J. of Basic Sciences, Cheju Nat'l Univ.
20(1), 17~31, 2007

기초과학연구 제주대학교
20(1), 17~31, 2007

LINEAR OPERATORS THAT PRESERVE EXTREMES OF THE INEQUALITIES OF BOOLEAN RANKS

Seok-Zun Song and Young-Shim Kang

Department of Mathematics, Cheju National University,

Jeju 690-756, Republic of Korea

Abstract

In this paper, we construct the sets of Boolean matrix pairs. These sets are naturally occurred at the extreme cases for the Boolean rank inequalities relative to the sum of Boolean matrices. These sets were constructed with the Boolean matrix pairs which are related with the ranks of the sums and difference of two Boolean matrices or compared between their Boolean ranks and their real ranks.

That is, we construct the following 3 sets ;

$$\mathcal{S}_1(\mathcal{B}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{B})^2 \mid r_B(X+Y) = r_B(X) + r_B(Y)\};$$

$$\mathcal{S}_2(\mathcal{B}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{B})^2 \mid r_B(X+Y) = 1\};$$

$$\mathcal{S}_3(\mathcal{B}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{B})^2 \mid r_B(X+Y) = r_B(X)\};$$

For these 3 sets, we consider the linear operators that preserve them. We characterize those linear operators as $T(X) = PXQ$ or $T(X) = PX^tQ$ with appropriate invertible Boolean matrices P and Q. We also obtain the equivalent conditions for these linear operators and prove their equivalence.

Keywords: Boolean linear operator, Boolean rank, semiring, (P,Q)-operator.

AMS Subject Classifications: 06E25, 15A03, 15A04.

1 Introduction

A *semiring* $\mathcal{S}$ consists of a set $\mathcal{S}$ and two binary operations, addition and multiplication, such that: (1) $\mathcal{S}$ is a monoid under addition (identity denoted by 0); (2) $\mathcal{S}$ is a semigroup under multiplication (identity, if any, denoted by 1); (3) multiplication is distributive over addition on both sides; (4) $s0 = 0s = 0$ for all $s \in \mathcal{S}$.

A semiring is called ***antinegative*** if the zero element is the only element with an additive inverse. For example, the set of nonnegative integers is an antinegative semiring with usual addition and multiplication.

Definition 1.1. A semiring $\mathcal{S}$ is called ***Boolean*** if $\mathcal{S}$ is equivalent to a set of subsets of a given set N, the sum of two subsets is their union, and the product is their intersection. The zero element is the empty set and the identity element is the whole set N.

It is straightforward to see that a Boolean semiring is commutative and antinegative. If $\mathcal{B}$ consists of only the empty subset and N then it is called a binary Boolean algebra (or $\{0,1\}$-semiring) and is denoted by $\mathcal{B}$.

Let $\mathcal{M}_{m,n}(\mathcal{B})$ denote the set of $m \times n$ matrices with entries from the binary Boolean algebra $\mathcal{B}$. Matrix theory over semirings is an object of intensive study during the last decades, see for example [5, 6] and references therein. In particular, many authors have investigated various rank functions for matrices over Boolean algebra and their properties, see [1, 9, 10, 13]. Among the rank functions that have the most interesting applications is the well-known notion of the factor rank.

Let $\mathcal{M}_{m,n}(\mathcal{B})$ be the set of $m \times n$ Boolean matrices. Throughout we assume that $m \leq n$. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J, and O, respectively. The matrix $E_{i,j}$, called a *cell*, denotes the matrix with exactly 1, that being a 1 in the (i,j) entry. Let R_i denote the matrix whose i^{th} row is all ones and is zero elsewhere, and C_j denote the matrix whose j^{th} column is all ones and is zero elsewhere. We let $|A|$ denote the number of nonzero entries in the matrix A.

Definition 1.2. The matrix $A \in \mathcal{M}_{m,n}(\mathcal{B})$ is said to be of *Boolean rank* k ($r_{\mathcal{B}}(A) = k$) if there exist matrices $B \in \mathcal{M}_{m,k}(\mathcal{B})$ and $C \in \mathcal{M}_{k,n}(\mathcal{B})$ such that $A = BC$ and k is the smallest positive integer such that such a factorization exists. By definition the only matrix with Boolean rank equal to 0 is the zero matrix, O.

If $\mathcal{B}$ is considered as a subsemiring of a real field R then there is a real rank function $\rho(A)$ for any Boolean matrix $A \in \mathcal{M}_{m,n}(\mathcal{B})$.

Example 1.3. Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \in \mathcal{M}_{4,4}(\mathcal{B}).$$

Then $r_{\mathcal{B}}(A) = 4$ from Example 2.3.1 [4]. But $\rho(A) = 3$.

■

The above example shows that the Boolean rank and real rank of A are not equal. However, the inequality $r_{\mathcal{B}}(A) \geq \rho(A)$ always holds.

The behavior of the function ρ with respect to matrix addition is given by the following inequalities:

The *rank-sum inequalities*:

$$| \rho(A) - \rho(B) | \leq \rho(A+B) \leq \rho(A) + \rho(B);$$

Sylvester's laws:

$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min\{\rho(A), \rho(B)\},$$

where A, B are real matrices (see [7]).

Arithmetic properties of Boolean rank is restricted by the following list of inequalities established from [3] because Boolean algebra is antinegative semiring .

1. $r_B(A+B) \leq r_B(A) + r_B(B)$;

2. $r_B(A+B) \geq \begin{cases} r_B(A) & \text{if} \quad B = O \\ r_B(B) & \text{if} \quad A = O \\ 1 & \text{if} \quad A \neq O \text{ and } B \neq O \end{cases}$

If $\mathcal{B}$ is considered as a subsemiring of $\Re^+$, the positive real numbers, we have:

3. $r_B(A+B) \geq |\rho(A) - \rho(B)|$.

As was proved in [3] the inequalities $1 \sim 3$ are sharp and the best possible.

The natural question is to characterize the equality cases in the above inequalities. Even over fields this is an open problem, see [2] for more details. In section 2, we present the concrete sets of matrix pairs which come from the the extreme cases of the inequalities of Boolean ranks.

In section 3 to 5, we characterize the linear operators that preserve the sets of matrix pairs which come from the the extreme cases of the inequalities of Boolean ranks.

2 Preliminaries

Let $\mathcal{B}$ be the binary Boolean algebra. Consider following notation in order to denote sets of Boolean matrices that arise as extremal cases in the inequalities listed above:

$$\mathcal{S}_1(\mathcal{B}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{B})^2 \mid r_B(X+Y) = r_B(X) + r_B(Y)\};$$

$$\mathcal{S}_2(\mathcal{B}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{B})^2 \mid r_B(X+Y) = 1\};$$

$$\mathcal{S}_3(\mathcal{B}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{B})^2 \mid r_B(X+Y) = r_B(X)\};$$

Definition 2.1. We say an operator, T, *preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ implies that $T(X) \in \mathcal{P}$, or, if $\mathcal{P}$ is a set of ordered pairs [triples], that $(X,Y) \in \mathcal{P}$ [(X,Y,Z)] $\in \mathcal{P}$] implies $(T(X),T(Y)) \in \mathcal{P}$ [$(T(X),T(Y),T(Z)) \in \mathcal{P}$].

Definition 2.2. An operator T *strongly preserves* the set $\mathcal{P}$ if $X \in \mathcal{P}$ if and only if $T(X) \in \mathcal{P}$, or, if $\mathcal{P}$ is a set of ordered pairs [triples], that $(X,Y) \in \mathcal{P}$ [$(X,Y,Z) \in \mathcal{P}$] if and only if $(T(X),T(Y)) \in \mathcal{P}$ [$(T(X),T(Y),T(Z)) \in \mathcal{P}$].

Definition 2.3. An operator $T : \mathcal{M}_{m,n}(\mathcal{B}) \to \mathcal{M}_{m,n}(\mathcal{B})$ is called a *(P,Q)-operator* if there exist permutation matrices P and Q of appropriate orders such that $T(X) = PXQ$ for all $X \in \mathcal{M}_{m,n}(\mathcal{B})$, or, if $m = n$, $T(X) = PX^tQ$ for all $X \in \mathcal{M}_{m,n}(\mathcal{B})$, where X^t denotes the transpose of X.

Definition 2.4. A mapping $T : \mathcal{M}_{m,n}(\mathcal{B}) \to \mathcal{M}_{m,n}(\mathcal{B})$ is called a *Boolean linear operator* if $T(O_{m,n}) = O_{m,n}$ and $T(X+Y) = T(X) + T(Y)$ for all $X, Y \in \mathcal{M}_{m,n}(\mathcal{B})$.

Definition 2.5. A matrix $A \in \mathcal{M}_{m,n}(\mathcal{B})$ is called *monomial* if it has exactly one nonzero element in each row and column.

Definition 2.6. A *line* of a matrix A is a row or a column of the matrix A.

Definition 2.7. We say that the matrix A *dominates* the matrix B if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$.

Definition 2.8. If A and B are Boolean matrices and $A \geq B$ we let $A \backslash B$ denote the matrix C where

$$c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} = 1 \\ 1 & \text{if } b_{i,j} = 0 \end{cases}.$$

Definition 2.9. The matrix $X \circ Y$ denotes the *Hadamard* or *Schur product*, i.e., the (i,j) entry of $X \circ Y$ is $x_{i,j}y_{i,j}$.

Lemma 2.10. *Let $A = (a_{i,j}) \in \mathcal{M}_{m,n}(\mathcal{B})$ where $m, n \geq 2$. Let (l, k) be any fixed pair of integers such that $2 \leq k \leq n$, $2 \leq l \leq m$. Assume that Boolean rank of each $l \times k$-submatrix of A is 1. Then the Boolean rank of each $(l+1) \times k$-submatrix (if any) is 1 and the Boolean rank of each $l \times (k+1)$-submatrix (if any) is 1.*

Proof. Let us consider any $l \times (k+1)$-submatrix of the matrix A. Applying a permutation of rows and columns, if necessary, it is possible to assume that this submatrix has the form $A' = (a_{i,j})$, where $1 \leq i \leq l$, $1 \leq j \leq k+1$. Let us denote $A_1 = (a_{i,j})$, where $1 \leq i \leq l$, $1 \leq j \leq k$, $A_2 = (a_{i,j})$, where $1 \leq i \leq l$, $2 \leq j \leq k+1$. By conditions, there are four vectors $\mathbf{s} = (s_1, \ldots, s_l) \in \mathcal{B}^l$, $\mathbf{t} = (t_1, \ldots, t_k) \in \mathcal{B}^k$, $\mathbf{u} = (u_1, \ldots, u_l) \in \mathcal{B}^l$, $\mathbf{v} = (v_1, \ldots, v_k) \in \mathcal{B}^k$ such that $A_1 = \mathbf{s}^t\,\mathbf{t}$ and $A_2 = \mathbf{u}^t\,\mathbf{v}$.

Consider the matrix $A'' = \mathbf{s}^t\,(t_1, t_2, \ldots, t_k, u_1 v_k)$. Let us check that $A' = A''$. The first k columns of these two matrices are equal by definitions of vectors $\mathbf{s}$ and $\mathbf{t}$. Consider the last column.

We have

$$a''_{i,k+1} = s_i u_1 v_k = \begin{cases} 0 & \text{if } s_i = 0 \\ u_1 v_k & \text{if } s_i = 1 \end{cases}.$$

i) If $s_i = 0$, $a_{i,k+1} = u_1 v_k = s_i t_{k+1} = 0$.

ii) If $s_i = 1$, $a_{i,k+1} = u_i v_k = u_1 v_k$.

(For all i, j, $s_i t_j = u_i v_{j-1}$ and $s_i = 1$, then $t_j = u_i v_{j-1}$.

That is, $t_j = u_1 v_{j-1}$, $t_j = u_2 v_{j-1}$, $\cdots$, $t_j = u_n v_{j-1}$. i.e. $u_1 v_{j-1} = u_i v_{j-1}$ ($\forall$ i)

Thus $u_1 v_k = u_i v_k$).

Thus $a''_{i,k+1} = a_{i,k+1}$.

i.e., $A' = A''$. Thus $r_B(A') = 1$. Similar considerations with an $(l+1) \times k$-matrix conclude the proof. ∎

The following two corollaries are straightforward.

Corollary 2.11. *Let $A = (a_{i,j}) \in \mathcal{M}_{m,n}(\mathcal{B})$ where $m, n \geq 2$. Let $r_B(A') = 1$ for any 2×2-submatrix A' of A. Then $r_B(A) = 1$.*

Proof. By Lemma 2.10. ∎

Corollary 2.12. *Let $A = (a_{i,j}) \in \mathcal{M}_{m,n}(\mathcal{B})$ where $m, n \geq 2$. Let $r_B(A) > 1$. Then there exists a 2×2-submatrix of A of Boolean rank 2.*

Proof. By Corollary 2.11. ∎

The following theorem implies the characterizations of the surjective linear operator on $\mathcal{M}_{m,n}(\mathcal{B})$.

Theorem 2.13. *Let $T : \mathcal{M}_{m,n}(\mathcal{B}) \to \mathcal{M}_{m,n}(\mathcal{B})$ be a Boolean linear operator. Then the following are equivalent:*

1. *T is bijective.*

2. *T is surjective.*

3. *There exists a permutation σ on $\{(i,j) \mid i = 1, 2, \cdots, m; j = 1, 2, \cdots, n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$.*

Proof. That 1) implies 2) and 3) implies 1) is straight forward. We now show that 2) implies 3).

We assume that T is surjective. Then, for any pair (i,j), there exists some X such that $T(X) = E_{i,j}$. Clearly $X \neq O$ by the linearity of T. Thus there is a pair of indices (r,s) such that $X = E_{r,s} + X'$ where (r,s) entry of X' is zero and $T(E_{r,s}) \neq O$. Indeed, if $T(E_{r,s}) = O$ for all pairs (r,s), then $T(X) = O$ by linearity of T. Thus we have a contradiction. But $T(X) = E_{i,j} \neq O$. Hence

$$T(E_{r,s}) \leq T(E_{r,s}) + T(X \setminus (E_{r,s})) = T(X) = E_{i,j}.$$

That is, $T(E_{r,s}) \leq E_{i,j}$ and $T(E_{r,s}) = E_{i,j}$. Since the set $\{(i,j) \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\}$ is a finite set, T is injective since it is surjective.

Therefore there is some permutation σ on $\{(i,j) \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$. ■

Henceforth we will always assume that $m,n \geq 2$.

Lemma 2.14. *Let* $T : \mathcal{M}_{m,n}(\mathcal{B}) \to \mathcal{M}_{m,n}(\mathcal{B})$ *be a Boolean operator which maps lines to lines and is defined by* $T(E_{i,j}) = E_{\sigma(i,j)}$, *where* σ *is a permutation on the set* $\{(i,j) \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\}$. *Then* T *is a* (P,Q)*-operator.*

Proof. Since no combination of a rows and b columns can dominate J where $a+b=m$ unless $b=0$ (or if $m=n$, if $a=0$) we have that either the image of each row is a row and the image of each column is a column, or $m=n$ and the image of each row is a column and the image of each column is a row. Thus, there are permutation matrices P and Q such that $T(R_i) \leq PR_iQ$ and $T(C_j) \leq PC_jQ$ or, if $m=n$, $T(R_i) \leq P(R_i)^tQ$ and $T(C_j) \leq P(C_j)^tQ$. Since each cell lies in the intersection of a row and a column and T maps nonzero cells to nonzero (weighted) cells, it follows that $T(E_{i,j}) = PE_{i,j}Q$, or, if $m=n$, $T(E_{i,j}) = PE_{j,i}Q = P(E_{i,j})^tQ$. ■

Lemma 2.15. *If $T(X) = X \circ A$ for all $X \in \mathcal{M}_{m,n}(\mathcal{B})$ and $r_{\mathcal{B}}(A) = 1$ then there exist diagonal matrices D and E such that $T(X) = DXE$ for all $X \in \mathcal{M}_{m,n}(\mathcal{B})$.*

Proof. If $r_{\mathcal{B}}(A) = 1$ then there exist vectors $\vec{\mathbf{d}} = [d_1, d_2, \cdots, d_m]$ and $\vec{\mathbf{e}} = [e_1, e_2, \cdots, e_n]$ such that $A = \vec{\mathbf{d}}^t\vec{\mathbf{e}}$ or $a_{i,j} = d_i e_j$. Let $D = diag\{d_1, d_2, \cdots, d_m\}$ and $E = diag\{e_1, e_2, \cdots, e_n\}$. Now the (i,j) entry of $T(X)$ is $x_{i,j}a_{i,j}$ and the (i,j) entry of DXE is $d_i x_{i,j} e_j = d_i e_j x_{i,j} = a_{i,j} x_{i,j}$. Thus the lemma follows. ∎

3 Linear preservers of $\mathcal{S}_1(\mathcal{B})$.

Recall that

$$\mathcal{S}_1(\mathcal{B}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{B})^2 \mid r_{\mathcal{B}}(X+Y) = r_{\mathcal{B}}(X) + r_{\mathcal{B}}(Y)\};$$

We begin with some general observations on Boolean linear operators of special types that preserve $\mathcal{S}_1(\mathcal{B})$.

Lemma 3.1. *Let σ be a permutation of the set $\{(i,j) \mid 1 \le i \le m, 1 \le j \le n\}$, and $T : \mathcal{M}_{m,n}(\mathcal{B}) \to \mathcal{M}_{m,n}(\mathcal{B})$ be defined by $T(E_{i,j}) = E_{\sigma(i,j)}$, $i = 1, \cdots, m; j = 1, \cdots, n$. If T preserves $\mathcal{S}_1(\mathcal{B})$, then T is a (P,Q)-operator.*

Proof. Consider the action of T on rows and columns of a matrix. Suppose that the image of two cells are in the same line, but the cells are not, say E, F then $r_{\mathcal{B}}(E+F) = 2$. If $r_{\mathcal{B}}(T(E+F)) = 1$, then $(E,F) \in \mathcal{S}_1(\mathcal{B})$ but $(T(E), T(F)) \notin \mathcal{S}_1(\mathcal{B})$. Then T does not preserve $\mathcal{S}_1(\mathcal{B})$ which is a contradiction. Thus T maps lines to lines. By Lemma 2.14 T is a (P,Q)-operator. ∎

Theorem 3.2. *Let $T : \mathcal{M}_{m,n}(\mathcal{B}) \to \mathcal{M}_{m,n}(\mathcal{B})$ be a surjective Boolean linear operator. Then T preserves $\mathcal{S}_1(\mathcal{B})$ if and only if T is a (P,Q)-operator.*

Proof. It is easy to see that multiplication with invertible matrices preserves Boolean rank, since permutation matrices are the only invertible Boolean matrices [9]. Hence (P,Q)-operator preserve the Boolean rank. For arbitrary $(X,Y) \in \mathcal{S}_1(\mathcal{B})$,

$$r_B(T(X)+T(Y)) = r_B(T(X+Y)) = r_B(P(X+Y)Q) = r_B(X+Y)$$

$$= r_B(X)+r_B(Y) = r_B(PXQ)+r_B(PYQ) = r_B(T(X))+r_B(T(Y)).$$

Thus $(T(X),T(Y)) \in \mathcal{S}_1(\mathcal{B})$ and T preserves $\mathcal{S}_1(\mathcal{B})$.

Conversely, if T is surjective then by Theorem 2.13 we have that T is defined by a permutation σ on the set $\{(i,j) \mid 1 \le i \le m, 1 \le j \le n\}$. i.e. $T(E_{i,j}) = E_{\sigma(i,j)}$.

By Lemma 3.1 we have that T is a (P,Q)-operator since T preserves $\mathcal{S}_1(\mathcal{B})$. ∎

Over a binary Boolean algebra the assumption of surjectivity from the previous theorem can be replaced with the assumption that T is a strong preserver.

Theorem 3.3. *Let $T : \mathcal{M}_{m,n}(\mathcal{B}) \to \mathcal{M}_{m,n}(\mathcal{B})$ be a Boolean linear operator that strongly preserves $\mathcal{S}_1(\mathcal{B})$. Then T is a (P,Q)-operator.*

Proof. It is proved in [4] that for a binary Boolean algebra there is a power of T which is idempotent. Thus only finite set of different matrices can be obtained by considering the powers of the matrix A. Hence, there are integers s and t such that for all $p,q > s$, $p \equiv q (\mathrm{mod}\, t)$ it holds that $A^p = A^q$. Thus $A^{st} = A^{2st}$. Hence for a certain power of any Boolean linear operator on binary Boolean algebra is idempotent. In both cases we denote $L = T^d$ and $L^2 = L$. One can easily check that L strongly preserves $\mathcal{S}_1(\mathcal{B})$.

If $X \in \mathcal{M}_{m,n}(\mathcal{B})$ and $(X,X) \in \mathcal{S}_1(\mathcal{B})$ then $r_B(X+X) = r_B(X)+r_B(X)$. Therefore $r_B(X) = 0$ and $X = O$.

Thus, if $A \ne O$ then we have that $(A,A) \notin \mathcal{S}_1(\mathcal{B})$. Then $(L(A),L(A)) \notin \mathcal{S}_1(\mathcal{B})$.

That is, $r_B(L(A)) + r_B(L(A)) \ne r_B(L(A))$. i.e.$L(A) \ne O$.

We examine the action of L on rows and columns. Suppose that $L(R_i)$ is not dominated by R_i. Then there is some (r,s) such that $E_{r,s} \leq L(R_i)$ while $E_{r,s} \nleq R_i$. Then we have that $(R_i, E_{r,s}) \in \mathcal{S}_1(\mathcal{B})$ and there exists a matrix $X = (x_{i,j}) \in \mathcal{M}_{m,n}(\mathcal{B})$ with $x_{r,s} = 0$ such that $L(R_i) = E_{r,s} + X$. Now,

$$L(R_i + E_{r,s}) = L(R_i) + L(E_{r,s}) = L(L(R_i)) + L(E_{r,s})$$

$$= L((E_{r,s} + X)) + L(E_{r,s}) = L(X) + L(E_{r,s}) + L(E_{r,s})$$

$$= L(X) + L(E_{r,s}) = L(X + E_{r,s}) = L(L(R_i)) = L(R_i).$$

Now, $(R_i, E_{r,s}) \in \mathcal{S}_1(\mathcal{B})$ but,

$$L(R_i) + L(E_{r,s}) = L(R_i + E_{r,s}) = L(R_i)$$

and hence, $(L(R_i), L(E_{r,s})) \notin \mathcal{S}_1(\mathcal{B})$, a contradiction.

We have established that $L(R_i) \leq R_i$ for all i. Similarly, $L(C_j) \leq C_j$ for all j. By considering that $E_{i,j}$ is dominated by both R_i and C_j we have that $L(E_{i,j}) \leq E_{i,j}$. Since $\mathcal{B}$ is a binary Boolean algebra, we have that T also maps a cell to a cell, or $|T(E_{i,j})| = 1$ for all i, j, and $T(J)$ has all nonzero entries.

So T induces a permutation σ, on the set of subscripts $\{1, 2, \cdots, m\} \times \{1, 2, \cdots, n\}$. That is, $T(E_{i,j}) = E_{\sigma(i,j)}$. Since T induces a permutation σ, on the set of subscripts $\{1, 2, \cdots, m\} \times \{1, 2, \cdots, n\}$ and T preserve $\mathcal{S}_1(\mathcal{B})$.

By Lemma 3.1 we have that T is a (P, Q)-operator. ■

4 Linear preservers of $\mathcal{S}_2(\mathcal{B})$.

Recall that

$$\mathcal{S}_2(\mathcal{B}) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{B})^2 \mid r_B(X+Y) = 1\};$$

Theorem 4.1. *Let $T : \mathcal{M}_{m,n}(\mathcal{B}) \to \mathcal{M}_{m,n}(\mathcal{B})$ be a surjective Boolean linear operator. Then T preserves $\mathcal{S}_2(\mathcal{B})$ if and only if T is a (P,Q)-operator.*

Proof. Let T be a (P,Q)-operator. For $(X,Y) \in \mathcal{S}_2(\mathcal{B})$, Since

$$1 = r_B(X+Y) = r_B(P(X+Y)Q) = r_B(T(X+Y)) = r_B(T(X)+T(Y)).$$

Hence $(T(X),T(Y)) \in \mathcal{S}_2(\mathcal{B})$. That is, T preserves $\mathcal{S}_2(\mathcal{B})$.

Conversely, assume that T preserves $\mathcal{S}_2(\mathcal{B})$. Hence if T is surjective and $\mathcal{B}$ is a binary Boolean algebra then by Theorem 2.13 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$. It is easy to see that the cells $E_{i,j}$ and $E_{r,s}$ are in the same line if and only if $r_B(E_{i,j}+E_{r,s}) = 1$ if and only if $(E_{i,j}, E_{r,s}) \in \mathcal{S}_2(\mathcal{B})$. Since T preserves $\mathcal{S}_2(\mathcal{B})$, if $(E_{i,j}, E_{r,s}) \in \mathcal{S}_2(\mathcal{B})$, then

$$(T(E_{i,j}), T(E_{r,s})) \in \mathcal{S}_2(\mathcal{B}).$$

That is,

$$r_B(T(E_{i,j}) + T(E_{r,s})) = 1.$$

Therefore $T(E_{i,j})$ and $T(E_{r,s})$ are in the same line. Thus lines are mapped to lines, and we have that T is a (P,Q)-operator by Lemma 2.14. ∎

We have another characterization of the linear operators that preserve $\mathcal{S}_2(\mathcal{B})$.

Theorem 4.2. *Let $T : \mathcal{M}_{m,n}(\mathcal{B}) \to \mathcal{M}_{m,n}(\mathcal{B})$ be a Boolean linear operator that preserves $\mathcal{S}_2(\mathcal{B})$. Then these are equivalent :*

1. *T is surjective*

2. T strongly preserves $\mathcal{S}_2(\mathcal{B})$

3. T is a (P, Q)-operator.

Proof. 3) implies 1) : For any $A \in \mathcal{M}_{m,n}(\mathcal{B})$, take $P^tAQ^t \in \mathcal{M}_{m,n}(\mathcal{B})$. Then $T(P^tAQ^t) = P(P^tAQ^t)Q = A$.

3) implies 2) : For any $(X, Y) \in \mathcal{S}_2(\mathcal{B})$. Since

$$1 = r_B(X+Y) = r_B(P(X+Y)Q) = r_B(T(X+Y)) = r_B(T(X)+T(Y)).$$

1) implies 3) : From Theorem 4.1, we have done.

2) implies 1) : Suppose that T strongly preserves $\mathcal{S}_2(\mathcal{B})$. In order to prove this it suffices to check that for each pair of indices (i, j) there exist $Y \in \mathcal{M}_{m,n}(\mathcal{B})$ such that $T(Y) = E_{i,j}$. Assume that this is not the case. Then $T(J) < J$. That is there exists a Boolean matrix N such that $n_{r,s} = 0$ for some (r, s) and $T(N) \geq T(J)$. Hence $T(J\backslash E_{r,s}) = T(J)$.

One has that $(J\backslash E_{r,s}, J\backslash E_{r,s}) \notin \mathcal{S}_2(\mathcal{B})$ since $rank(J\backslash E_{r,s}) \neq 1$. While $(J, J) \in \mathcal{S}_2(\mathcal{B})$, since $r_B(J) = 1$. Hence, $(T(J\backslash E_{r,s}), T(J\backslash E_{r,s})) \notin \mathcal{S}_2(\mathcal{B})$ while $(T(J), T(J)) \in \mathcal{S}_2(\mathcal{B})$, a contradiction with $T(J) = T(J\backslash E_{r,s})$. Thus there is no such a matrix N with a zero entry such that $T(N) \geq T(J)$. It follows that the image of a cell dominates only one cell. Thus T is surjective on $\mathcal{M}_{m,n}(\mathcal{B})$. ■

5 Linear preservers of $\mathcal{S}_3(\mathcal{B})$.

Recall that

$$\mathcal{S}_3(\mathcal{B}) = \{(X, Y) \in \mathcal{M}_{m,n}(\mathcal{B})^2 \mid r_B(X+Y) = r_B(X)\};$$

Theorem 5.1. *Let $T : \mathcal{M}_{m,n}(\mathcal{B}) \rightarrow \mathcal{M}_{m,n}(\mathcal{B})$ be a surjective Boolean linear operator. Then T preserves $\mathcal{S}_3(\mathcal{B})$ if and only if T is a (P, Q)-operator.*

Proof. One can easily see that (P, Q)-operators preserve the set $\mathcal{S}_3(\mathcal{B})$:

For $(X, Y) \in \mathcal{S}_3(\mathcal{B})$, we have $r_B(X+Y) = r(X)$. Using T on both sides, $r_B(P(X+Y)Q) = r_B(PXQ)$. Then

$$r_B(T(X+Y)) = r_B(T(X)).$$

That is,

$$r_B(T(X)+T(Y)) = r_B(T(X)).$$

Conversely, let T preserve $\mathcal{S}_3(\mathcal{B})$. If T is surjective and $\mathcal{B}$ is a binary Boolean algebra then by Theorem 2.13 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$. It is easy to see that the cells $E_{i,j}$ and $E_{r,s}$ are in the same line if and only if $r_B(E_{i,j}+E_{r,s}) = r_B(E_{i,j})$ if and only if $(E_{i,j}, E_{r,s}) \in \mathcal{S}_3(\mathcal{B})$. Since T preserves $\mathcal{S}_3(\mathcal{B})$ and $(E_{i,j}, E_{r,s}) \in \mathcal{S}_3(\mathcal{B})$, we have $(T(E_{i,j}), T(E_{r,s})) \in \mathcal{S}_3(\mathcal{B})$. That is,

$$r_B(T(E_{i,j})+T(E_{r,s})) = r_B(T(E_{i,j})).$$

Therefore $T(E_{i,j})$ and $T(E_{r,s})$ are in the same line. Thus lines are mapped to lines, and we have that T is a (P, Q)-operator by Lemma 2.14. ∎

References

[1] D. A. Gregory, N. J. Pullman, Semiring rank: Boolean rank and nonnegative rank factorization, *J. Combin. Inform. System Sci.* 8 (1983), 223-233.

[2] G. Marsaglia, P. Styan, Equalities and inequalities for ranks of matrices, *Linear and Multilinear Algebra*, **2** (1974), 269-292.

[3] L. B. Beasley, A. E. Guterman, Rank inequalities over semirings, J.Korean Math. Soc. 42(2) (2005), 223-242.

[4] L. B. Beasley, D. A. gregory and N. J. Pullman, Nonnegative rank-preserving operators, *Linear Algebra Appl.* **65** (1985), 207-223.

[5] K. Glazek, *A Guide to the Literature on Semirings and their Applications in Mathematics and Information Sciences*, Kluwer Academic Publishers, (2002).

[6] K. H. Kim, *Boolean Matrix Theory and Applications*, Pure and Applied Mathematics, V.70, Marcel Dekker, New York, (1982).

[7] L. B. Beasley, A. E. Guterman, C. L. Neal, Linear preservers for Sylvester and Frobenius bounds on matrix rank, *Rocky Mountains J. of Math.*,36(2006), 67-80.

[8] L. B. Beasley, S.-G. Lee, S.-Z. Song, Linear operators that preserve pairs of matrices which satisfy extreme rank properties, *Linear Algebra Appl.*, **350**(2002), 263-272.

[9] L. B. Beasley, S.-G. Lee, S.-Z. Song, Linear operators that preserve zero-term rank of Boolean matrices, *J. Korean Math. Soc.*, 36, (1999), 1181-1190.

[10] L. B. Beasley, N. J. Pullman, Semiring rank versus column rank, *Linear Algebra Appl.* **101** (1988), 33-48.

[11] S.-G Hwang and S.-Z. Song, Spanning column ranks and there preservers of nonnegative matrices, *Linear Algebra Appl.*, 254, (1997), 485-495.

[12] P. Pierce and others, A Survey of Linear Preserver Problems, *Linear and Multilinear Algebra*, **33** (1992), 1-119.

[13] V. L. Watts, Boolean rank of Kronecker products, *Linear Algebra Appl.*, 336, (2001), 261-264.