states whereas there are only two independent securities. Therefore, there are infinitive number of equilibrium martingale measure. However, as long as the payoff of the option is attainable, we can determine the unique initial value of the option although we adopt the different equilibrium martingale measures. To get Q-martingale probabilities, equation (2) and (3) are applied in this payoff matrix.

$$\sum_{i=1}^{4} Q = 1 \quad \text{-------------------- (8)}$$

$$\frac{e^{0.1}}{1.088} Q_1 + \frac{e^{-0.1}}{1.088} Q_2 + \frac{e^{0.1}}{1.073} Q_3 + \frac{e^{-0.1}}{1.073} Q_4 = 1 \quad \text{-------------------- (9)}$$

One of the Q's which satisfying equation (8) an (9) is $Q_1 = 0.8$, $Q_2 = 0$, $Q_3 = 0.1$ and $Q_4 = 0.1$.

Then the initial value of the stock option can be determined by discounted expected value of the payoff of that option.

$$\frac{(55.26 - 50)}{1.088} (0.8) + \frac{(55.26 - 50)}{1.073} (0.1) = 4.6$$

since the payoff of the option is 5.26 at states w_1 and w_3 or nothing otherwise.

This payoff is always attainable because the relation among the rows of payoff matrix is

$\frac{w1}{w_3} = \frac{w_2}{w_4}$ and the payoff of the option satisfies this relation.

Now, Dothan's method is used to price the initial value of the option for the same example. Because the natures probability P measure (equally likely) is used, we need to determine **z** factors to adjust the payoffs of the option. According to equation (7), **z** factors are determined as $(1 - l_1 s)$, $(1 + l_1 s)$, $(1 - l_2 s)$, and $(1 + l_2 s)$ where

$$l_1 = \frac{(m - r_1)}{s^2} = \frac{(0.10 - 0.088)}{0.1^2} = 1.2, \text{ and } l_2 = \frac{(m - r_2)}{s^2} = \frac{(0.10 - 0.073)}{0.1^2} = 2.73$$

Thus, $z_1 = 1 - (1.2)(0.1) = 0.88$, $z_2 = 1 + (1.2)(0.1) = 1.12$, $z_3 = 1 - (2.73)(0.1) = 0.727$ and z_4

= 1 + (2.73) (0.1) = 1.273, respectively.

In this case, U = (1 + 0.1 + 0.1) = 1.2 and D = (1+ 0.1- 0.1) = 1 for nature's payoff.

Therefore, stock prices, option payoff and **z** factors under P probability measure are as follows:

S	payoffs	z factors	probabilities
50*1.2 = 60	(60 - 50) = 10	0.88	0.25
50* 1 = 50	(50- 50) = 0	1.12	0.25
50*1.2 = 60	(60- 50) = 10	0.727	0.25
50* 1 = 50	(50- 50) = 0	1.273	0.25

The initial value of the option is calculated as follows:

$$C_0 = \frac{(0.25)\,(10*0.88)}{1.088} + \frac{(0.25)\,(10*0.727)}{1.0723} = 3.72$$

Of course, the values that is calculated by different methods are not same each other. However, as we increase the number of subperiod, two values approache each other and with the infinite number of subperiod (continuous case), two values will be exactly same.

Summary

In this paper, I tried to use a different method to value the stock option in a simple discrete one-period model. Although the assumptions used here are extremely simple so that it is far from the real world and it is not so new idea , it provides us another view about securities price behavior and valuing the derivative securities. Also, by comparing with risk-neutral valuation based on Q-martingale measure (no arbitrage condition), we can have more intuitive sights of the valuation of the option. This model, I guess, can be easily extended to continuous time model and the resulting valuation formula would not be different from those of other researcher's.

The only difference is that in this paper, I did not include the bond price behavior with different maturities to avoid complexities.

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