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ON THE NORMAL FUZZY PROBABILITY

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ABSTRACT. A fuzzy set A on a probability space $(\Omega, \mathfrak{F}, P)$ is called a fuzzy event. We calculate exactly the normal fuzzy probability for some triangle fuzzy numbers. Furthermore, we study the normal probability for some operations of two triangle fuzzy numbers.

1. INTRODUCTION

The operations of two fuzzy numbers (A, μ_A) and (B, μ_B) are based on the Zadeh's extension principle([6], [7], [8]). We consider the following four operations.

1. Addition $A(+)B : \mu_{A(+)B}(z) = \sup\limits_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}$, $x \in A, y \in B$.
2. Subtraction $A(-)B : \mu_{A(-)B}(z) = \sup\limits_{z=x-y} \min\{\mu_A(x), \mu_B(y)\}$, $x \in A, y \in B$.
3. Multiplication $A(\cdot)B : \mu_{A(\cdot)B}(z) = \sup\limits_{z=x\cdot y} \min\{\mu_A(x), \mu_B(y)\}$, $x \in A, y \in B$.
4. Division $A(/)B : \mu_{A(/)B}(z) = \sup\limits_{z=x/y} \min\{\mu_A(x), \mu_B(y)\}$, $x \in A, y \in B$.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, where Ω denotes the sample space, $\mathfrak{F}$ the $\sigma-$algebra on Ω, and P a probability measure. A fuzzy set A on Ω is called a fuzzy event. Let $\mu_A(\cdot)$ be the membership function of the fuzzy event A. Then the probability of the fuzzy event A is defined by Zadeh([5]) as

$$\widetilde{P}(A) = \int_\Omega \mu_A(\omega)\, dP(\omega), \qquad \mu_A(\omega) : \Omega \to [0, 1].$$

In this paper, we define the normal fuzzy probability using the normal distribution, and then we calculate exactly the normal fuzzy probability for some triangle fuzzy numbers. Furthermore, we study the normal probabilities for the four operations of two fuzzy numbers.

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2. OPERATIONS OF TWO FUZZY NUMBERS

Let X be a set of elements, called the universe, whose elements are denoted x. Membership in a classical subset A of X is often viewed as a characteristic function μ_A from X to [0,1] such that $\mu_A(x) = 1$ iff $x \in A$, and $\mu_A(x) = 0$ iff $x \notin A$. [0,1] is called a valuation set.

Definition 2.1. If the valuation set is allowed to be the real interval [0,1], A is called a fuzzy set. $\mu_A(x)$ is to 1, the more x belongs to A.

Clearly, A is a subset of X that has no sharp boundary. A is completely characterized by the set of pairs

$$A = \{(x, \mu_A(x)), x \in X\}.$$

When X is a finite set $\{x_1, \cdots, x_n\}$, a fuzzy set A on X is expressed as

$$A = \mu_A(x_1)/x_1 + \cdots + \mu_A(x_n)/x_n = \sum_{i=1}^{n} \mu_A(x_i)/x_i.$$

When X is not finite, we write

$$A = \int_X \mu_A(x)/x.$$

Two fuzzy sets A and B are said to be equal(denoted $A = B$) if and only if $\mu_A(x) = \mu_B(x), \forall x \in X$.

Example 2.2. $X = \{1, 2, 3, 4\}$. Membership function for $A = \{\text{two or so}\}$ is given as follows; $\mu_A(1) = 0, \mu_A(2) = 1, \mu_A(3) = 0.5, \mu_A(4) = 0$, i.e., $A = 0/1 + 1/2$ $+0.5/3 + 0/4$.

Example 2.3. $X = \mathbb{R}$. Let $\mu_A(x) = \dfrac{1}{1 + (x-5)^2}$, i.e., $A = \displaystyle\int_{\mathbb{R}} \frac{1}{1 + (x-5)^2}/x$. A is a fuzzy set of real numbers clustered around 5.

Definition 2.4. The set $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$ is said to the α-cut set of fuzzy set A.

The membership function of a fuzzy set A can be expressed in terms of the characteristic functions of its α-cuts according to the formula

$$\mu_A(x) = \sup_{\alpha \in (0,1]} \min(\alpha, \mu_{A_\alpha}(x)),$$

where

$$\mu_{A_\alpha}(x) = \begin{cases} 1 & \text{iff} \quad x \in A_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily checked that the following properties hold

$$(A \cup B)_\alpha = A_\alpha \cup B_\alpha, \quad (A \cap B)_\alpha = A_\alpha \cap B_\alpha.$$

Definition 2.5. A fuzzy set A is convex if

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)), \quad \forall x_1, x_2 \in X, \ \forall \lambda \in [0,1].$$

Alternatively, a fuzzy set is convex if all α-cuts are convex.

Definition 2.6. A fuzzy number A is a convex normalized fuzzy set A of the real line $\mathbb{R}$ such that

1. It exists exactly one $x_0 \in \mathbb{R}$ such that $\mu_A(x_0) = 1$.
2. $\mu_A(x)$ is piecewise continuous.

Definition 2.7. A triangular fuzzy number is a fuzzy number A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x < a_3, \\ 0, & a_3 \leq x. \end{cases}$$

The above triangular fuzzy number is denoted by $A = (a_1, a_2, a_3)$.

Definition 2.8. The addition, subtraction, multiplication, and division of two fuzzy sets are defined as

1. Addition $A(+)B : \mu_{A(+)B}(z) = \sup\limits_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}, x \in A, y \in B.$
2. Subtraction $A(-)B : \mu_{A(-)B}(z) = \sup\limits_{z=x-y} \min\{\mu_A(x), \mu_B(y)\}, \ x \in A, y \in B.$
3. Multiplication $A(\cdot)B : \mu_{A(\cdot)B}(z) = \sup\limits_{z=x\cdot y} \min\{\mu_A(x), \mu_B(y)\}, \ x \in A, y \in B.$
4. Division $A(/)B : \mu_{A(/)B}(z) = \sup\limits_{z=x/y} \min\{\mu_A(x), \mu_B(y)\}, \ x \in A, y \in B.$

Example 2.9. Let $A = \{(2,1), (3,0.5)\}$ and $B = \{(3,1), (4,0.5)\}$.

1. Addition :

(i) If $z < 5$, since $x + y \geq 5$(for all $x \in A, y \in B$), $\mu_{A(+)B}(z) = 0$.

(ii) If $z = 5$, since $\mu_A(2) \wedge \mu_B(3) = 1 \wedge 1 = 1$, $\mu_{A(+)B}(5) = \sup\limits_{2+3}(1) = 1$.

(iii) If $z = 6$, since $\mu_A(3) \wedge \mu_B(3) = 0.5 \wedge 1 = 0.5$ and $\mu_A(2) \wedge \mu_B(4) = 1 \wedge 0.5 = 0.5$, we have $\mu_{A(+)B}(6) = \sup\limits_{3+3,2+4}(0.5, 0.5) = 0.5$.

(iv) If $z = 7$, since $\mu_A(3) \wedge \mu_B(4) = 0.5 \wedge 0.5 = 0.5$, we have $\mu_{A(+)B}(7) = \sup_{3+4}(0.5) = 0.5$.

(v) If $z > 7$, since $x + y \leq 7$(for all $x \in A, y \in B$), $\mu_{A(+)B}(z) = 0$.

Thus we have $A(+)B = \{(5,1),(6,0.5),(7,0.5)\}$.

By the same way, we have

2. Subtraction : $A(-)B = \{(-2,0.5),(-1,1),(0,0.5)\}$.
3. Multiplication : $A(\cdot)B = \{(6,1),(8,0.5),(9,0.5),(12,0.5)\}$.
4. Division : $A(/)B = \{\frac{1}{2},0.5),(\frac{2}{3},1),(\frac{3}{4},0.5),(1,0.5)\}$.

For the following two triangular fuzzy numbers

$$\mu_A(x) = \begin{cases} 0 & (x < 1) \\ x - 1 & (1 \leq x < 2) \\ -\frac{1}{2}x + 2 & (2 \leq x < 4) \\ 0 & (4 \leq x) \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0 & (x < 2) \\ \frac{1}{2}x - 1 & (2 \leq x < 4) \\ -x + 5 & (4 \leq x < 5) \\ 0 & (5 \leq x), \end{cases}$$

we calculate exactly the above four operations using $\alpha-$ cuts.

Let A_α and B_α be the α-cuts of A and B, respectively. Put $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$. Since $\alpha = a_1^{(\alpha)} - 1$ and $\alpha = -\frac{a_2^{(\alpha)}}{2} + 2$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha + 1, -2\alpha + 4]$. Since $\alpha = \frac{b_1^{(\alpha)}}{2} - 1$ and $\alpha = -b_2^{(\alpha)} + 5$, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [2\alpha + 2, -\alpha + 5]$.

1. Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [3\alpha + 3, -3\alpha + 9]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[3,9]^c$ and $\mu_{A(+)B}(x) = 1$ at $x = 6$. Therefore

$$\mu_{A(+)B}(x) = \begin{cases} 0 & (x < 3) \\ \frac{1}{3}x - 1 & (3 \leq x < 6) \\ -\frac{1}{3}x + 3 & (6 \leq x < 9) \\ 0 & (9 \leq x). \end{cases}$$

2. Subtraction :

Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [2\alpha - 4, -4\alpha + 2]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-4, 2]^c$ and $\mu_{A(-)B}(x) = 1$ at $x = -2$. Therefore

$$\mu_{A(-)B}(x) = \begin{cases} 0 & (x < -4) \\ \frac{1}{2}x + 2 & (-4 \le x < -2) \\ -\frac{1}{4}x + \frac{1}{2} & (-2 \le x < 2) \\ 0 & (2 \le x). \end{cases}$$

3. Multiplication :

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_1^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] = [2\alpha^2+4\alpha+2, 2\alpha^2-14\alpha+20]$, $\mu_{A(\cdot)B}(x) = 0$ on the interval $[2,20]^c$ and $\mu_{A(\cdot)B}(x) = 1$ at $x = 8$. Therefore

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0 & (x < 2) \\ \frac{-2+\sqrt{2x}}{2} & (2 \le x < 8) \\ \frac{7-\sqrt{9+2x}}{2} & (8 \le x < 20) \\ 0 & (20 \le x). \end{cases}$$

4. Division :

Since $A_\alpha(/)B_\alpha = [\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}] = [\frac{\alpha+1}{-\alpha+5}, \frac{-\alpha+2}{\alpha+1}]$, $\mu_{A(/)B}(x) = 0$ on the interval $[\frac{1}{5}, 2]^c$ and $\mu_{A(/)B}(x) = 1$ at $x = \frac{1}{2}$. Therefore

$$\mu_{A(/)B}(x) = \begin{cases} 0 & (x < \frac{1}{5}) \\ \frac{5x-1}{x+1} & (\frac{1}{5} \le x < \frac{1}{2}) \\ \frac{-x+2}{x+1} & (\frac{1}{2} \le x < 2) \\ 0 & (2 \le x). \end{cases}$$

3. THE NORMAL FUZZY PROBABILITY

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, and X be a random variable defined on it. Let g be a real-valued Borel-measurable function on $\mathbb{R}$. Then $g(X)$ is also a random variable.

Definition 3.1. We say that the mathematical expectation (or simply,the expectation) of $g(X)$ exists if $Eg(X)$ of the random variable $g(X)$ by

$$Eg(X) = \int_\Omega g(X(\omega))\, dP(\omega) = \int_\Omega g(X)\, dP.$$

We note that a random variable X defined on $(\Omega, \mathfrak{F}, P)$ induces a measure P_X on $\mathcal{B}$ defined by the relation $P_X(A) = P\{X^{-1}(A)\}, A \in \mathcal{B}$. Then P_X becomes a probability measure on $\mathcal{B}$ and is called the (probability) distribution of X. Suppose that $Eg(X)$ exist. Then it follow([3]) that g is also integrable over $\mathbb{R}$ with respect to P_X. Moreover, the relation

(3.1) $$\int_{\Omega} g(X)\, dP = \int_{\mathbf{R}} g(t)\, dP_X(t)$$

holds. We note that the integral on the right-hand side of (3.1) is the Lebesgue-Stieltjes integral of g with respect to P_X.

In particular, if g is continuous on $\mathbb{R}$ and $Eg(X)$ exists, we can rewrite (3.1) as follows

(3.2) $$\int_{\Omega} g(X)\, dP = \int_{\mathbf{R}} g\, dP_X = \int_{-\infty}^{\infty} g(x)\, dF(x),$$

where F is the distribution function corresponding to P_X, and the last integral is a Riemann-Stieltjes integral. An important special case of (3.2) is follows.

Let F be absolutely continuous on $\mathbb{R}$ with probability density function $f(x) = F'(x)$. Then $Eg(X)$ exists if and only if $\int_{-\infty}^{\infty} |g(x)|f(x)dx < \infty$, and in that case we have

(3.3) $$Eg(X) = \int_{-\infty}^{\infty} g(x)f(x)\, dx.$$

Example 3.2. Let the random variable X(denoted $X \sim N(m, \sigma^2)$) have the normal distribution given by the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-m)^2}{2\sigma^2}}, \quad x \in \mathbb{R},$$

where $\sigma^2 > 0$ and $m \in \mathbb{R}$. Then $E|X|^\gamma < \infty$ for every $\gamma > 0$, and we have

$$EX = m \quad \text{and} \quad E(X-m)^2 = \sigma^2.$$

The induced measure P_X is called the normal distribution.

A fuzzy set A on Ω is called a *fuzzy event*. Let $\mu_A(\cdot)$ be the membership function of the fuzzy event(set) A. Then the probability of the fuzzy event A is defined by Zadeh([1], [5]) as

$$\widetilde{P}(A) = \int_{\Omega} \mu_A(\omega)\, dP(\omega), \quad \mu_A(\omega) : \Omega \to [0, 1].$$

Definition 3.3. The normal fuzzy probability $\widetilde{P}(A)$ of a fuzzy set A on $\mathbb{R}$ is defined by

$$\widetilde{P}(A) = \int_{\mathbf{R}} \mu_A(x)\, dP_X,$$

where P_X is a normal distribution of $X \sim N(m, \sigma^2)$.

4. Main results

In this section, we calculate the normal fuzzy probability for a triangle fuzzy number and the results of the operations of two fuzzy sets calculated in section 2.

Theorem 4.1. The normal fuzzy probability $\widetilde{P}(A)$ of a triangle fuzzy number $A = (a_1, a_2, a_3)$ is

$$\begin{aligned}\widetilde{P}(A) = &\frac{m-a_1}{a_2-a_1}\Big(N(\frac{a_2-m}{\sigma}) - N(\frac{a_1-m}{\sigma})\Big)\\ &+ \frac{\sigma}{\sqrt{2\pi}(a_2-a_1)}\Big(e^{-\frac{(a_1-m)^2}{2\sigma^2}} - e^{-\frac{(a_2-m)^2}{2\sigma^2}}\Big)\\ &+ \frac{m-a_3}{a_2-a_3}\Big(N(\frac{a_3-m}{\sigma}) - N(\frac{a_2-m}{\sigma})\Big)\\ &+ \frac{\sigma}{\sqrt{2\pi}(a_2-a_3)}\Big(e^{-\frac{(a_2-m)^2}{2\sigma^2}} - e^{-\frac{(a_3-m)^2}{2\sigma^2}}\Big),\end{aligned}$$

where $N(a)$ is the standard normal distribution, that is,

$$N(a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-\frac{t^2}{2}}\,dt.$$

Proof. Since

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \le x < a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \le x < a_3, \\ 0, & a_3 \le x, \end{cases}$$

by (3.3), we have

$$\begin{aligned}\widetilde{P}(A) &= \int_{\mathbf{R}} \mu_A(x)\,dP_X \\ &= \int_{a_1}^{a_2} g_1(x)f(x)\,dx + \int_{a_2}^{a_3} g_2(x)f(x)\,dx,\end{aligned}$$

where $g_1(x) = \dfrac{x-a_1}{a_2-a_1}$, $g_2(x) = \dfrac{a_3-x}{a_3-a_2}$, and $f(x) = \dfrac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}$. Put $\frac{x-m}{\sigma} = t$, then by the change of variables,

$$\begin{aligned}\widetilde{P}(A) &= \int_{a_1}^{a_2} \frac{x-a_1}{a_2-a_1}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx + \int_{a_2}^{a_3} \frac{a_3-x}{a_3-a_2}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx \\ &= \frac{1}{\sqrt{2\pi}(a_2-a_1)}\int_{\frac{a_1-m}{\sigma}}^{\frac{a_2-m}{\sigma}} (m+\sigma t)e^{-\frac{t^2}{2}}\,dt - \frac{a_1}{\sqrt{2\pi}(a_2-a_1)}\int_{\frac{a_1-m}{\sigma}}^{\frac{a_2-m}{\sigma}} e^{-\frac{t^2}{2}}\,dt \\ &\quad + \frac{1}{\sqrt{2\pi}(a_2-a_3)}\int_{\frac{a_2-m}{\sigma}}^{\frac{a_3-m}{\sigma}} (m+\sigma t)e^{-\frac{t^2}{2}}\,dt - \frac{a_3}{\sqrt{2\pi}(a_2-a_3)}\int_{\frac{a_2-m}{\sigma}}^{\frac{a_3-m}{\sigma}} e^{-\frac{t^2}{2}}\,dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{m}{\sqrt{2\pi}(a_2-a_1)}\int_{\frac{a_1-m}{\sigma}}^{\frac{a_2-m}{\sigma}} e^{-\frac{t^2}{2}}dt + \frac{\sigma}{\sqrt{2\pi}(a_2-a_1)}\int_{\frac{a_1-m}{\sigma}}^{\frac{a_2-m}{\sigma}} te^{-\frac{t^2}{2}}dt \\
&\quad - \frac{a_1}{\sqrt{2\pi}(a_2-a_1)}\int_{\frac{a_1-m}{\sigma}}^{\frac{a_2-m}{\sigma}} e^{-\frac{t^2}{2}}dt + \frac{m}{\sqrt{2\pi}(a_2-a_3)}\int_{\frac{a_2-m}{\sigma}}^{\frac{a_3-m}{\sigma}} e^{-\frac{t^2}{2}}dt \\
&\quad + \frac{\sigma}{\sqrt{2\pi}(a_2-a_3)}\int_{\frac{a_2-m}{\sigma}}^{\frac{a_3-m}{\sigma}} te^{-\frac{t^2}{2}}dt - \frac{a_3}{\sqrt{2\pi}(a_2-a_3)}\int_{\frac{a_2-m}{\sigma}}^{\frac{a_3-m}{\sigma}} e^{-\frac{t^2}{2}}dt \\
&= \frac{m-a_1}{a_2-a_1}\Big(N(\frac{a_2-m}{\sigma}) - N(\frac{a_1-m}{\sigma})\Big) \\
&\quad + \frac{\sigma}{\sqrt{2\pi}(a_2-a_1)}\Big(e^{-\frac{(a_1-m)^2}{2\sigma^2}} - e^{-\frac{(a_2-m)^2}{2\sigma^2}}\Big) \\
&\quad + \frac{m-a_3}{a_2-a_3}\Big(N(\frac{a_3-m}{\sigma}) - N(\frac{a_2-m}{\sigma})\Big) \\
&\quad + \frac{\sigma}{\sqrt{2\pi}(a_2-a_3)}\Big(e^{-\frac{(a_2-m)^2}{2\sigma^2}} - e^{-\frac{(a_3-m)^2}{2\sigma^2}}\Big).
\end{aligned}
$$

Example 4.2. In the case of the triangular fuzzy number $A = (1, 4, 6)$, the normal fuzzy probability with respect to $X \sim N[3, 2^2]$ is 0.3965. Putting $\frac{x-3}{2} = t$,

$$
\begin{aligned}
\widetilde{P}(A) &= \int_1^4 \frac{x-1}{3}\cdot\frac{1}{2\sqrt{2\pi}}e^{-\frac{(x-3)^2}{8}}dx + \int_4^6 \frac{6-x}{2}\cdot\frac{1}{2\sqrt{2\pi}}e^{-\frac{(x-3)^2}{8}}dx \\
&= \frac{1}{3\sqrt{2\pi}}\int_{-1}^{1/2}(2t+2)e^{-\frac{t^2}{2}}dt + \frac{1}{2\sqrt{2\pi}}\int_{1/2}^{1}(3-2t)e^{-\frac{t^2}{2}}dt \\
&= \frac{2}{3\sqrt{2\pi}}\int_{-1}^{1/2} te^{-\frac{t^2}{2}}dt + \frac{2}{3\sqrt{2\pi}}\int_{-1}^{1/2} e^{-\frac{t^2}{2}}dt + \frac{3}{2\sqrt{2\pi}}\int_{1/2}^{1} e^{-\frac{t^2}{2}}dt \\
&\quad - \frac{1}{\sqrt{2\pi}}\int_{1/2}^{1} te^{-\frac{t^2}{2}}dt \\
&= \frac{2}{3\sqrt{2\pi}}(e^{-\frac{1}{2}} - e^{-\frac{1}{8}}) + \frac{2}{3}\Big(N(\frac{1}{2}) - N(-1)\Big) + \frac{3}{2}\Big(N(1) - N(\frac{1}{2})\Big) \\
&\quad - \frac{1}{\sqrt{2\pi}}(e^{-\frac{1}{8}} - e^{-\frac{1}{2}}) \\
&= \frac{5}{3\sqrt{2\pi}}(e^{-\frac{1}{2}} - e^{-\frac{1}{8}}) + \frac{3}{2}N(1) - \frac{2}{3}N(-1) - \frac{5}{6}N(\frac{1}{2}) \\
&= 0.3965.
\end{aligned}
$$

Now, we calculate the normal fuzzy probability for the four operations of two triangle fuzzy numbers. Since the two operations(Addition, Subtraction) of two triangle fuzzy numbers are triangle fuzzy number, the normal fuzzy probability of the operations can be calculated by the same way. But, in the calculating of the normal fuzzy probability for the other operations(Multiplication, Division) of two triangle fuzzy numbers, we have to calculate the integral of the following two forms.

Form 1. $\mu_{A(\cdot)B}(x) = \sqrt{ax+b}$

$$\begin{aligned}\widetilde{P} &= \int \sqrt{ax+b}\,\frac{1}{\sqrt{2\pi}\sigma}\,e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int \sqrt{ax+b}\,e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int \sqrt{ax+b}\,e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\left(-\frac{\sqrt{ax+b}\sqrt{\frac{\pi}{2}}\,\mathrm{Erf}(\frac{\sigma(m-x)}{\sqrt{2}})}{\sigma}\right),\end{aligned}$$

where

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz.$$

Form 2. $\mu_{A(/)B}(x) = \dfrac{cx+d}{ax+b} = \dfrac{p}{ax+b} + q$

$$\begin{aligned}\widetilde{P} &= \int \left(\frac{p}{ax+b} + q\right)\frac{1}{\sqrt{2\pi}\sigma}\,e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx \\ &= \frac{p}{\sqrt{2\pi}\sigma}\int \frac{e^{-\frac{(x-m)^2}{2\sigma^2}}}{ax+b}\,dx + \frac{q}{\sqrt{2\pi}\sigma}\int e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx \\ &= \frac{p}{\sqrt{2\pi}\sigma}\left(\frac{\sqrt{\frac{\pi}{2}}\,\mathrm{Erf}(\frac{\sigma(m-x)}{\sqrt{2}})}{(ax+b)\sigma}\right) + \frac{q}{\sqrt{2\pi}\sigma}\left(-\frac{\sqrt{\frac{\pi}{2}}\,\mathrm{Erf}(\frac{\sigma(m-x)}{\sqrt{2}})}{\sigma}\right).\end{aligned}$$

Example 4.3. We calculate the normal fuzzy probability for only two operations(Multiplication, Division) of two triangle fuzzy numbers calculated in section 2 with respect to $X \sim N[5,4]$.

1. Multiplication

$$\begin{aligned}\widetilde{P} &= \int \mu_{A(\cdot)B}(x)dP_X \\ &= \int_2^8 \frac{-2+\sqrt{2x}}{2}\,\frac{1}{2\sqrt{2\pi}}\,e^{-\frac{(x-5)^2}{2\cdot 4}}\,dx \\ &\quad + \int_8^{20} \frac{7-\sqrt{9+2x}}{2}\cdot\frac{1}{2\sqrt{2\pi}}\,e^{-\frac{(x-5)^2}{2\cdot 4}}\,dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{-2}{4\sqrt{2\pi}} \int_2^8 e^{-\frac{(x-5)^2}{8}} dx + \frac{1}{4\sqrt{2\pi}} \int_2^8 \sqrt{2x}\, e^{-\frac{(x-5)^2}{8}} dx \\
&\quad + \frac{7}{4\sqrt{2\pi}} \int_8^{20} e^{-\frac{(x-5)^2}{8}} dx - \frac{1}{4\sqrt{2\pi}} \int_8^{20} \sqrt{9+2x}\, e^{-\frac{(x-5)^2}{8}} dx \\
&= -\frac{1}{\sqrt{2\pi}} \int_{-\frac{3}{2}}^{\frac{3}{2}} e^{-\frac{t^2}{2}} dt + \frac{1}{4\sqrt{2\pi}} \int_2^8 \sqrt{2x}\, e^{-\frac{(x-5)^2}{8}} dx \\
&\quad + \frac{7}{2\sqrt{2\pi}} \int_{\frac{3}{2}}^{\frac{15}{2}} e^{-\frac{t^2}{2}} dt - \frac{1}{4\sqrt{2\pi}} \int_8^{20} \sqrt{9+2x}\, e^{-\frac{(x-5)^2}{8}} dx \\
&= N(-\frac{3}{2}) - N(\frac{3}{2}) + \frac{1}{4\sqrt{2\pi}} \int_2^8 \sqrt{2x}\, e^{-\frac{(x-5)^2}{8}} dx \\
&\quad + \frac{7}{2}\Big((N(\frac{15}{2}) - N(\frac{3}{2})\Big) - \frac{1}{4\sqrt{2\pi}} \int_8^{20} \sqrt{9+2x}\, e^{-\frac{(x-5)^2}{8}} dx \\
&= -0.8664 + 1.3538 + 0.2338 - 0.1727 = 0.5485.
\end{aligned}
$$

2. Division

$$
\begin{aligned}
\widetilde{P} &= \int \mu_{A(/)B}(x) dP_X \\
&= \int_{\frac{1}{5}}^{\frac{1}{2}} \frac{5x-1}{x+1} \cdot \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-5)^2}{2\cdot 4}} dx + \int_{\frac{1}{2}}^{2} \frac{-x+2}{x+1} \cdot \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-5)^2}{2\cdot 4}} dx \\
&= \frac{1}{2\sqrt{2\pi}} \int_{\frac{1}{5}}^{\frac{1}{2}} \Big(-\frac{6}{x+1} + 5\Big) e^{-\frac{(x-5)^2}{8}} dx + \frac{1}{2\sqrt{2\pi}} \int_{\frac{1}{2}}^{2} \Big(\frac{3}{x+1} - 1\Big) e^{-\frac{(x-5)^2}{8}} dx \\
&= \frac{-3}{\sqrt{2\pi}} \int_{\frac{1}{5}}^{\frac{1}{2}} \frac{e^{-\frac{(x-5)^2}{8}}}{x+1} dx + \frac{5}{2\sqrt{2\pi}} \int_{\frac{1}{5}}^{\frac{1}{2}} e^{-\frac{(x-5)^2}{8}} dx \\
&\quad + \frac{3}{2\sqrt{2\pi}} \int_{\frac{1}{2}}^{2} \frac{e^{-\frac{(x-5)^2}{8}}}{x+1} dx - \frac{1}{2\sqrt{2\pi}} \int_{\frac{1}{2}}^{2} e^{-\frac{(x-5)^2}{8}} dx \\
&= \frac{-3}{\sqrt{2\pi}} \int_{\frac{1}{5}}^{\frac{1}{2}} \frac{e^{-\frac{(x-5)^2}{8}}}{x+1} dx + \frac{5}{\sqrt{2\pi}} \int_{-\frac{12}{5}}^{-\frac{9}{4}} e^{-\frac{t^2}{2}} dt \\
&\quad + \frac{3}{2\sqrt{2\pi}} \int_{\frac{1}{2}}^{2} \frac{e^{-\frac{(x-5)^2}{8}}}{x+1} dx - \frac{1}{\sqrt{2\pi}} \int_{-\frac{9}{4}}^{-\frac{3}{2}} e^{-\frac{t^2}{2}} dt \\
&= \frac{-3}{\sqrt{2\pi}} \int_{\frac{1}{5}}^{\frac{1}{2}} \frac{e^{-\frac{(x-5)^2}{8}}}{x+1} dx + 5\Big(N(-\frac{9}{4}) - N(-\frac{12}{5})\Big) + \frac{3}{2\sqrt{2\pi}} \int_{\frac{1}{2}}^{2} \frac{e^{-\frac{(x-5)^2}{8}}}{x+1} dx \\
&\quad - \Big(N(-\frac{3}{2}) - N(-\frac{9}{4})\Big) \\
&= -0.0178 + 0.0201 + 0.0700 - 0.0546 = 0.0177.
\end{aligned}
$$

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