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Numerical Approach of Equation of Motion with Positional Constraints

Yong-Soo Kang*, Hee-Chang Eun** and Ill-Gyo Suh**

상태에 관한 구속 조건을 갖는 운동방정식의 수치 해석법

강 용 수* · 은 희 창** · 서 일 교**

ABSTRACT

In 1992, Udwadia and Kalaba proposed an explicit equation of motion for constrained systems based on Gausss principle and elementary linear algebra without any multipliers or complicated intermediate process. However, numerical results to integrate the equation of motion gradually veer away the constraint equations with time. Thus, an objective of this study is to provide a numerical integration scheme, which modifies the generalized inverse method to reduce the errors. The modified equation of motion for constrained systems includes the positional constraints of index 3 systems and their first time derivative besides their second time derivative. Its effectiveness is established by means of numerical examples.

Key Words : Errors in the satisfaction of constraints, constraints, control, eigenvalue, generalized inverse method

Ⅰ. INTRODUCTION

The motion of mechanical or structural systems can be sometimes constrained by any given trajectories or conditions. The constrained motion requires the constaint force provided by Nature for satisfying the constraints. Gauss's Principle defines the constraint force as the minimum force of all forces to need for satisfying the constraints or pulling the state variables into the given trajectories. The constraint force must be explicitly calculated and provided such that the state variables do not violate the constraints. But most of methods to describe the constrained motion depend on numerical approaches like Lagrange multiplier method expressed by differential/algebraic system [3-6].

Mathematically, the equation of motion for constrained systems using the Lagrange formulation can be expressed by differential/algebraic systems $F(t, y, \dot{y}) = 0$, y are n-dimensional vector. They also involve the Lagrange multiplier functions. The

* 제주대학교 건축공학과
Dept. of Architectural Engineering, Cheju Nat'l Univ.
** 제주대학교 건축공학과, 첨단기술연구소
Dept. of Architectural Engineering, Research Institute of Advanced Technology, Cheju Nat'l Univ.

formulations are basically based on an over-determined system of equations including time derivative of the constraints and stabilization with respect to those differential constraints via additional Lagrange multipliers. These methods have difficulties in numerically determining the multipliers.

Gibbs-Appell[1,7] method requires a felicitous choice of quasi-coordinates and is also difficult to use, when dealing with systems having several tens of degrees of freedom and several non-integrable constraints. Kane[8] introduced an analytical method for nonholonomic systems based upon the develop- ment of Lagrange equations from D'Alembert's Principle. Though his method is usually less tedious than the computation associated with Lagrange multiplier, it is difficult to compute vector components of acceleration. It also gets more complicated with increasing numbers of degrees of freedom. Passerello and Huston[9] introduced a computer-oriented method similar to the method of the orthogonal component of the matrix associated with the constraint equations, which reduces the dimension of the dynamical equations by elimination of constraint forces. In 1992, Udwadia and Kalaba[10] proposed an explicit equation of motion for constrained mechanical and structural systems. The generalized inverse method by Udwadia and Kalaba was the first work to present the explicit equation of motion for constrained systems since Lagrange. This method has advantages which do not require any linearization process for the control of nonlinear systems and can explicitly describe the motion of holonomically and/or nonholonomically constrained systems.

The constrained motion can be described by numerically integrating the differential equation by Udwadia and Kalaba, and must satisfy the constraints during the integration time. However, the numerical results gradually veer away the given constraints with time. In a viewpoint of numerical integration, it is necessary to devise numerical methods to pull the deviated state variables into the given paths. Because the generalized inverse method was based on the second time derivatives of positional constraints, the errors in the satisfaction of constraints are caused by the neglect of positional constraints of index 3 systems as well as their first time derivatives. The index, which is a mathematical term, is the number of times one must differentiate to get a system of ordinary differential equations. Accordingly, an objective of this paper is to present a numerical method which modifies the generalized inverse method to reduce the errors in the satisfaction of the constraints. The modified equation of motion for constrained systems includes the effects of positional constraints, their first and second time derivatives in the differential equation. Numerical examples illustrate the effectiveness of the proposed numerical method.

II. EQUATION OF MOTION FOR CONSTRAINED SYSTEMS

The matrix equation of motion of a system modeled by an n-degree-of-freedom lumped mass-spring-dashpot system can be written as

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = Ef(t), \tag{1}$$

where M, C, and K are, respectively, the $n \times n$ mass, damping, and stiffness matrices, $x(t)$ is the n-dimensional displacement vector, and $f(t)$ is an r-vector representing applied load or external excitation. The $n \times r$ matrix E is location matrix which defines locations of the excitation.

Assume that this n-degree-of-freedom system is constrained by the m consistent constraints

$$\phi_i(x,t) = 0, \qquad i = 1, 2, \cdots, m, \tag{2}$$

of which $m < n$. The constrained motion requires the constraint force such that the state variables satisfy the constraint sets. Therefore, the general equation of motion at time t of constrained system can be expressed as

$$M\ddot{x} = F(x, \dot{x}, t) + F^c(x, \dot{x}, t), \quad (3)$$

where, $F(x, \dot{x}, t) = -C\dot{x}(t) - Kx(t) + Ef(t)$, and $F^c(x, \dot{x}, t)$ is the n-dimensional constraint force vector.

Assuming that the constraint equations are sufficiently smooth, the proper differentiation of Eq. (2) with respect to time t leads to the linear set of equations

$$A(x, \dot{x}, t)\ddot{x} = b(x, \dot{x}, t), \quad (4)$$

where A is an $m \times n$ matrix, and b is an vector. Using Gauss's Principle and elementary linear algebra, and combining Eqs. (1) and (4), the generalized inverse method gives a constrained equation of motion written by

$$\ddot{x} = a + M^{-\frac{1}{2}}(AM^{-\frac{1}{2}})^{+}(b - Aa), \quad (5)$$

where $a = M^{-1}F(x, \dot{x}, t)$. This is the first work to present an explicit equation of motion for constrained systems and has advantages which do not require any linearization process for the control of nonlinear systems and can explicitly describe the motion of holonomically and/or nonholonomically constrained systems.

However, the numerical results to integrate the differntial equation (5) by any numerical integration schemes veer away the constraints. The integration of constrained equation of motion based on the second time derivatives of positional constraints leads to the errors in the satisfaction of the constraints caused by the neglect of the positional constraints and their first time derivatives. Thus, starting from the generalized inverse method, this study presents a modified equation of motion for constrained systems to include all three constraint sets.

III. ERRORS IN THE SATISFACTION OF CONSTRAINTS

To investigate the errors developed during numerical integration of the differential equation, consider a three-DOF system subjected to a constraint. As shown by Fig. 1, the state variable vector, which describes configuration space of the system, is denoted by $q = [\,q_1\ q_2\ q_3]^T$. The unconstrained equation of motion for this system is given by a constraint

$$M\ddot{q} + C\dot{q} + Kq = P(t). \quad (6)$$

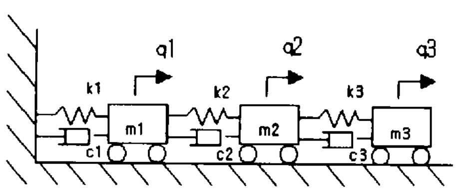


Fig. 1. A Three-DOF system.

Assuming that this system is constrained by a constraint

$$\phi_1 = q_1 - 3q_2 = 0, \quad (7)$$

and differentiating Eq. (7) twice and expressing $A\ddot{x} = b$ of Eq. (4), we can write it as

$$\ddot{\phi}_1 = \ddot{q}_1 - 3\ddot{q}_2 = 0. \quad (8)$$

The physical values for numerical application were selected by

$$m_1 = m_2 = 3\text{units}, \quad m_3 = 1\text{unit},$$
$$k_1 = 300\text{units}, \quad k_2 = 200\text{units} \quad (9)$$
$$k_3 = 100\text{units}.$$

The damping coefficients were selected by the values which the damping ratio of each mode is 0.02 and the external excitation vector was assumed as $P_{(t)} = [\,300\sin 6t\ \ 500\cos 3t\ \ 0\,]^T$. For numerical integration of the equation ofelected motion for this system, the local tolerance for the Runge-Kutta scheme was set to 10^{-6}. When the differential equation is integrated by any numerical integration scheme, the state variables must satisfy the constraint equation (7) at all times. To investigate the errors in the satisfaction of constraints, which are the positional constraint and its first time derivative, we defined the errors as

$$\text{Error1} = q_1 - 3q_2$$

and $$\text{Error2} = \dot{q}_1 - 3\dot{q}_2 \qquad (10)$$

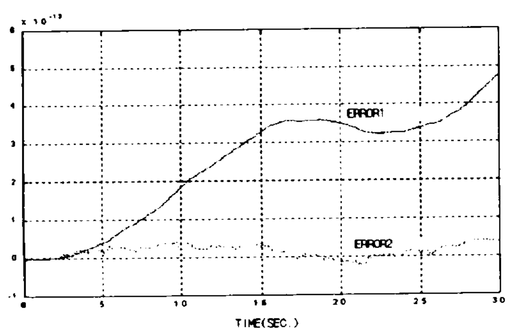


Fig. 2. Errors in the satisfaction of the constraints.

Fig. 2 shows the errors given by Eq. (10). It is observed that the numerical solutions are found to gradually veer away the constraints and the errors increase with time. Recognizing that the genealized inverse method is based on the second time derivatives of positional constraints, it can be interpreted that the errors are due to the neglect of positional constraints and their first time derivatives. The errors can be reduced by the action of additional force which needs to pull the deviated state variables into the positional constraints and their first time derivatives. Thus, the modified equation of motion includes the effects of the positional constraints and their first time derivatives.

Ⅳ. NUMERICAL INTEGRATION SCHEME

Assuming that the constraint equations are sufficiently smooth and taking the total derivatives of the set (2), and using the chain rule, we obtain these equations

$$\dot{\phi}_i = \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_j} \dot{x}_j + \frac{\partial \phi_i}{\partial t} = 0,$$
$$i = 1, 2, \cdots, m. \qquad (11)$$

These equations are differentiated, provided the functions $\frac{\partial \phi_i}{\partial x_j}$ and $\frac{\partial \phi_i}{\partial t}$ are sufficiently smooth, to yield the set of equations

$$\ddot{\phi}_i = \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_j} \ddot{x}_j + \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial \phi_i}{\partial x_j}\right) \dot{x}_k \dot{x}_j + \sum_{j=1}^{n} \frac{\partial}{\partial t} \dot{x}_j + \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(\frac{\partial \phi_i}{\partial t}\right) \dot{x}_k + \frac{\partial^2 \phi_i}{\partial t^2} = 0$$
$$i = 1, 2, \cdots, m \qquad (12)$$

Eq. (12) can be cast into $A\ddot{x} = b$. To determine the constraint force on the basis of all constraint equation sets, all three constraint equations are combined as

$$\ddot{\phi}_i + \alpha_i \dot{\phi}_i + \beta_i \phi_i = 0,\ i = 1, 2, \cdots, m. \qquad (13)$$

or $$\ddot{H} + R\dot{H} + SH = 0, \qquad (14)$$

where the α_i s and β_is are positive values.

$$H = [\,\phi_1\ \phi_2 \cdots \phi_m\,]^T,$$

$$R = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_m \end{bmatrix},$$

and
$$S=\begin{bmatrix}\beta_1 & 0 & \cdots & 0\\ 0 & \beta_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \beta_m\end{bmatrix} \tag{15}$$

Also, $A\ddot{x}=b$ is rewritten as

$$A\ddot{x}=b-R\dot{H}-SH \tag{16}$$

where

$$A=\begin{bmatrix}\frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_n}\\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \frac{\partial\phi_2}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial\phi_m}{\partial x_2} & \frac{\partial\phi_m}{\partial x_2} & \cdots & \frac{\partial\phi_m}{\partial x_n}\end{bmatrix} \tag{17}$$

$$b=[b_1\ \ b_2]^T$$
$$b_1=\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial\phi_1}{\partial x_j}\right)\dot{x}_k\dot{x}_j+\sum_{j=1}^{n}\frac{\partial}{\partial t}\left(\frac{\partial\phi_1}{\partial x_j}\right)\dot{x}_j+\sum_{j=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial\phi_1}{\partial t}\right)\dot{x}_k+\frac{\partial^2\phi_1}{\partial t^2}$$
$$b_2=\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial\phi_1}{\partial x_j}\right)\dot{x}_k\dot{x}_j+\sum_{j=1}^{n}\frac{\partial}{\partial t}\left(\frac{\partial\phi_1}{\partial x_j}\right)\dot{x}_j \tag{18}$$

The original equation of motion (5) for constrained systems is modified as

$$\ddot{x}=a+M^{-\frac{1}{2}}\left(AM^{-\frac{1}{2}}\right)^{+}(b-R\dot{H}-SH-Aa) \tag{19}$$

We can alternssatively think of Eq. (13) as the equations of motion of m second-order dynamic systems. The α_i's and β_i's are damping coefficient and stiffness of the i-th oscillator, respectively. Let us call Eq. (13) the i-th dynamical error equation. The terms $\alpha_i\dot{\phi}_1+\beta_i\phi_i$ in Eq. (13) play the role for reducing the errors in the satisfaction of the constraints. The coefficients α_i and β_i need to be selected in such a way that the errors in the satisfaction of the constraints ϕ_i and $\dot{\phi}_i$ are damped out rapidly.

Baumgarte (1972) discussed the proper choice of the values of the coefficients α_i and β_i in Eq. (13) for reducing numerical errors and suggested positive values for the parameters α_i and β_i corresponding to the i-th oscillator. This method considered the dynamical error equations as decoupled equations that the unknown coefficients involved in each of the m dynamical error equations are independently selected.

Each of the m dynamical error equations can be looked upon as an oscillator and shows three types of motion depending on the values of the coefficients α_i and β_i : critically damped motion, underdamped motion, and overdamped motion. These three types of motion depend on the quantity of $\alpha_i^2-4\beta_i$ corresponding to the i-th oscillator. If $\alpha_i^2-4\beta_i<0$, this is called an underdamped system. If $\alpha_i^2-4\beta_i=0$, this is called a critically damped system. And if $\alpha_i^2-4\beta_i>0$, this is called an overdamped system. Baumgarte normally selected the values of the unknowns corresponding to the critically damped motion with values of $\alpha_1<20$.

In order to investigate the variations of the errors according to the selection of α_i and β_i on the above system, let us define the magnitude of the errors in the satisfaction of constraints caused by numerical procedure as

$$E1=\frac{1}{T_f}\sqrt{\int_0^{T_f}(q_1-3q_2)^2dt} \tag{20}$$

and

$$E2=\frac{1}{T_f}\sqrt{\int_0^{T_f}(\dot{q}_1-3\dot{q}_2)^2dt}, \tag{21}$$

where T_f is 30 seconds.

Figs. 3 and 4 show the variations of E1 and E2 according to the coefficients α and β, where the values of α range from 0 to 20 in increments of 2 and the values of β corresponds to the underdamped, critically damped, and overdamped system. The minimum value of E1 occurs at $\alpha=20$ and $\beta=200$, while the minimum value of E2 occurs at $\alpha=18.0$ and $\beta=16.2$. The parameter values α and β to minimize E1 and E2 correspond to an underdamped and an overdamped system, respectively.

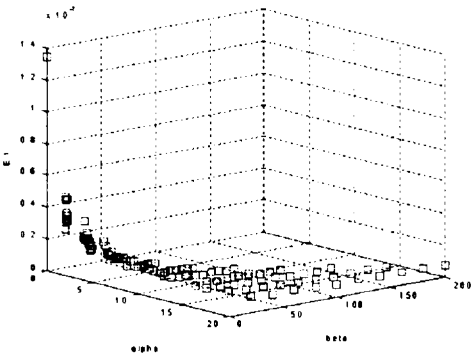


Fig. 3. Variation of the magnitude of error 1.

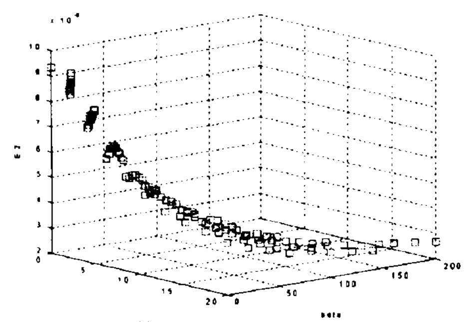
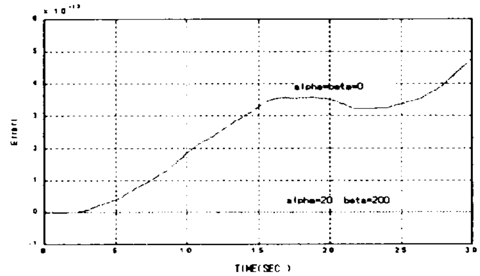


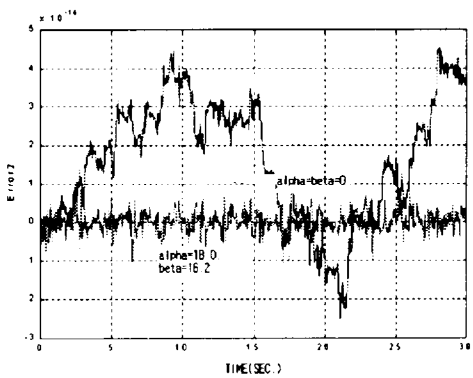
Fig. 4. Variation of the magnitude of error 2.

Fig. 5 compares the error E1 to be taken by the values of $\alpha=\beta=0$ and $\alpha=20$, $\beta=200$ and E2 at the values of $\alpha=\beta=0$ and $\alpha=18.0$, $\beta=16.2$.

To take = =0 becomes the original equation of motion for constrained systems and the minimum values of E1 and E2 do not take the same coefficient values. From this plot, it is observed that the action of the additional force by consideration of the positional constraints and their first time derivatives leads to the reduction of the errors. Also, it is exhibited that the error is not totally damped out and the reduction of the errors largely depends on the selection of the parameter values.



(a)



(b)

Fig. 5. Comparison of the errors according to the selected parameter values: (a) Error 1, (b) Error 2.

With the object of reducing the errors in the satisfaction of multiple constraints, assume that the above system is constrained by another additional constraint

$$\phi_2 = q_1 + q_2 + q_3 = 0. \tag{22}$$

Properly differentiating two constraints (7) and (22) with respect to time *t*, these constraint equations in the form (14) can be expressed as

$$A=\begin{bmatrix}1 & -3 & 0\\ 1 & 1 & 1\end{bmatrix},$$

$b-R\dot{H}-SH=$

$$\begin{bmatrix}-\alpha_1(\dot{q}_1-3\dot{q}_2) & -\beta_1(q_1-3q_2)\\ -\alpha_2(\dot{q}_1+\dot{q}_2+\dot{q}_3) & -\beta_2(q_1+q_2+q_3)\end{bmatrix} \quad (23)$$

where the parameters α_1, α_2, β_1, and β_2 are positive values.

Fig. 6 shows the magnitude of the errors defined by Eqs. (20) and (21), and

$$E3=\frac{1}{T_f}\sqrt{\int_0^{T_f}(q_1+q_2+q_3)^2dt} \quad (24)$$

and

$$E2=\frac{1}{T_f}\sqrt{\int_0^{T_f}(\dot{q}_1+\dot{q}_2+\dot{q}_3)^2dt}. \quad (25)$$

As shown by Fig. 6, the minimum values of both E1 and E2 occurred at the same values of $\alpha=20$ and $\beta=200$, while the minimum values of E3 and E4 occurs at $\alpha=20$, $\beta=180$ and $\alpha=18$, $\beta=145.8$, respectively. The difference of Figs. 3 and 6(a), or Figs. 4 and 6(b) is due to the coupling of dynamical error equations by the parameters. The errors in the satisfaction of the i-th constraint are not independently affected by the parameter values α_i and β_i, but correlatively. This means that the error in the satisfaction of each constraint is a function of all the parameters present. Generalizing this scheme, the matrices R and S in Eq. (14) are replaced by

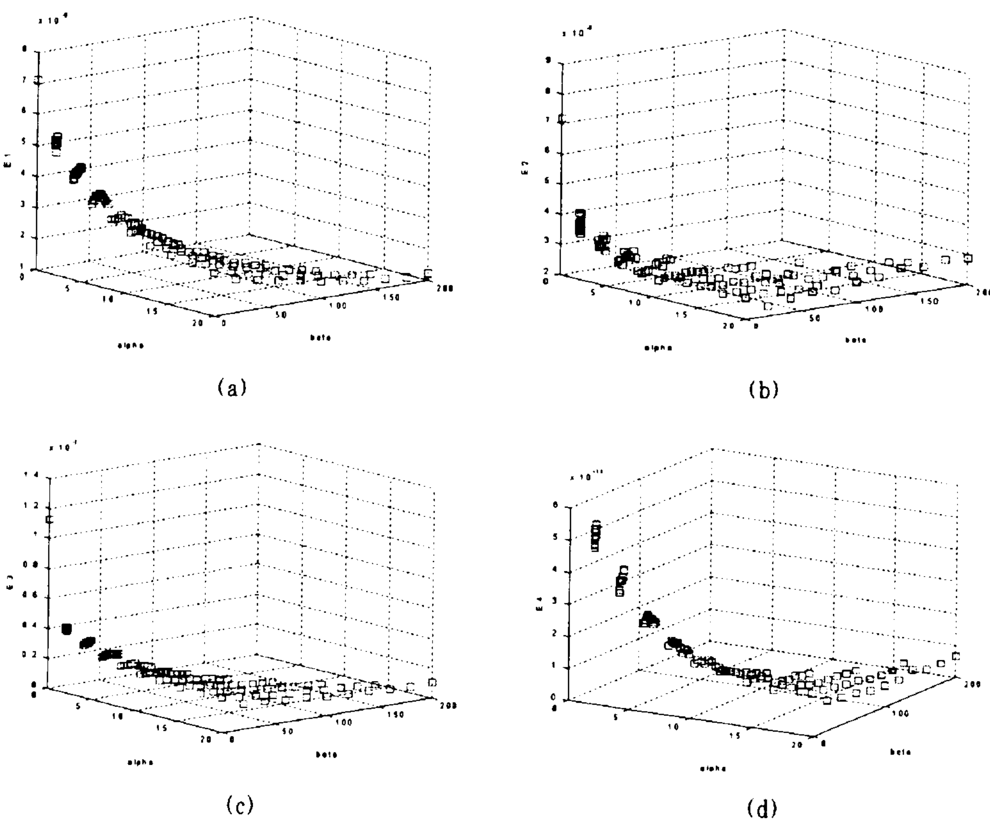


Fig. 6. Comparison of the magnitude of errors: (a) E1, (b) E2, (c) E3, (d) E4.

$$R = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mm} \end{bmatrix}$$

and

$$S = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mm} \end{bmatrix}, \qquad (26)$$

respectively.

The dynamical error equations are coupled by the coefficient matrices, and the matrices are selected such that the errors are rapidly damped out. Substituting $H = e^{\lambda t}U$ into the dynamical error equation (14), we obtain

$$(\lambda^2 + \lambda R + S)U e^{\lambda t} = 0, \qquad (27)$$

where λ are eigenvalues. Because $U \neq 0$, the eigenvalues, λ, satisfying $\lambda^2 I + \lambda R + S = 0$ must be negative real part so that $H \to 0$ with $t \to \infty$. It is convinced that the errors in the satisfaction of multiple constraints can be reduced by inserting the positional constraints and their first time derivatives into the original equation of motion and selecting the proper coefficient matrices with the eigenvalues of negative real part.

V. CONCLUSIONS

Most of methods to describe the constrained motion depend on numerical approaches like Lagrange multiplier method expressed by differential/ algebraic system. The equation of motion for constrained systems proposed by Udwadia and Kalaba has a great advantage to explicitly describe the constrained motion. However, the numerical results to integrate the differential equation gradually veer away the given constraints with time as a result that the generalized inverse method was based on the second time derivatives of positional constraints. The errors in the satisfaction of constraints are due to the neglect of the positional constraints and their first time derivatives. Thus, starting from the generalized inverse method, this study presented a numerical method to reduce the errors by inserting the effects of the positional constraints and their first time derivatives into the original constrained equation of motion. The modified equation of motion for constrained systems could more precisely describe the constrained motion by reducing the errors in the satisfaction of constraints.

요 약

구속된 시스템의 운동방정식은 generalized inverse method에 의해 명확히 계산될 수 있으나, 이 미분방정식을 수치 적분을 실시할 경우에 구속 조건을 만족하지 않는 오차들이 발생하며, 수치 적분의 시간이 증가할수록 점점 벗어나는 경향이 있다. 따라서, 본 연구에서는 수치 적분을 위해 구속 조건과 그 미분을 추가로 사용하여 보다 정밀한 해를 얻는 방법을 제시하였으며, 예를 통해 입증하였다.

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