J. of Basic Sciences, Cheju Nat'l Univ.
20(2), 29~39, 2007

기초과학연구 제주대학교
20(2), 29~39, 2007

# Regular matrices and their invertible preservers over chain semirings

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**Abstract**

An $n \times n$ matrix $A$ over a chain semiring $\mathbb{K}$ is called *regular* if there exists an $n \times n$ matrix $G$ over $\mathbb{K}$ such that $AGA = A$. We study the problem of characterizing those invertible linear operators $T$ on the matrices over $\mathbb{K}$ such that $T(X)$ is regular if and only if $X$ is regular.

*Keywords*: Semiring; chain semiring, generalized inverse of a matrix; regular matrix; linear operator.

*AMS Subject Classifications*: 15A04, 15A09, 16Y60.

## 1 Introduction

A *semiring* ([4]) consists of a set $\mathbb{S}$ and two binary operations on $\mathbb{S}$, addition(+) and multiplication($\cdot$), such that:

(1) $(\mathbb{S}, +)$ is an Abelian monoid (identity denoted by 0);

(2) $(\mathbb{S}, \cdot)$ is a monoid (identity denoted by 1);

(3) multiplication distributes over addition;

(4) $s \cdot 0 = 0 \cdot s = 0$ for all $s \in \mathbb{S}$; and

(5) $1 \neq 0$.

Usually $\mathbb{S}$ denotes both the semiring and the set. Thus all rings with identity are semirings.

One of the most active and fertile subjects in matrix theory during the past one hundred years is the linear preserver problem, which concerns the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. Although the linear operators concerned are mostly linear operators on matrix spaces over some fields or rings, the same problem has been extended to matrices over various semirings([2, 10] and therein).

Regular matrices play a central role in the theory of matrices, and they have many applications in network and switching theory and information theory([3, 4, 7]). Recently, a number of authors have studied characterizations of regular matrices over semirings([1, 3, 4, 7, 8]). But there are no known results on characterizing those linear operators that (strongly) preserve regular matrices over semirings.

In this paper, we study the invertible linear operators that preserve regular matrices over chain semiring including the Boolean algebra and the fuzzy scalars.

# 2 Preliminaries

Let $\mathbb{B} = \{0, 1\}$, then $(\mathbb{B}, +, \cdot)$ is a semiring (the *Boolean algebra*) if

$$0 + 0 = 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0 \quad \text{and} \quad 1 + 1 = 1 \cdot 1 = 1.$$

Let $\mathbb{K}$ be any set of two or more elements. If $\mathbb{K}$ is totally ordered by $<$ (i.e., $x < y$ or $y < x$ for all distinct elements $x, y \in \mathbb{K}$), then define $x + y$ and $xy$ as

$$x + y = \max(x, y) \quad \text{and} \quad xy = \min(x, y)$$

for all $x, y \in \mathbb{K}$. If $\mathbb{K}$ has a universal lower bound and a universal upper bound, then $\mathbb{K}$ becomes a semiring, and called a *chain semiring*. The following are interesting examples of a chain semiring.

Let $\mathbb{H}$ be any nonempty family of sets nested by inclusion, $0 = \cap_{x\in\mathbf{H}} x$, and $1 = \cup_{x\in\mathbf{H}} x$. Then $\mathbb{S} = \mathbb{H} \cup \{0,1\}$ is a chain semiring.

Let $\alpha, w$ be real numbers with $\alpha < w$. Define $\mathbb{S} = \{\beta \in \mathbb{R} : \alpha \leq \beta \leq w\}$. Then $\mathbb{S}$ is a chain semiring with $\alpha = 0$ and $w = 1$. It is isomorphic to the chain semiring in the previous example with $\mathbb{H} = \{[\alpha, \beta] : \alpha \leq \beta \leq w\}$. Furthermore, if we choose the real numbers 0 and 1 as $\alpha$ and $w$ in the previous example, then $\mathbb{F} \equiv \{\beta : 0 \leq \beta \leq 1\}$ is called *fuzzy semiring.*

In particular, if we take $\mathbb{H}$ to be a singleton set, say $\{a\}$, and denote $\emptyset$ by 0 and $\{a\}$ by 1, the resulting chain semiring becomes the binary Boolean algebra $\mathbb{B} = \{0,1\}$, and it is a subsemiring of every chain semiring.

Let $\mathcal{M}_n(\mathbb{K})$ denote the set of all $n \times n$ matrices with entries in a chain semring $\mathbb{K}$. Algebraic operations on $\mathcal{M}_n(\mathbb{K})$ and such notions as *linearity* and *invertibility* are also defined as if the underlying scalars were in a field.

The matrix $I_n$ is the $n \times n$ identity matrix, $J_n$ is the $n \times n$ matrix all of whose entries are 1, and $O_n$ is the $n \times n$ zero matrix. We will suppress the subscripts on these matrices when the orders are evident from the context and we write $I, J$ and $O$, respectively. For any matrix $A$, $A^t$ is denoted by the transpose of $A$. A zero-one matrix in $\mathcal{M}_n(\mathbb{K})$ with only one equal to 1 are called a *cell.* If the nonzero entry occurs in the $i^{\text{th}}$ row and the $j^{\text{th}}$ column, we denote the cell by $E_{i,j}$.

A matrix $A$ in $\mathcal{M}_n(\mathbb{K})$ is said to be *invertible* if there is a matrix $B$ in $\mathcal{M}_n(\mathbb{K})$ such that $AB = BA = I$.

The notion of generalized inverse of an arbitrary matrix apparently originated in the work of Moore (see [6]). Let $A$ be a matrix in $\mathcal{M}_n(\mathbb{K})$. Consider a matrix $X \in \mathcal{M}_n(\mathbb{K})$ in the equation

$$AXA = A. \tag{2.1}$$

If (2.1) has a solution $X$, then $X$ is called a *generalized inverse* of $A$. Furthermore $A$ is called *regular* if there is a solution of (2.1).

Clearly, $J$ and $O$ are regular in $\mathcal{M}_n(\mathbb{K})$ because $JGJ = J$ and $OGO = O$, where $G$ is any cell in $\mathcal{M}_n(\mathbb{K})$. Thus in general, a solution of (2.1), although it exists, is not necessarily unique. Characterizations of regular matrices over semirings have been obtained by several authors([1, 3, 4, 7, 8]). Furthermore Plemmons [7] have obtained an algorithm for computing generalized inverses of Boolean matrices under certain conditions.

The following Proposition is an immediate consequence of definitions of regular matrix and invertible matrix.

**Proposition 2.1.** *Let $A$ be a matrix in $\mathcal{M}_n(\mathbb{K})$. If $U$ and $V$ are invertible matrices in $\mathcal{M}_n(\mathbb{K})$, then the following are equivalent:*

(i) *$A$ is regular in $\mathcal{M}_n(\mathbb{K})$;*

(ii) *$UAV$ is regular in $\mathcal{M}_n(\mathbb{K})$;*

(iii) *$A^t$ is regular in $\mathcal{M}_n(\mathbb{K})$.*

Also we can easily show that for a matrix $A \in \mathcal{M}_n(\mathbb{K})$

$$A \text{ is regular if and only if } \begin{bmatrix} A & O \\ O & B \end{bmatrix} \text{ is regular} \tag{2.2}$$

for all regular matrices $B \in \mathcal{M}_{p,q}(\mathbb{K})$. In particular, all idempotent matrices in $\mathcal{M}_n(\mathbb{K})$ are regular. Let

$$\Lambda_n = [\lambda_{i,j}] = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \\ & 1 & \cdots & 1 & 1 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \in \mathcal{M}_n(\mathbb{K}).$$

Then the following Proposition shows that $\Lambda_n$ is not regular for $n \geq 3$.

**Proposition 2.2.** *$\Lambda_n$ is regular in $\mathcal{M}_n(\mathbb{K})$ if and only if $n \leq 2$.*

**Proof.** Clearly $\Lambda_n$ is regular for $n \leq 2$ because $\Lambda_n I_n \Lambda_n = \Lambda_n$.

Conversely, assume that $\Lambda_n$ is regular for some $n \geq 3$. Then there is a nonzero $B \in \mathcal{M}_n(\mathbb{K})$ such that $\Lambda_n = \Lambda_n B \Lambda_n$. From $0 = \lambda_{1,n} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} b_{i,j}$, all entries of the $2^{th}$ column of $B$ are zero except for $b_{n,2}$. From $0 = \lambda_{2,1} = \sum_{i=2}^{n} b_{i,1}$, all entries of the $1^{st}$ column of $B$ are zero except for $b_{1,1}$. Also, from $0 = \lambda_{3,2} = \sum_{i=3}^{n}\sum_{j=1}^{2} b_{i,j}$, we have $b_{n,2} = 0$. If we combine these three results, we conclude that all entries of the first two columns are zero except for $b_{1,1}$. But then $1 = \lambda_{2,2} = \sum_{i=2}^{n}\sum_{j=1}^{2} b_{i,j} = 0$, a contradiction. Hence $\Lambda_n$ is not regular for all $n \geq 3$. □

The *(factor) rank*, $fr(A)$, of a nonzero $A \in \mathcal{M}_n(\mathbb{K})$ is defined as the least integer $r$ for which there are $B \in \mathcal{M}_{m,r}(\mathbb{K})$ and $C \in \mathcal{M}_{r,n}(\mathbb{K})$ such that $A = BC$, see ([2]). The rank of a zero matrix is zero. Also we can easily obtain

$$0 \leq fr(A) \leq \min\{m,n\} \quad \text{and} \quad fr(AB) \leq \min\{fr(A), fr(B)\} \tag{2.3}$$

for all $A \in \mathcal{M}_n(\mathbb{K})$ and for all $B \in \mathcal{M}_{n,q}(\mathbb{K})$.

**Lemma 2.3.** *Let $A$ be a matrix in $\mathcal{M}_n(\mathbb{K})$ with $fr(A) = 1$. Then $A$ is regular.*

**Proof.** Since $fr(A) = 1$, there exist permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} a_1 \\ \vdots \\ a_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} [b_1 \ \cdots \ b_s \ 0 \ \cdots \ 0],$$

where $0 < a_1 \leq \cdots \leq a_r$ and $0 < b_1 \leq \cdots \leq b_s$. Let $m = \max\{a_r, b_s\}$. Then we have $(PAQ)(mJ)(PAQ) = PAQ$ so that $PAQ$ and hence $A$ is regular. □

The number of nonzero entries of a matrix $A$ is denoted by $|A|$.

**Corollary 2.4.** *Let $A$ be a matrix in $\mathcal{M}_n(\mathbb{K})$ with $|A| \leq 2$. Then $A$ is regular.*

**Proof.** If $|A| = 0$ or $1$, clearly $A$ is regular because $AA^tA = A$. If $|A| = 2$, by Proposition 2.3, we may assume $fr(A) = 2$. Furthermore, by Proposition 2.1, we may assume $A = \begin{bmatrix} B & O \\ O & O \end{bmatrix}$, where $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with $ab \neq 0$. Then we can easily show that $A$ is idempotent and hence $A$ is regular. □

If $A = \begin{bmatrix} 1 & .5 \\ .5 & 0 \end{bmatrix} \in \mathcal{M}_2(\mathbb{F})$, then the below Proposition shows that $A$ is not regular.

**Lemma 2.5.** *Let $A$ be a matrix in $\mathcal{M}_2(\mathbb{K})$ with $|A| = 3$. Then $A$ is regular if and only if there exist permutation matrices $P$ and $Q$, and nonzero $a, b, c \in \mathbb{K}$ such that $PAQ = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, where $a \leq b \leq c$ or $b < a \leq c$.*

**Proof.** ($\Leftarrow$) Suppose that there exist permutation matrices $P$ and $Q$, and nonzero $a, b, c \in \mathbb{K}$ such that $PAQ = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, where $a \leq b \leq c$ or $b \leq a \leq c$. Then we have $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$. By Proposition 2.1, $A$ is regular.

($\Rightarrow$) Let $A$ be regular. By Proposition 2.1, we may assume $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, where $abc \neq 0$ and $b \leq c$. Thus, there exists a nonzero matrix $G = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that $AGA = A$ so that

$$\begin{bmatrix} ax + abz + acy + bw & abx + bz \\ acx + cy & bx \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}. \tag{2.4}$$

From $bx = 0$, we have $x = 0$ and hence (2.4) becomes

$$\begin{bmatrix} abz + acy + bw & bz \\ cy & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}. \tag{2.5}$$

If $a \leq b$, then we are done. Assume $b < a$. If $c < a$, then we have $abz + acy + bw = bz + cy + bw \leq b + c < a$. This contradicts (2.5). Hence we have $a \leq c$ and so $b < a \leq c$. □

**Lemma 2.6.** *Let $A$ be a matrix in $\mathcal{M}_2(\mathbb{K})$ with $|A| = 4$. Then $A$ is regular if and only if there exist permutation matrices $P$ and $Q$, and nonzero $p, q, r, s \in \mathbb{K}$ such that the form of $PAQ$ is just one of the following:*

(1) $\begin{bmatrix} p & q \\ p & r \end{bmatrix}$;

(2) $\begin{bmatrix} p & q \\ r & p \end{bmatrix}$, *where $q \geq p$ or $r \leq p$;*

(3) $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$, *where $p < q < r$ and $p < s < r$ with $q \neq s$.*

**Proof.** ($\Leftarrow$) (1) Let $X = \begin{bmatrix} p & q \\ p & r \end{bmatrix}$. Without loss of generality, we may assume $q \leq r$. If $p \leq q \leq r$, then $XGX = X$, where $G = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. If $q \leq p \leq r$ or $q \leq r \leq p$, then $X$ is idempotent. Hence $X$ is regular and so $A$ is regular.

(2) Let $Y = \begin{bmatrix} p & q \\ r & p \end{bmatrix}$, where $q \geq p$ or $r \leq p$. Without loss of generality, we may assume $q \leq r$. If $r \leq p$, then $Y$ is idempotent and if $q \geq p$, then $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} Y$ is idempotent. Hence $Y$ is regular so that $A$ is regular.

(3) Let $Z = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, where $p < q < r$ and $p < s < r$ with $q \neq s$. Then we have $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ so that $Z$ is regular. Therefore $A$ is regular.

($\Rightarrow$) Let $A$ be regular. First suppose that all entries of $A$ are distinct. By Proposition 2.1, we assume $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ with $p < q < r$ and $p < s$. Since $A$ is regular, there exists a nonzero matrix $G = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that $AGA = A$ so that

$$\begin{bmatrix} p(x+y+z)+qw & p(x+y)+q(z+w) \\ p(x+z)+r(y+w) & q(x+z)+ry+sw \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

If $s > r$, from the $(2,2)^{th}$ entries of $AGA$ and $A$, we have $w \geq s$. Therefore the $(1,1)^{th}$ entry of $AGA$ is $q$, a contradiction. Hence $s < r$ and so (3) is satisfied.

Next, suppose that at least two entries of $A$ are same. By Proposition 2.1, we loss of generality in assuming that

$$A = \begin{bmatrix} p & q \\ p & r \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} p & q \\ r & p \end{bmatrix}.$$

If $A$ is the former form, then (1) is satisfied. If $A = \begin{bmatrix} p & q \\ r & p \end{bmatrix}$, without loss of generality, we may assume $q \leq r$. Suppose neither $q \geq p$ nor $r \leq p$. Then we have $q < p < r$. Since $A$ is regular, there exists a nonzero matrix $G = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that $AGA = A$ so that

$$\begin{bmatrix} p(x+y) & q(x+z+w)+py \\ ry & p(y+w) \end{bmatrix} = \begin{bmatrix} p & q \\ r & p \end{bmatrix}.$$

From the equality of $(2,1)^{\text{th}}$ entries of $AGA$ and $A$, we have $y \geq r$. But then the $(1,2)^{\text{th}}$ entry of $AGA$ is $p$, a contradiction. Hence $q \geq p$ or $r \leq p$. This shows that (2) is satisfied. □

# 3 Invertible regular preservers

In this section, we have characterizations of invertible linear operators that preserve regular matrices over chain semiring $\mathbb{K}$ including the fuzzy scalars $\mathbb{F}$.

A mapping $T : \mathcal{M}_n(\mathbb{K}) \to \mathcal{M}_n(\mathbb{K})$ is called a *linear operator* if $T(aA + bB) = aT(A) + bT(B)$ for all $A, B \in \mathcal{M}_n(\mathbb{K})$ and for all $a, b \in \mathbb{K}$. A linear operator on $\mathcal{M}_n(\mathbb{K})$ is completely determined by its behavior on the set of cells in $\mathcal{M}_n(\mathbb{K})$. An operator $T$ is called *invertible* if it is surjective and injective.

Let $\Delta_n = \{(i,j) : 1 \le i,j \le n\}$. For $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathcal{M}_n(\mathbb{K})$, $B$ *dominates* $A$, denoted by $A \sqsubseteq B$, if $a_{i,j} \neq 0$ implies $b_{i,j} \neq 0$ for all $(i,j) \in \Delta_n$.

**Lemma 3.1.** *Let $T$ be a liner operator on $\mathcal{M}_n(\mathbb{K})$. If $T$ is invertible, then there exists a permutation $\theta$ on $\Delta_n$ such that $T(E_{i,j}) = E_{\theta(i,j)}$ for all $(i,j) \in \Delta_n$.*

**Proof.** Suppose that $T$ is invertible on $\mathcal{M}_n(\mathbb{K})$. Let $E_{r,s}$ be an arbitrary cell in $\mathcal{M}_n(\mathbb{K})$. Since $T$ is surjective, there exists a nonzero matrix $X = [x_{i,j}] \in \mathcal{M}_n(\mathbb{K})$ such that $T(X) = E_{r,s}$. Since $T$ is linear, it follows that there exists $x_{i,j} \neq 0$ such that $T(E_{i,j}) \sqsubseteq E_{r,s}$. This shows $T(E_{i,j}) = b_{i,j}E_{r,s}$ for some nonzero $b_{i,j} \in \mathbb{K}$. Let

$$C_{r,s} = \{E_{i,j} : T(E_{i,j}) = b_{i,j}E_{r,s} \text{ for some nonzero } b_{i,j} \in \mathbb{K}\}.$$

By the above, $C_{r,s} \neq \emptyset$ for all $(r,s) \in \Delta_n$. Suppose $T(E_{k,l}) = b_{k,l}E_{r,s}$ for a cell $E_{k,l}$ different from $E_{i,j}$ with $b_{k,l} \neq 0$. Then we have

$$T(b_{k,l}E_{i,j}) = b_{k,l}T(E_{i,j}) = b_{k,l}b_{i,j}E_{r,s} = b_{i,j}T(E_{k,l}) = T(b_{i,j}E_{k,l}),$$

a contradiction to the fact that $T$ is injective. Hence $C_{r,s}$ is a singleton set for all $(r,s)$. Thus, there exists a permutation $\theta$ on $\Delta_n$ such that $T(E_{i,j}) = b_{i,j}E_{\theta(i,j)}$ for some nonzero $b_{i,j} \in \mathbb{K}$. It remains to show $b_{i,j} = 1$ for all $(i,j) \in \Delta_n$. Since $T$ is surjective and $T(E_{r,s}) \not\sqsubseteq E_{\theta(i,j)}$ for $(r,s) \neq (i,j)$, $T(cE_{i,j}) = E_{\theta(i,j)}$ for some nonzero $c \in \mathbb{K}$. By the linearity of $T$, we have

$$E_{\theta(i,j)} = T(cE_{i,j}) = cT(E_{i,j}) = cb_{i,j}E_{\theta(i,j)}.$$

That is, $cb_{i,j} = 1$ and hence $c = b_{i,j} = 1$.

The converse is immediate. □

A matrix $L \in \mathcal{M}_n(\mathbb{K})$ is called a *line matrix* if $L = \sum_{k=1}^{n} E_{i,k}$ or $\sum_{l=1}^{n} E_{l,j}$ for some $i \in \{1,\ldots,n\}$ or $j \in \{1,\ldots,n\}$; $R_i = \sum_{k=1}^{n} E_{i,k}$ is an $i^{\text{th}}$ *row matrix* and $C_j = \sum_{l=1}^{n} E_{l,j}$ is a $j^{\text{th}}$ *column matrix*.

**Lemma 3.2.** *If $T$ is an invertible linear operator on $\mathcal{M}_n(\mathbb{K})$ that preserves regular matrices, then $T$ preserves all line matrices.*

**Proof.** Suppose that $T$ does not map some line matrix into a line matrix. By Lemma 3.2 and Proposition 2.1, without loss of generality, we may assume $T(E_{1,1}) = E_{1,2}$ and $T(E_{1,2}) = E_{2,1}$. Since $T$ is surjective, $T(E_{i,j}) = E_{1,1}$ for some $(i,j) \in \Delta_n - \{(1,1),(1,2)\}$. Let $X = E_{i,j} + pE_{1,1} + pE_{1,2}$, where $p \neq 0, 1$. Then $X$ is regular: for, if $i = 1$ and $j \geq 3$, it follows from Lemma 2.3 that $X$ is regular; if $i \geq 2$ and $j \geq 3$, $X$ is regular by (2.2); otherwise, $X$ is regular by Lemma 2.5. Notice $T(X) = \begin{bmatrix} A & O \\ O & O \end{bmatrix}$, where $A = \begin{bmatrix} 1 & p \\ p & 0 \end{bmatrix}$. It follows from Lemma 2.5 that $A$ is not regular and so $T(X)$ is not regular by (2.2). This contradicts to the fact that $T$ preserves regular matrices. Hence $T$ preserves all line matrices. □

Now, we are ready to prove the main Theorem.

**Theorem 3.3.** *Let $T$ be an invertible linear operator on $\mathcal{M}_n(\mathbb{K})$. Then $T$ preserves regular matrices if and only if there exist permutation matrices $P$ and $Q$ such that $T(X) = PXQ$ or $T(X) = PX^tQ$ for all $X \in \mathcal{M}_n(\mathbb{K})$.*

**Proof.** Suppose that $T$ is an invertible linear operator on $\mathcal{M}_n(\mathbb{K})$ that preserves regular matrices. Then $T$ is bijective on the set of cells by Lemma 3.1 and $T$ preserves all line matrices by Lemma 3.2. Since no combination of $s$ row matrices and $t$ column matrices can dominate $J_n$ where $s + t = n$ unless $s = 0$ or $t = 0$, we have that either

(1) the image of $T$ of each row matrix is a row matrix and the image of $T$ of each column matrix is a column matrix, or

(2) the image of $T$ of each row matrix is a column matrix and the image of $T$ of each column matrix is a row matrix.

If (1) holds, then there are permutations $\sigma$ and $\tau$ of $\{1, \ldots, n\}$ such that $T(R_i) = R_{\sigma(i)}$ and $T(C_j) = C_{\tau(j)}$ for all $i, j = 1, \ldots, n$. Let $P$ and $Q$ be permutation matrices corresponding to $\sigma$ and $\tau$, respectively. Then we have

$$T(E_{i,j}) = E_{\sigma(i),\tau(j)} = P(E_{i,j})Q$$

for all cells $E_{i,j}$. By the action of $T$ on the cells, we have $T(X) = PXQ$.

If (2) holds, then a parallel argument shows that there are permutation matrices $P$ and $Q$ such that $T(X) = PX^tQ$ for all $X \in \mathcal{M}_n(\mathbb{K})$. □

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