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Transversally conformal geometry on a Riemannian foliation

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Abstract

In this paper, we study the transversally conformal metric on a foliation. Also we study the transversal Dirac operator of transversally conformal metric.

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1 Known facts on a Riemannian foliation

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$.

We recall the exact sequence

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0$$

determined by the tangent bundle L and the normal bundle $Q = TM/L$ of $\mathcal{F}$. The assumption of g_M to be a bundle-like metric means that the induced metric g_Q on the normal bundle $Q \cong L^\perp$ satisfies the holonomy invariance condition $\overset{\circ}{\nabla} g_Q = 0$, where $\overset{\circ}{\nabla}$ is the Bott connection in Q.

For a distinguished chart $\mathcal{U} \subset M$ the leaves of $\mathcal{F}$ in $\mathcal{U}$ are given as the fibers of a Riemannian submersion $f : \mathcal{U} \to \mathcal{V} \subset N$ onto an open subset $\mathcal{V}$ of a model Riemannian manifold N.

For overlapping charts $U_\alpha \cap U_\beta$, the corresponding local transition functions $\gamma_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$ on N are isometries. Further, we denote by ∇ the canonical connection of the normal bundle Q of $\mathcal{F}$. It is defined ([6]) by

$$\nabla_X s = \begin{cases} \pi([X, Y_s]) & \forall X \in \Gamma L \\ \pi(\nabla_X^M Y_s) & \forall X \in \Gamma L^\perp, \end{cases} \tag{1.1}$$

where $s \in \Gamma Q$ and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $Q \cong L^\perp$. The connection ∇ is metric and torsion free. It corresponds to the Riemannian connection of the model space N. The curvature R^∇ of ∇ is defined by

$$R^\nabla(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \text{for } X,\ Y \in TM.$$

Since $i(X)R^\nabla = 0$ for any $X \in \Gamma L$([11]), we can define the (transversal) Ricci curvature $\rho^\nabla : \Gamma Q \to \Gamma Q$ and the (transversal) scalar curvature σ^∇ of $\mathcal{F}$ by

$$\rho^\nabla(s) = \sum_a R^\nabla(s, E_a)E_a, \quad \sigma^\nabla = \sum_a g_Q(\rho^\nabla(E_a), E_a),$$

where $\{E_a\}_{a=1,\cdots,q}$ is an orthonormal basis of Q. $\mathcal{F}$ is said to be (transversally) *Einsteinian* if the model space N is Einsteinian, that is,

$$\rho^\nabla = \frac{1}{q}\sigma^\nabla \cdot id \tag{1.2}$$

with constant transversal scalar curvature σ^∇. The *second fundamental form* of α of $\mathcal{F}$ is given by

$$\alpha(X,Y) = \pi(\nabla_X^M Y) \quad \text{for } X, Y \in \Gamma L. \tag{1.3}$$

It is trivial that α is Q-valued, bilinear and symmetric. The *mean curvature vector field* of $\mathcal{F}$ is then defined by

$$\tau = \sum_i \alpha(E_i, E_i), \tag{1.4}$$

where $\{E_i\}_{i=1,\cdots,p}$ is an orthonormal basis of L. The dual form κ, the *mean curvature form* for L, is then given by

$$\kappa(X) = g_Q(\tau, X) \quad \text{for } X \in \Gamma Q. \tag{1.5}$$

The foliation $\mathcal{F}$ is said to be *minimal* (or *harmonic*) if $\kappa = 0$.

Let $\Omega_B^r(\mathcal{F})$ be the space of all *basic r-forms*, i.e.,

$$\Omega_B^r(\mathcal{F}) = \{\phi \in \Omega^r(M) |\, i(X)\phi = 0,\ \theta(X)\phi = 0,\ \text{for } X \in \Gamma L\}.$$

The foliation $\mathcal{F}$ is said to be *isoparametric* if $\kappa \in \Omega_B^1(\mathcal{F})$. We already know that κ is closed, i.e., $d\kappa = 0$ if $\mathcal{F}$ is isoparametric ([11]). Since the exterior derivative preserves the basic forms (that is, $\theta(X)d\phi = 0$ and $i(X)d\phi = 0$ for $\phi \in \Omega_B^r(\mathcal{F})$), the restriction $d_B = d|_{\Omega_B^*(\mathcal{F})}$ is well defined. Let δ_B the adjoint operator of d_B. Then it is well-known([1,5]) that

$$d_B = \sum_a \theta_a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_a i(E_a)\nabla_{E_a} + i(\kappa_B), \tag{1.6}$$

where κ_B is the basic component of κ, $\{E_a\}$ is a local orthonormal basic frame in Q and $\{\theta_a\}$ its g_Q-dual 1-form.

The *basic Laplacian* acting on $\Omega_B^*(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B. \tag{1.7}$$

If $\mathcal{F}$ is the foliation by points of M, the basic Laplacian is the ordinary Laplacian. In the more general case, the basic Laplacian and its spectrum provide information about the transverse geometry of $(M, \mathcal{F})$([10]).

2 Curvatures of transversally conformal metrics

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. Now, we consider, for any real basic function u on M, the transversally conformal metric $\bar{g}_Q = e^{2u} g_Q$. Let $\bar{\nabla}$ be the metric and torsion free connection corresponding to $\bar{g}_Q$. Then we have the following proposition.

Proposition 2.1 *On a Riemannian foliation, we have that for $X, Y \in \Gamma TM$,*

$$\bar{\nabla}_X \pi(Y) = \nabla_X \pi(Y) + X(u)\pi(Y) + Y(u)\pi(X) - g_Q(\pi(X), \pi(Y)) grad_\nabla(u), \quad (2.1)$$

where $grad_\nabla(u) = \sum_a E_a(u) E_a$ is a transversal gradient of u and $X(u)$ is the Lie derivative of the function u in the direction of X.

Proof. Since $\bar{\nabla}$ is the metric and torsion free connection with respect to $\bar{g}_Q$ on Q, we have

$$\begin{aligned} 2\bar{g}_Q(\bar{\nabla}_X s, t) &= X\bar{g}_Q(s,t) + Y\bar{g}_Q(\pi(X), t) - Z_t \bar{g}_Q(\pi(X), s) \\ &= \bar{g}_Q(\pi[X, Y_s], t) + \bar{g}_Q(\pi[Z_t, X], s) - \bar{g}_Q(\pi[Y_s, Z_t], \pi(X)), \end{aligned}$$

where $\pi(Y_s) = s$ and $\pi(Z_t) = t$. From this formula, the proof is completed. □

Proposition 2.2 *On a Riemannian foliation $\mathcal{F}$, the curvature tensor associated with $\bar{g}_Q$ is given by*

$$\begin{aligned} R^{\bar{\nabla}}(X,Y)Z =& R^{\nabla}(X,Y)Z - g_Q(\pi(Y), Z)\nabla_X d_B u + g_Q(\pi(X), Z)\nabla_Y d_B u \\ &+ \{Y(u)Z(u) - g_Q(\pi(Y), Z)|d_B u|^2 - g_Q(\nabla_Y d_B u, Z)\}\pi(X) \\ &- \{X(u)Z(u) - g_Q(\pi(X), Z)|d_B u|^2 - g_Q(\nabla_X d_B u, Z)\}\pi(Y) \\ &+ \{X(u) g_Q(\pi(Y), Z) - Y(u) g_Q(\pi(X), Z)\} d_B u \end{aligned}$$

for $X, Y \in TM$ and $Z \in \Gamma Q$. Here $d_B u := grad_\nabla(u)$.

Proof. By long calculation with (2.1), we obtain the result. □

Lemma 2.3 *On a Riemannian foliation $\mathcal{F}$, the mean curvature form $\kappa_{\bar g}$ associated with $\bar g_Q = e^{2u} g_Q$ satisfies*

$$\kappa_{\bar g} = e^{-2u}\kappa. \tag{2.2}$$

Proof. From (1.3) and (1.4), we have

$$g_M(\tau, X) = g_M(\nabla^M_{E_i} E_i, X), \quad \forall X \in \Gamma Q, \tag{2.3}$$

where $\{E_i\}$ is an orthonormal basis of L. Let $\bar g_M = g_L + \bar g_Q$ be a transversally conformal metric of g_M. So $\bar Y = Y$ for any $Y \in L$. Hence we have

$$\begin{aligned}
\bar g_M(\tau_{\bar g}, X) &= \bar g_M(\bar\nabla^M_{\bar E_i} \bar E_i, X) = \bar g_M(\bar\nabla^M_{E_i} E_i, X)\\
&= \frac{1}{2}\{E_i \bar g_M(E_i, X) + E_i \bar g_M(X, E_i) - X \bar g_M(E_i, E_i)\\
&\quad - \bar g_M([E_i, X], E_i) - \bar g_M([E_i, X], E_i) + \bar g_M([E_i, E_i], X)\\
&= g_M(\nabla^M_{E_i} E_i, X) = g_M(\tau, X).
\end{aligned}$$

In the last equality of the above equation, we used the fact that $g_M(X,Y) = 0$ for $X \in L$, $Y \in Q$ and $g_L = \bar g_L$. Hence

$$e^{2u} g_Q(\tau_{\bar g}, X) = \bar g_M(\tau_{\bar g}, X) = g_M(\tau, X) = g_Q(\tau, X),$$

which implies $\tau_{\bar g} = e^{-2u}\tau$ and so $\kappa_{\bar g} = e^{-2u}\kappa$. □

Lemma 2.4 *On a Riemannian foliation $\mathcal{F}$, the basic Laplacian $\bar\Delta_B$ associated with $\bar g_Q = e^{-2u} g_Q$ satisfies*

$$\bar\Delta f = e^{-2u}\{\Delta_B f - (q-2) g_Q(d_B f, d_B u)\} \tag{2.4}$$

for any basic function f.

Proof. By the definition, we have

$$\bar\Delta_B f := \bar\delta_B \bar d_B f = -\sum_a \bar E_a \bar E_a(f) - \kappa_{\bar g}(f) - \sum_{a,b} \bar E_b(f) \bar g_Q(\bar\nabla_{\bar E_a} \bar E_b, \bar E_a),$$

where $\{\bar{E}_a\}$ is an orthonormal basic frame associated to $\bar{g}_Q$. Note that from (2.1)

$$\bar{\nabla}_{E_a}\bar{E}_b = \bar{E}_b(u)E_a - e^{-u}\delta_{ab}d_Bu. \tag{2.5}$$

So we have

$$\bar{\Delta}_B f = e^{-2u}\{\Delta_B f - (q-2)\sum_a E_a(f)E_a(u)\},$$

which proves (2.4). □

A direct calculation gives

$$\Delta_B(h^{-1}f) = -fh^{-2}\Delta_B h + h^{-1}\Delta_B f - 2fh^{-3}|d_Bh|^2 + 2h^{-2}g_Q(d_Bh, d_Bf). \tag{2.6}$$

From (2.4) and (2.6), we have the following corollary.

Corollary 2.5 *On a Riemannian foliation $\mathcal{F}$, we have the following. For any conformal change $\bar{g}_Q = e^{2u}g_Q = h^{\frac{4}{q-2}}g_Q$*

$$e^{2u}\bar{\Delta}_B(h^{-1}f) = h^{-1}\Delta_B f - fh^{-2}\Delta_B h. \tag{2.7}$$

The transversal Ricci curvature $\rho^{\bar{\nabla}}$ of $\bar{g}_Q = e^{2u}g_Q$ and the transversal scalar curvature $\sigma^{\bar{\nabla}}$ of $\bar{g}_Q$ are related to the transversal Ricci curvature ρ^{∇} of g_Q and the transversal scalar curvature σ^{∇} of g_Q by the following lemma.

Proposition 2.6 *On a Riemannian foliation $\mathcal{F}$, we have that for any $X \in Q$,*

$$\begin{aligned} e^{2u}\rho^{\bar{\nabla}}(X) =& \rho^{\nabla}(X) + (2-q)\nabla_X grad_{\nabla}(u) + (2-q)|grad_{\nabla}(u)|^2X \\ &+ (q-2)X(u)grad_{\nabla}(u) + \{\Delta_B u - \kappa(u)\}X. \end{aligned} \tag{2.8}$$

$$e^{2u}\sigma^{\bar{\nabla}} = \sigma^{\nabla} + (q-1)(2-q)|grad_{\nabla}(u)|^2 + 2(q-1)\{\Delta_B u - \kappa(u)\}. \tag{2.9}$$

Proof. Let $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ with the property that $(\nabla E_a)_x = 0$ for all a. Then

$$\begin{aligned} \rho^{\bar{\nabla}}(X) &= \sum_a R^{\bar{\nabla}}(X, \bar{E}_a)\bar{E}_a \\ &= \sum_a \bar{\nabla}_X\bar{\nabla}_{E_a}\bar{E}_a - \sum_a \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_X\bar{E}_a - \sum_a \bar{\nabla}_{[X,\bar{E}_a]}\bar{E}_a. \end{aligned}$$

By a direct calculation, we have

$$\begin{aligned} e^{2u}\sum_a \bar\nabla_X\bar\nabla_{\bar E_a}\bar E_a &= (1-q)\{\nabla_X grad_\nabla(u) + |grad_\nabla(u)|^2 X \\ &\quad -2X(u)grad_\nabla(u)\} + \sum_a \nabla_X\nabla_{E_a}E_a. \end{aligned}$$

Similarly,

$$\begin{aligned} e^{2u}\sum_a \bar\nabla_{\bar E_a}\bar\nabla_X\bar E_a &= \sum_a \nabla_{E_a}\nabla_X E_a + \sum_a E_aE_a(u)X \\ &\quad +\nabla_{grad_\nabla(u)}X - \sum_a g(\nabla_{E_a}X, E_a)grad_\nabla(u) \\ &\quad -\nabla_X grad_\nabla(u) - |grad_\nabla(u)|^2X - X(u)grad_\nabla(u). \end{aligned}$$

and

$$\begin{aligned} e^{2u}\sum_a \bar\nabla_{[X,\bar E_a]}\bar E_a &= \sum_a \nabla_{[X,E_a]}E_a + X(u)(q-1)grad_\nabla(u) \\ &\quad -\nabla_{grad_\nabla(u)}X + \sum_a g(\nabla_{E_a}X, E_a)grad_\nabla(u). \end{aligned}$$

Since $\Delta_B u = \delta_B d_B u = -\sum_a E_aE_a(u) + i(\kappa)d_B u$, the above equations give (2.8).

On the other hand,

$$\sigma^{\bar\nabla} = \sum_a \bar g_Q(\rho^{\bar\nabla}(\bar E_a), \bar E_a) = \sum_a g_Q(\rho^{\bar\nabla}(E_a), E_a).$$

From (2.8) we have

$$\begin{aligned} e^{2u}\sigma^{\bar\nabla} &= \sum_a g_Q(e^{2u}\rho^{\bar\nabla}(E_a), E_a) \\ &= \sigma^\nabla + (2-q)\sum_a g_Q(\nabla_{E_a}grad_\nabla(u), E_a) \\ &\quad +(q-1)(2-q)|grad_\nabla(u)|^2 + q\{\Delta_B u - \kappa(u)\}. \end{aligned}$$

Since $\sum_a g_Q(\nabla_{E_a}grad_\nabla(u), E_a) = \sum_a E_aE_a(u) = -\Delta_B u + \kappa(u)$, we have

$$e^{2u}\sigma^{\bar\nabla} = \sigma^\nabla + (q-1)(2-q)|grad_\nabla(u)|^2 + 2(q-1)\{\Delta_B u - \kappa(u)\},$$

which proves (2.9). □

Corollary 2.7 *On a Riemannian foliation $\mathcal{F}$, the scalar curvature $\sigma^{\bar\nabla}$ associated with $\bar g_Q = e^{2u} g_Q = h^{\frac{4}{q-2}} g_Q$ is simplied as*

$$h^{\frac{q+2}{q-2}}\sigma^{\bar\nabla} = 4\frac{q-1}{q-2}\{\Delta_B h - \kappa(h)\} + \sigma^\nabla h. \tag{2.10}$$

Proof. For $q \geq 3$, $u = \frac{2}{q-2}\ln h$. Hence we have

$$\Delta_B u = \frac{2}{q-2}\{h^{-2}|grad_\nabla(h)|^2 + h^{-1}\Delta_B h\}, \tag{2.11}$$

$$|grad_\nabla(u)|^2 = (\frac{2}{q-2})^2 h^{-2}|grad_\nabla(h)|^2. \tag{2.12}$$

From (2.9), the proof is completed. □

So we define the *basic Yamabe operator* Y_b by

$$Y_b = 4\frac{q-1}{q-2}\Delta_B + \sigma^\nabla. \tag{2.13}$$

Theorem 2.8 *On a Riemannian foliation $\mathcal{F}$ of codimension $q \geq 3$, the basic Yamabe operator of the transversally conformal metric satisfies the following equation: For $\bar g_Q = h^{\frac{4}{q-2}} g_Q$,*

$$\bar Y_b(h^{-1}f) = h^{\frac{-q-2}{q-2}} Y_b f - 4\frac{q-1}{q-2}h^{\frac{-2q}{q-2}}\kappa(h)f. \tag{2.14}$$

Proof. From (2.7) and (2.10), we have

$$\begin{aligned}\bar Y_b(h^{-1}f) =& 4\frac{q-1}{q-2}\bar\Delta_B(h^{-1}f) + \sigma^{\bar\nabla}(h^{-1}f)\\ =& h^{\frac{-q-2}{q-2}}\{4\frac{q-1}{q-2}\Delta_B f + \sigma^\nabla f\} - 4\frac{q-1}{q-2}h^{\frac{-2q}{q-2}}\kappa(h)f,\end{aligned}$$

which implies (2.14). □

Corollary 2.9 *On a Riemannian foliation $\mathcal{F}$ of codimension $q \geq 3$, the basic Yamabe operator of the transversally conformal metric $\bar g_Q = h^{\frac{4}{q-2}} g_Q$ such that $\kappa(h) = 0$ satisfies*

$$\bar Y_b(h^{-1}f) = h^{\frac{-q-2}{q-2}} Y_b f. \tag{2.15}$$

Definition 2.10 *For any vectors $X, Y \in TM$ and $s \in \Gamma Q$, the transversal Weyl conformal curvature tensor W^∇ is defined by*

$$\begin{aligned} W^\nabla(X,Y)s &= R^\nabla(X,Y)s \\ &+\frac{1}{q-2}\{g_Q(\rho^\nabla(\pi(X)),s)\pi(Y) - g_Q(\rho^\nabla(\pi(Y)),s)\pi(X) \\ &+g_Q(\pi(X),s)\rho^\nabla(\pi(Y)) - g_Q(\pi(Y),s)\rho^\nabla(\pi(X))\} \\ &-\frac{\sigma^\nabla}{(q-1)(q-2)}\{g_Q(\pi(X),s)\pi(Y) - g_Q(\pi(Y),s)\pi(X)\}. \end{aligned} \tag{2.16}$$

By a direct calculation, the transversal Weyl conformal curvature tensor W^∇ vanishes identically for $q = 3$, where $q = codim\mathcal{F}$. Moreover, we have the following theorem.

Theorem 2.11 *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then the transversal Weyl conformal curvature tensor is invariant under any transversally conformal change of g_M.*

Proof. By a long calculation with Proposition 2.2 and 2.6, we have that $W^{\bar\nabla} = W^\nabla$. □

3 Transversal Dirac operators of transversally conformal metrics

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. Let $S(\mathcal{F})$ be the foliated spinor bundle([4,5]) associated with $P_{spin}(\mathcal{F})$.

Proposition 3.1 ([7]) *The spinorial covariant derivative on $S(\mathcal{F})$ is given locally by:*

$$\nabla\Psi_\alpha = \frac{1}{4}\sum_{a,b} g_Q(\nabla E_a, E_b)E_a \cdot E_b \cdot \Psi_\alpha, \tag{3.1}$$

where Ψ_α *is an orthonormal basis of* S_q. *And the curvature transform* R^S *on* $S(\mathcal{F})$ *is given as*

$$R^S(X,Y)\Phi = \frac{1}{4}\sum_{a,b} g_Q(R^\nabla(X,Y)E_a, E_b)E_a \cdot E_b \cdot \Phi \quad \textit{for } X, Y \in TM. \tag{3.2}$$

where $\{E_a\}$ *is an orthonormal basis of the normal bundle* Q.

We now define a canonical section $\mathcal{R}^\nabla$ of $Hom(S(\mathcal{F}), S(\mathcal{F}))$ by the formula

$$\mathcal{R}^\nabla(\Psi) = \sum_{a<b} E_a \cdot E_b \cdot R^S(E_a, E_b)\Psi. \tag{3.3}$$

Theorem 3.2 *On the foliated spinor bundle* $S(\mathcal{F})$, *we have the following equations*

$$\mathcal{R}^\nabla = \frac{1}{4}\sigma^\nabla, \tag{3.4}$$

$$\sum_a E_a \cdot R^S(X, E_a)\Psi = -\frac{1}{2}\rho^\nabla(X) \cdot \Psi \quad \textit{for } X \in \Gamma Q. \tag{3.5}$$

Proof. From (3.2), we have

$$\begin{aligned}
\sum_a E_a R^S(X, E_a) =& \frac{1}{4}\sum_{a,b,c} g_Q(R^\nabla(X, E_a)E_b, E_c)E_aE_bE_c\} \\
=& \frac{1}{4}\sum_{a\neq b\neq c\neq a} g_Q(R^\nabla(X, E_a)E_b, E_c)E_aE_bE_c \\
&+ \frac{1}{4}\sum_{a=b,c} g_Q(R^\nabla(X, E_a)E_b, E_c)E_aE_bE_c \\
&+ \frac{1}{4}\sum_{a=c,b} g_Q(R^\nabla(X, E_a)E_b, E_c)E_aE_bE_c \\
=& \frac{1}{4}\sum_{b,c} g_Q(R^\nabla(X, E_b)E_b, E_c)E_c \\
&+ \frac{1}{4}\sum_{b,c} g_Q(R^\nabla(X, E_c)E_b, E_c)E_cE_bE_c.
\end{aligned}$$

In the above equation, the first term of the second equation zero. In fact, the first Bianchi identity implies

$$\begin{aligned}
&\sum_{a\neq b\neq c\neq a} g_Q(R^\nabla(X,E_a)E_b,E_c)E_aE_bE_c \\
&=-\sum_{a\neq b\neq c\neq a} g_Q(R^\nabla(E_b,E_c)E_a,X)E_aE_bE_c \\
&=\sum_{a\neq b\neq c\neq a}\{g_Q(R^\nabla(E_c,E_a)E_b,X)+g_Q(R^\nabla(E_a,E_b)E_c,X)\}E_aE_bE_c \\
&=2\sum_{a\neq b\neq c\neq a} g_Q(R^\nabla(E_c,E_a)E_b,X)E_bE_cE_a \\
&=2\sum_{a\neq b\neq c\neq a} g_Q(R^\nabla(E_b,E_c)E_a,X)E_aE_bE_c,
\end{aligned}$$

which implies zero. From the Clifford multiplication, we have

$$\sum_a E_aR^S(X,E_a)=-\frac{1}{2}\sum_b R^\nabla(X,E_b)E_b=-\frac{1}{2}\rho^\nabla(X).$$

The proof of (3.4) is followed by (3.5) directly. □

The transversal Dirac operator D_{tr} is locally defined ([4,5]) by

$$D_{tr}\Psi=\sum_a E_a\cdot\nabla_{E_a}\Psi-\frac{1}{2}\kappa\cdot\Psi \quad \text{for } \Psi\in\Gamma S(\mathcal{F}), \tag{3.6}$$

where $\{E_a\}$ is a local orthonormal basic frame of Q. We define the subspace $\Gamma_B(S(\mathcal{F}))$ of *basic* or *holonomy invariant* sections of $S(\mathcal{F})$ by

$$\Gamma_B(S(\mathcal{F}))=\{\Psi\in\Gamma S(\mathcal{F})|\nabla_X\Psi=0 \quad \text{for } X\in\Gamma L\}.$$

Trivially, we see that D_{tr} leaves $\Gamma_B(S(\mathcal{F}))$ invariant if and only if the foliation $\mathcal{F}$ is isoparametric, i.e., $\kappa\in\Omega_B^1(\mathcal{F})$. Let $D_b=D_{tr}|_{\Gamma_B(S(\mathcal{F}))}:\Gamma_B(S(\mathcal{F}))\to\Gamma_B(S(\mathcal{F}))$. This operator D_b is called the *basic Dirac operator* on (smooth) basic sections. On an isoparametric transverse spin foliation $\mathcal{F}$ with $\delta\kappa=0$, it is well-known([4,5]) that

$$D_{tr}^2\Psi=\nabla_{tr}^*\nabla_{tr}\Psi+\frac{1}{4}K_\sigma^\nabla\Psi, \tag{3.7}$$

where $K_\sigma^\nabla = \sigma^\nabla + |\kappa|^2$ and

$$\nabla_{tr}^*\nabla_{tr}\Psi = -\sum_a \nabla^2_{E_a,E_a}\Psi + \nabla_\kappa\Psi. \tag{3.8}$$

The operator $\nabla_{tr}^*\nabla_{tr}$ is non-negative and formally self-adjoint ([4]). Now, we consider, for any real basic function u on M, the transversally conformal metric $\bar g_Q = e^{2u}g_Q$. Let $P_{so}(\mathcal{F})$ and $\bar P_{so}(\mathcal{F})$ be the principal bundles of g_Q- and $\bar g_Q$-orthogonal frames, respectively. Locally, the section $\bar s$ of $\bar P_{so}(\mathcal{F})$ corresponding a section $s = (E_1, \cdots, E_q)$ of $P_{so}(\mathcal{F})$ is $\bar s = (\bar E_1, \cdots, \bar E_q)$, where $\bar E_a = e^{-u}E_a$ $(a = 1, \cdots, q)$. This isometry will be denoted by I_u. Thanks to the isomorphism I_u one can define a transverse spin structure $\bar P_{spin}(\mathcal{F})$ on $\mathcal{F}$ in such a way that the diagram

$$\begin{array}{ccc} P_{spin}(\mathcal{F}) & \xrightarrow{\tilde I_u} & \bar P_{spin}(\mathcal{F}) \\ \downarrow & & \downarrow \\ P_{so}(\mathcal{F}) & \xrightarrow{I_u} & \bar P_{so}(\mathcal{F}) \end{array}$$

commutes.

Let $\bar S(\mathcal{F})$ be the foliated spinor bundles associated with $\bar P_{spin}(\mathcal{F})$. For any section Ψ of $S(\mathcal{F})$, we write $\bar\Psi \equiv I_u\Psi$. If $<,>_{g_Q}$ and $<,>_{\bar g_Q}$ denote respectively the natural Hermitian metrics on $S(\mathcal{F})$ and $\bar S(\mathcal{F})$, then for any $\Phi,\ \Psi \in \Gamma S(\mathcal{F})$

$$<\Phi, \Psi>_{g_Q} = <\bar\Phi, \bar\Psi>_{\bar g_Q}, \tag{3.9}$$

and the Clifford multiplication in $\bar S(\mathcal{F})$ is given by

$$\bar X \mathbin{\bar\cdot} \bar\Psi = \overline{X\cdot\Psi} \quad \text{for } X \in \Gamma Q. \tag{3.10}$$

From (2.1), we have the following proposition.

Proposition 3.3 *The connection ∇ and $\bar\nabla$ acting respectively on the sections of $S(\mathcal{F})$ and $\bar S(\mathcal{F})$, are related, for any vector field X and any spinor field Ψ by*

$$\bar\nabla_X\bar\Psi = \overline{\nabla_X\Psi} - \frac{1}{2}\overline{\pi(X)\cdot grad_\nabla(u)\cdot\Psi} - \frac{1}{2}g_Q(grad_\nabla(u), \pi(X))\bar\Psi. \tag{3.11}$$

Proof. Let $\{E_a\}$ be an orthonormal basis of Q and denote by ω and $\bar\omega$, the connection forms corresponding to g_Q and $\bar g_Q$. That is, for any vector field $X \in TM$,

$$\nabla_X E_b = \sum_c \omega_{bc}(\pi(X))E_c, \quad \bar\nabla_X \bar E_b = \sum_c \bar\omega_{bc}(\pi(X))\bar E_c. \tag{3.12}$$

From (2.1), we have

$$\bar\omega_{bc}(\pi(X)) = \omega_{bc}(\pi(X)) + g_Q(\pi(X), E_c)E_b(u) - g_Q(\pi(X), E_b)E_c(u). \tag{3.13}$$

Let $\{\Psi_A\}(A = 1, \cdots, 2^{[\frac{q}{2}]})$ be a local frame field of $S(\mathcal{F})$. Then the spinor covariant derivative of Ψ_A is given ([7]) by

$$\nabla_X \Psi_A = \frac{1}{2}\sum_{b<c} \omega_{bc}(\pi(X))E_b \cdot E_c \cdot \Psi_A. \tag{3.14}$$

With respect to $\bar g_Q$, we have

$$\begin{aligned}
\bar\nabla_X \bar\Psi_A &= \frac{1}{2}\sum_{b<c} \bar\omega_{bc}(\pi(X))\bar E_b \bar\cdot \bar E_c \bar\cdot \bar\Psi_A \\
&= \frac{1}{2}\sum_{b<c}\{\omega_{bc}(\pi(X)) + g_Q(\pi(X), E_c)E_b(u) - g_Q(\pi(X), E_b)E_c(u)\}\bar E_b \bar\cdot \bar E_c \bar\cdot \bar\Psi_A \\
&= \overline{\nabla_X \Psi_A} - \frac{1}{2}\sum_{b\neq c} g_Q(\pi(X), E_c)E_b(u)\bar E_c \bar\cdot \bar E_b \bar\cdot \bar\Psi_A \\
&= \overline{\nabla_X \Psi_A} - \frac{1}{2}\overline{\pi(X) \cdot grad_\nabla(u) \cdot \Psi_A} - \frac{1}{2} g_Q(grad_\nabla(u), \pi(X))\bar\Psi_A. \quad \square
\end{aligned}$$

Let $\bar D_{tr}$ be the transversal Dirac operator associated with the metric $\bar g_Q = e^{2u} g_Q$ and acting on the sections of the foliated spinor bundle $\bar S(\mathcal{F})$. Let $\{E_a\}$ be a local frame of $P_{so}(\mathcal{F})$ and $\{\bar E_a\}$ a local frame of $\bar P_{so}(\mathcal{F})$.

Locally, $\bar D_{tr}$ is expressed by

$$\bar D_{tr}\bar\Psi = \sum_a \bar E_a \bar\cdot \bar\nabla_{\bar E_a}\bar\Psi - \frac{1}{2}\kappa_{\bar g} \bar\cdot \bar\Psi. \tag{3.15}$$

Using (3.10), we have that for any Ψ,

$$\bar D_{tr}\bar\Psi = e^{-u}\{\overline{D_{tr}\Psi} + \frac{q-1}{2}\overline{grad_\nabla(u) \cdot \Psi}\}. \tag{3.16}$$

Now, for any function f, we have $D_{tr}(f\Psi) = grad_\nabla(f)\cdot\Psi + fD_{tr}\Psi$. Hence we have

$$\bar{D}_{tr}(f\bar{\Psi}) = e^{-u}\overline{grad_\nabla(f)\cdot\Psi} + f\bar{D}_{tr}\bar{\Psi}. \tag{3.17}$$

From (3.15) and (3.16), we have the following proposition.

Proposition 3.4 *Let $\mathcal{F}$ be the transverse spin foliation of codimension q. Then the transverse Dirac operators D_{tr} and $\bar{D}_{tr}$ satisfy*

$$\bar{D}_{tr}(e^{-\frac{q-1}{2}u}\bar{\Psi}) = e^{-\frac{q+1}{2}u}\overline{D_{tr}\Psi} \tag{3.18}$$

for any spinor field $\Psi \in S(\mathcal{F})$.

From Proposition 3.4, if $D_{tr}\Psi = 0$, then $\bar{D}_{tr}\bar{\Phi} = 0$, where $\Phi = e^{-\frac{q-1}{2}u}\Psi$, and conversely. So we have the following corollary.

Corollary 3.5 *On the transverse spin foliation $\mathcal{F}$, the dimension of the space of the foliated harmonic spinors is a transversally conformal invariant.*

Let the mean curvature form κ of $\mathcal{F}$ be basic- harmonic, i.e., $\kappa \in \Omega^1_B(\mathcal{F})$ and $\delta_B\kappa = 0$. Then by direct calculation, we have the Lichnerowicz type formula.

Theorem 3.6 *On the transverse spin foliation with the basic harmonic mean curvature form κ, we have on $\bar{S}(\mathcal{F})$*

$$\bar{D}^2_{tr}\bar{\Psi} = \bar{\nabla}^*_{tr}\bar{\nabla}_{tr}\bar{\Psi} + \mathcal{R}^{\bar{\nabla}}(\bar{\Psi}) + K^{\bar{\nabla}}\bar{\Psi}, \tag{3.19}$$

where

$$\bar{\nabla}^*_{tr}\bar{\nabla}_{tr}\bar{\Psi} = -\sum_a \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_{\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\sum\bar{\nabla}_{\bar{E}_a}\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \tag{3.20}$$

$$K^{\bar{\nabla}} = \frac{1}{2}(q-2)\kappa_{\bar{g}}(u) + \frac{1}{4}|\bar{\kappa}|^2, \tag{3.21}$$

$$\mathcal{R}^{\bar{\nabla}}(\bar{\Psi}) = \sum_{a<b} \bar{E}_a\cdot\bar{E}_b\cdot\bar{R}^S(\bar{E}_a,\bar{E}_b)\bar{\Psi}. \tag{3.22}$$

Proof. Fix $x \in M$ and choose a local orthonormal basic frame $\{E_a\}$ satisfying $(\nabla E_a)_x = 0$ at $x \in M$. Then by definition,

$$\begin{aligned}
\bar{D}_{tr}^2 \bar{\Psi} &= \bar{D}_{tr}\{\sum_a \bar{E}_a \bar{\cdot} \bar{\nabla}_{\bar{E}_a}\bar{\Psi} - \frac{1}{2}\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi}\} \\
&= -\sum_a \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_{\bar{E}_a}\bar{\Psi} + \sum_{a<b} \bar{E}_a \bar{\cdot} \bar{E}_b \bar{\cdot} \bar{R}^S(\bar{E}_a, \bar{E}_b)\bar{\Psi} \\
&\quad + \sum_{a<b} \bar{E}_a \bar{\cdot} \bar{E}_b \bar{\cdot} \bar{\nabla}_{[\bar{E}_a,\bar{E}_b]}\bar{\Psi} + \sum_{a,b} \bar{E}_b \bar{\cdot} \bar{\nabla}_{\bar{E}_b}\bar{E}_a \bar{\cdot} \bar{\nabla}_{\bar{E}_a}\bar{\Psi} \\
&\quad -\frac{1}{2}\sum_b \bar{E}_b \bar{\cdot} (\bar{\nabla}_{\bar{E}_b}\kappa_{\bar{g}}) \bar{\cdot} \bar{\Psi} + \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi} + \frac{1}{4}\kappa_{\bar{g}} \bar{\cdot} \kappa_{\bar{g}} \bar{\cdot} \bar{\Psi}.
\end{aligned}$$

From (2.5), we have

$$\begin{aligned}
&\sum_{a<b} \bar{E}_a \bar{\cdot} \bar{E}_b \bar{\cdot} \bar{\nabla}_{[\bar{E}_a,\bar{E}_b]}\bar{\Psi} = e^{-u}\{\sum_a \overline{E_a \cdot grad_\nabla(u)} \bar{\cdot} \bar{\nabla}_{\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\overline{grad_\nabla(u)}}\bar{\Psi}\}, \\
&\sum_{a,b} \bar{E}_b \bar{\cdot} \bar{\nabla}_{\bar{E}_b}\bar{E}_a \bar{\cdot} \bar{\nabla}_{\bar{E}_a}\bar{\Psi} = -e^{-u}\{q\bar{\nabla}_{\overline{grad_\nabla(u)}}\bar{\Psi} + \sum_a \bar{E}_a \bar{\cdot} \overline{grad_\nabla(u)} \bar{\cdot} \bar{\nabla}_{\bar{E}_a}\bar{\Psi}\}, \\
&\sum_a \bar{E}_a \bar{\cdot} (\bar{\nabla}_{\bar{E}_a}\kappa_{\bar{g}}) \bar{\cdot} \bar{\Psi} = e^{-2u}\{\sum_a \overline{E_a \cdot \nabla_{E_a}\kappa \cdot \Psi} + (2-q)\kappa(u)\bar{\Psi}\}.
\end{aligned}$$

From the above equations, we have

$$\begin{aligned}
\bar{D}_{tr}^2 \bar{\Psi} &= -\sum_a \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_{\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\sum_a \bar{\nabla}_{\bar{E}_a}\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi} \\
&\quad + \sum_{a<b} \bar{E}_a \bar{\cdot} \bar{E}_b \bar{\cdot} \bar{R}^S(\bar{E}_a, \bar{E}_b)\bar{\Psi} + \frac{1}{2}(q-2)\kappa_{\bar{g}}(u)\Psi + \frac{1}{4}|\bar{\kappa}|^2\bar{\Psi}.
\end{aligned}$$

This completes the proof. □

Lemma 3.7 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then*

$$\ll \bar{\nabla}_{tr}^* \bar{\nabla}_{tr}\bar{\Psi}, \bar{\Phi} \gg_{\bar{g}_Q} = \ll \bar{\nabla}_{tr}\bar{\Psi}, \bar{\nabla}_{tr}\bar{\Phi} \gg_{\bar{g}_Q}$$

for all $\Phi, \Psi \in S(\mathcal{F})$, where $< \bar{\nabla}_{tr}\bar{\Psi}, \bar{\nabla}_{tr}\bar{\Phi} >_{\bar{g}_Q} = \sum_a < \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\nabla}_{\bar{E}_a}\bar{\Phi} >_{\bar{g}_Q}$.

Proof. Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ such that $(\nabla E_a)_x = 0$ for all a. Then we have that at x

$$\bar\nabla_{\bar E_a}\bar E_b = e^{-2u}\{E_b(u)E_a - \delta_{ab}grad_\nabla(u)\}. \tag{3.23}$$

Hence we have

$$\begin{aligned} <\bar\nabla_{tr}^*\bar\nabla_{tr}\bar\Psi,\bar\Phi>_{\bar g_Q} &= -\sum_a <\bar\nabla_{\bar E_a}\bar\nabla_{\bar E_a}\bar\Psi,\bar\Phi>_{\bar g_Q} \\ &\quad +(1-q)e^{-2u}<\bar\nabla_{grad_\nabla(u)}\bar\Psi,\bar\Phi>_{\bar g_Q} + <\bar\nabla_{\kappa_{\bar g}}\bar\Psi,\bar\Phi>_{\bar g_Q} \\ &= -\sum_a \bar E_a <\bar\nabla_{\bar E_a}\bar\Psi,\bar\Phi>_{\bar g_Q} + \sum_a <\bar\nabla_{\bar E_a}\bar\Psi,\bar\nabla_{\bar E_a}\bar\Phi>_{\bar g_Q} \\ &\quad +(1-q)e^{-2u}<\bar\nabla_{grad_\nabla(u)}\bar\Psi,\bar\Phi>_{\bar g_Q} + <\bar\nabla_{\kappa_{\bar g}}\bar\Psi,\bar\Phi>_{\bar g_Q} \\ &= -div_{\bar\nabla}(V) + \sum_a <\bar\nabla_{\bar E_a}\bar\Psi,\bar\nabla_{\bar E_a}\bar\Phi>_{\bar g_Q} + <\bar\nabla_{\kappa_{\bar g}}\bar\Psi,\bar\Phi>_{\bar g_Q}, \end{aligned}$$

where $V \in \Gamma Q\otimes\mathbb{C}$ are defined by $\bar g_Q(V,Z) = <\bar\nabla_Z\bar\Psi,\bar\Phi>_{\bar g_Q}$ for all $Z\in\Gamma Q$. The last line is proved as follows: At $x \in M$,

$$\begin{aligned} div_{\bar\nabla}(V) &= \sum_a \bar g_Q(\bar\nabla_{\bar E_a}V,\bar E_a) = \sum_a \bar E_a \bar g_Q(V,\bar E_a) - \bar g_Q(V,\sum_a\bar\nabla_{\bar E_a}\bar E_a) \\ &= \sum_a \bar E_a <\bar\nabla_{\bar E_a}\bar\Psi,\bar\Phi>_{\bar g_Q} - (1-q)e^{-2u}<\bar\nabla_{grad_\nabla(u)}\bar\Psi,\bar\Phi>_{\bar g_Q}. \end{aligned}$$

By Green's theorem on the foliated Riemannian manifold([12])

$$\int_M div_{\bar\nabla}(V)v_{\bar g} = \int_M \bar g_Q(\kappa_{\bar g},V)v_{\bar g} = \int_M <\bar\nabla_{\kappa_{\bar g}}\bar\Psi,\bar\Phi>_{\bar g_Q} v_{\bar g},$$

where $v_{\bar g}$ is the volume form associated to the metric $\bar g_M = g_L + \bar g_Q$. By integrating, we obtain our result. □

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