碩士學位 請求論文

# 行列上에서 Permanent函數의 極小값

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1988年度

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이 論文을 教育學 碩士學位 論文으로 提出함.



濟州大學校 教育大學院 數學教育專攻

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1988年 月 日

# 趙龍玉의 碩士學位 論文을 認准함

1988年　　月　　日



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# 감사의 글

이 논문이 완성되기까지 연구에 바쁘신 가운데에도 많은 시간을 할애하며 지도를 하여주신 송석준 교수님께 감사의 말씀을 드립니다. 아울러, 교육과정을 이수하는데 강좌를 담당하여 주신 수학교육과의 여러 교수님들께도 감사의 말씀을 드립니다.

그리고 저에게 사랑과 격려를 해 주신 가족들과 자료를 정리하는데 도움을 주신 양영근 선생님, 또 여러 친구분들께도 감사의 말씀을 드립니다.

1988 년 6 월 일

조 용 옥

# CENTENTS

# I. INTRODUCTION

The theory of permanent function have been extended by the proofs of the van der Waerden's conjecture.

The proof of the van der Waerden's conjecture for permanent function have intensified the celebrated van der Waerden's conjecture for permanent function, recently proved by Egorycev asserts that if S is a doubly stochastic $n \times n$ matrix, then

$$\mathrm{Per}(s) \geq \frac{n!}{n^n} \quad \cdots\cdots\cdots\cdots\cdots\cdots \ (1)$$

and that equality can hold in (1) iff S is $J_n$, the matrix all of whose entries are $\frac{1}{n}$.

After the appearence of the proof of the van der Waerden's conjecture, many efforts have been made to exploit their techniques in problems of determination of the minimum permanents in various faces of the polyhedron $\Omega_n$ of all n-square doubly stochastic matrices. Without any doubt, one of the most interesting and important problem concerning the faces of $\Omega_n$ is that of determining the minimum values of the permanent function and the set of all minimizing matrices on them.



Knopp and Sinkhorn [5] determined the minimum permanent in a face of $\Omega_n$ with one prescribed zero.

Minc [7] found the minimum permanent in all faces of $\Omega_n$, in which the zeros are restricted to two rows or two columns.

Hwang [4] determined the minimum permanents in a face of $\Omega_n$ with zeros in staircase type matrices.

Song [9] found the minimum permanents in a face of $\Omega_n$ which is determined by the (0, 1)-matrices that contains identity matrix as a submatrix.

In this paper, we will investigate some minimum permanents on certain faces of the polyhedron of doubly stochastic matrices, and have some partial solutions on a conjecture of E. T. Wang for permanent function [8, 10].

# II. DEFINITIONS AND PRELIMINARIES

A nonnegative matrix is called *doubly stochastic* if all its row sums and coulum soms equal 1. The set of all $n \times n$ doubly stochastic matrices, denoted by $\Omega_n$, forms a convex polyhedron with permutation matrices as vertices [6].

For arbitrary n-square matrix $A=(a_{ij})$, the *permanent* of A is defined as

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where $S_n$ denotes the symmetric group of degree n.

Let $D=[d_{ij}]$ be an n-square matrix and

$$\Omega(D)=\{X=[x_{ij}] \in \Omega_n \mid x_{ij}=0 \text{ whenever } d_{ij}=0\}$$

Then, $\Omega(D)$ is called *the face* of the polyhedron $\Omega_n$ determined by D. Since $\Omega(D)$ is a compact subset of a finite dimensional Euclidean space, there exist a matrix A in $\Omega(D)$ such that

$$\operatorname{per}(A) \leqq \operatorname{per}(X) \text{ for all } X \in \Omega(D)$$

Such a matrix A will be called a *minimizing matrix* on $\Omega(D)$. The recent solution [2, 8] of the van der Waerden conjecture for the minimum permanent of matrices in $\Omega_n$ suggests the possibility of determining the minimum permanent of matrices in faces of $\Omega_n$.

An n-square (0, 1)-matrix D is called *cohesive* [1] if there is a matrix Z in the interior of $\Omega(D)$ for which

$$per(Z)=\min\{perX \mid X\in \Omega(D)\}$$

An n-square (0, 1)-matrix D is called *bary centric* [8] if

$$per\ b(D)=\min\{perX \mid X\in \Omega(D)\}$$

where the bary center b(D) of $\Omega(D)$ is given by

$$b(D)=\frac{1}{per(D)}\sum_{P\in\Omega(D)} P$$

such that the summation extends over the set of all permutation matrices P with $P\in\Omega(D)$ and per(D) is their number.

An $n\times n$ matrix A with nonnegative entries is called *partly decomposable* if it contains an $s\times(n-s)$ zero submatrix, otherwise it is *fully indecomposable* [6].

If $A=[a_{ij}]$ is an n-square matrix, then $A(i\mid j)$ is the $(n-1)\times(n-1)$ submatrix obtained from A by deleting $i$th row and $j$th column. The $k$th column of A is denoted by $\mathbf{a}_k$, $k=1, 2, \cdots n$. It is sometimes convenient to denote the permanent of A by $per(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)$.

If column $k$ of n-square matrix A contains exactly two nonzero entries, say in rows $i$th and $j$th, then the $(n-1)$-square matrix c(A) obtained from A by replacing row $i$th with the sum of rows $i$th and $j$th and deleting row $j$th and column $k$th is called a *contraction* of A. If A is fully indecomposable, so is c(A) [3].

Now, we start with known results.

**Lemma 2.1 ([9]).** Let $D=[d_{ij}]$ be a fully indecomposable n-square matrix, and let $A=[a_{ij}]$ be a minimizing matrix on $\Omega(D)$. Then A is fully indecomposable, and for $(i, j) \ni d_{ij} \neq 0$

$$\text{per } A(i \mid j) = \text{per}(A) \qquad \text{if } a_{ij} > 0$$

$$\text{per } A(i \mid j) \geqq \text{per}(A) \qquad \text{if } a_{ij} = 0$$

**Lemma 2.2 ([3]).** Suppose $A \epsilon \ \Omega_n$ is fully indecomposable and has a column (row) with exactly two positive entries. Then $\overline{c(\bar{A})}$ is $(n-1)$-square doubly stochastic and fully indecomposable and

$$2 \text{ per}(A) \geqq 2 \text{ per}(\bar{A}) \geqq \text{per } c(\bar{A}) \geqq \text{per } \overline{c(\bar{A})}$$

where $\bar{A}$ is a minimizing matrix on $\Omega(A)$ and $c(\bar{A})$ is a contraction of $\bar{A}$.



Now, Lemma 2.1 has been strengthened with the aid of Egorychev's reformulation [2] of the Alexandrov's inequality :

$$(\text{per}(A))^2 \geqq \text{per}[\mathbf{a}_1, \cdots \mathbf{a}_{n-1}, \mathbf{a}_{n-1}] \times \text{per}[\mathbf{a}_1, \cdots \mathbf{a}_{n-2}, \mathbf{a}_n, \mathbf{a}_n]$$

for any nonnegativ matrix $A=[\mathbf{a}_1, \cdots \mathbf{a}_n]$

**Lemma 2.3 ([9]).** Let $A=[a_{ij}]$ be a minimizing matrix on $\Omega(D)$, where $D=[\mathbf{d}_1, \cdots \mathbf{d}_n]$ is an n-square (0, 1)-matrix. If, for some $k \leqq n$, $\mathbf{d}_{j1} = \cdots\cdots = \mathbf{d}_{jk}$, and if, some $i$, $a_{ij1} + \cdots + a_{ijk} \neq 0$ then,

$$\text{per } A(i \mid j_t) = \text{per}(A)$$

for $t=1, \cdots, k$

If $A=[\mathbf{a}_1, \cdots \mathbf{a}_n]$ is a minimizing matrix on $\Omega(D)$, $D=[\mathbf{d}_1, \cdots \mathbf{d}_n]$, and if $\mathbf{d}_1=\mathbf{d}_2$, then

$$\text{per}[U\mathbf{a}_1+V\mathbf{a}_2, V\mathbf{a}_1+U\mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_n]=\text{per}(A)$$

for any U, $V \geqq 0$ with $U+V=1$

In 1967, M, Marcus and H, Minc suggested a conjecture as follows,

*Conjecture of Marcus and Minc* [8] : If S is a doubly stochastic $n \times n$ matrix, $n \geqq 2$ then

$$\text{per}(S) \geqq \text{per}\left(\frac{nJ_n - S}{n-1}\right)$$

if $n \geqq 4$, equality can hold if and only if $S=J_n$.



Wang [10] proved inequality for all doubly stochastic $3 \times 3$ matrices, and conjectured that

$$\text{per}(S) \geqq \text{per}\left(\frac{nJ_n + S}{n+1}\right)$$

for all $S \epsilon \Omega_n$. But the two conjectures do not solved completely.

In chapter IV, we have two partial solutions for the conjecture of Wang.

## III. MINIMUM PERMANENTS ON CERTAIN DOUBLY STOCHASTIC MATRICES.

**Proposition 3.1.** Let

$$D = \begin{pmatrix} 0 & 0 & 0 & 1 & \cdots\cdots & 1 \\ 0 & 0 & 0 & 1 & \cdots\cdots & 1 \\ 0 & 0 & 1 & 1 & \cdots\cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots\cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & & \\ 1 & 1 & 1 & 1 & \cdots\cdots & 1 \end{pmatrix}$$

be an $(m+3)$-square $(0, 1)$-matrix. Then the minimum permanent on the face $\Omega(D)$ is

$$(m-2)!\left\{\frac{(m-2)^2}{m^2(m-1)}\right\}^{m-3}\frac{(m-1)(m-2)^2}{m^4}$$



**Proof,** Let

$$A = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{m} & \cdots\cdots & \frac{1}{m} \\ 0 & 0 & 0 & \frac{1}{m} & \cdots\cdots & \frac{1}{m} \\ 0 & 0 & \chi & \frac{1-\chi}{m} & \cdots\cdots & \frac{1-\chi}{m} \\ \frac{1}{m} & \frac{1}{m} & \frac{1-\chi}{m} & Z & \cdots\cdots & Z \\ & \cdots\cdots & & \vdots & & \vdots \\ \frac{1}{m} & \frac{1}{m} & \frac{1-\chi}{m} & Z & \cdots\cdots & Z \end{pmatrix}$$

be a minimizing matrix on the face $\Omega$ (D). Then

$$mZ + \frac{2}{m} + \frac{1-\chi}{m} = 1$$

from doubly stochastic matrix A. Hence

$$Z = \frac{(m-3)+\chi}{m^2}$$

Now, we calculate:

$$\mathrm{per}(A) = \left(\frac{m-1}{m}\right)^2 \left\{ \chi \cdot (m-2)!\ Z^{m-2} + (m-2) \left(\frac{1-\chi}{m}\right)^2 (m-2)(m-3)!Z^{m-3} \right\}$$

$$= \left(\frac{m-1}{m}\right)^2 (m-2)!Z^{m-3} \left\{ \chi Z + (m-2) \left(\frac{1-\chi}{m}\right)^2 \right\}$$

$$= \left(\frac{m-1}{m}\right)^2 (m-2)!\ Z^{m-3} \left\{ \chi Z + \frac{m-2}{m^2} (1-\chi)^2 \right\}$$

Consider

$$\frac{d(\mathrm{per}(A))}{d\chi} = \left(\frac{m-1}{m}\right)^2 (m-2)!(m-3)Z^{m-4} \cdot \frac{1}{m^2} \left\{ \chi Z + \frac{m-2}{m^2} (1-\chi)^2 \right\}$$

$$+ \left(\frac{m-1}{m}\right)^2 (m-2)!Z^{m-3} \left\{ Z + \frac{\chi}{m^2} - \frac{m-2}{m^2} \cdot 2(1-\chi) \right\}$$

$$= \left(\frac{m-1}{m}\right)^2 (m-2)!\ Z^{m-4} \left[ \frac{m-3}{m^2} \left\{ \chi Z + \frac{m-2}{m^2} (1-\chi)^2 \right\} + Z^2 + \frac{\chi Z}{m^2} \right.$$

$$\left. - \left(\frac{m-2}{m^2}\right) 2(1-\chi)Z \right\} = 0$$

If $Z=0$, then A is partly decomposable. This contradicts to Lemma 2.1. thus $Z \neq 0$ so,

$$\frac{m-3}{m^2}\left\{\chi Z+\frac{m-2}{m^2}(1-\chi)^2+Z^2+\frac{\chi Z}{m^2}-\left(\frac{m-2}{m^2}\right)2(1-\chi)Z\right\}$$

$$=\left(\frac{m-3}{m^2}\chi\right)\left(\frac{(m-3)+\chi}{m^2}\right)+\frac{(m-3)(m-2)}{m^4}(1-\chi)^2+\left\{\frac{(m-3)+\chi}{m^2}\right\}^2$$

$$+\frac{\chi(m-3+\chi)}{m^4}-\frac{2(m-2)(1-\chi)(m-3+\chi)}{m^4}=0$$

Then,

$$(m-3)x^2+(m-3)^2x+(m-3)(m-2)(x^2-2x+1)+x^2+2(m-3)x+(m-3)^2+x^2+$$

$$(m-3)x+2(m-2)\{x^2+(m-4)x-(m-3)\}=0$$

*i, e,* $(m^2-2m+1)x^2+(m^2-5m+4)x-(m-3)=0$

From this, we have

$$\chi=\frac{1}{m-1} \quad \text{and} \quad Z=\frac{(m-3)+\frac{1}{m-1}}{m^2}=\frac{(m-2)^2}{m^2(m-1)}$$

Therefore

$$\text{per}(A)=\left(\frac{m-1}{m}\right)^2(m-2)!\left(\frac{(m-2)^2}{m^2(m-1)}\right)^{m-3}\left\{\frac{(m-2)^2}{m^2(m-1)^2}+\frac{m-2}{m^2}\left(\frac{m-2}{m-1}\right)^2\right\}$$

$$=(m-2)!\left\{\frac{(m-2)^2}{m^2(m-1)}\right\}^{m-3}\frac{(m-1)(m-2)^2}{m^4} \qquad \square$$

Consider the derangement matrix of degree 4 and we find the minimum permanent on $\Omega(R_4)$ such that

$$R_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

**Proposition 3.2.** If the minimizing matrix A in $\Omega(R_4)$ is symmetric, then

$$A = \frac{1}{3} R_4$$

**Proof,** Let

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{12} & 0 & a_{23} & a_{24} \\ a_{13} & a_{23} & 0 & a_{34} \\ a_{14} & a_{24} & a_{34} & 0 \end{pmatrix}$$

be the symmetric minimizing matrix in $\Omega(R_4)$. Then, doubly stochastic property of A implies that



$$A = \begin{pmatrix} 0 & a_{12} & 1-a_{12}-a_{23} & a_{23} \\ a_{12} & 0 & a_{23} & 1-a_{12}-a_{23} \\ 1-a_{12}-a_{23} & a_{23} & 0 & a_{12} \\ a_{23} & 1-a_{12}-a_{23} & a_{12} & 0 \end{pmatrix}$$

with nonzero $a_{12}$ and $a_{23}$. Since A is a minimizing matrix in $\Omega(R_4)$. We have per $A(1 \mid 2)$ = per $A(1 \mid 3)$ = per (A) from Lemma 2.1.

Calculating

$$\text{per } A(1 \mid 2) = \text{per} \begin{pmatrix} a_{12} & a_{23} & 1-a_{12}-a_{23} \\ 1-a_{12}-a_{23} & 0 & a_{12} \\ a_{23} & a_{12} & 0 \end{pmatrix}$$

$$= a_{12}{}^{3} + a_{12}a_{23}{}^{2} + a_{12}(1-a_{12}-a_{13})^{2} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (3.1)$$

and

$$\text{per } A(1 \mid 3) = \text{per} \begin{pmatrix} a_{12} & 0 & 1-a_{12}-a_{23} \\ 1-a_{12}-a_{23} & a_{23} & a_{12} \\ a_{23} & 1-a_{12}-a_{23} & 0 \end{pmatrix}$$

$$= a_{12}{}^{2}(1-a_{12}-a_{23}) + (1-a_{12}-a_{23})^{3} + a_{23}{}^{2}(1-a_{12}-a_{23}) \quad \cdots\cdots\cdots \quad (3.2)$$

Then,

$$0 = (3.1) - (3.2)$$

$$= a_{12}{}^{2}(a_{12}-1+a_{12}+a_{23}) + a_{23}{}^{2}(a_{12}-1+a_{12}+a_{23}) + (1-a_{12}-a_{23})^{2}(a_{12}-1+a_{12}+a_{23})$$

$$= (2a_{12}+a_{23}-1)\{a_{12}{}^{2}+a_{23}{}^{2}+(1-a_{23}-a_{23})^{2}\}$$



Hence,

$$2a_{12}+a_{23}-1=0, \qquad a_{23}=1-2a_{12}$$

Then,

$$A = \begin{pmatrix} 0 & a_{12} & a_{12} & 1-2a_{12} \\ a_{12} & 0 & 1-2a_{12} & a_{12} \\ a_{12} & 1-2a_{12} & 0 & a_{12} \\ 1-2a_{12} & a_{12} & a_{12} & 0 \end{pmatrix}$$

$$\mathrm{per}(A) = a_{12}\,\mathrm{per}\begin{pmatrix} a_{12} & 1-2a_{12} & a_{12} \\ a_{12} & 0 & a_{12} \\ 1-2a_{12} & a_{12} & 0 \end{pmatrix} + a_{12}\,\mathrm{per}\begin{pmatrix} a_{12} & 0 & a_{12} \\ a_{12} & 1-2a_{12} & a_{12} \\ 1-2a_{12} & a_{12} & 0 \end{pmatrix}$$

$$+(1-2a_{12})\,\mathrm{per}\begin{pmatrix} a_{12} & 0 & 1-2a_{12} \\ a_{12} & 1-2a_{12} & 0 \\ 1-2a_{12} & a_{12} & a_{12} \end{pmatrix}$$

$$= a_{12}\{ a_{12}^3 + a_{12}(1-2a_{12})^2 + a_{12}^3\} \times 2 + (1-2a_{12})\{ a_{12}^2(1-2a_{12}) + (1-2a_{12})a_{12}^2$$

$$+ (1-2a_{12})^3\}$$

$$= 2a_{12}\{2a_{12}^3 + a_{12}(1-2a_{12})^2\} + 2(1-2a_{12})^2 a_{12}^2 + (1-2a_{12})^4$$

$$= 4a_{12}^4 + (1-2a_{12})^4 + 4a_{12}^2(1-2a_{12})^2$$

$$= \{2a_{12}^2 + (1-2a_{12})^2\}^2$$

$$= \{6(a_{12} - \tfrac{1}{3})^2 + \tfrac{1}{3}\}^2$$



Hence, the minimum value of A is 1/9 when $a_{12}=1/3$. And that $A=1/3R_4$ is the minimizing matrix in $\Omega(R_4)$ ▣

Consider the toeplitz matrix of the degree 4 such that

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

and we find the minimum permanent on the face $\Omega(A_4)$.

**Lemma 3.3.** Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

be a minimizing matrix of the form $a_{11}=a_{44}=a$ on $\Omega(A_4)$. Then, the minimum permanent in $\Omega(A_4)$ is

$$\frac{7}{4}a^4 - 3a^3 + \frac{5}{2}a^2 - a + \frac{1}{4}$$

where, $a$ is a real root of $7a^3-9a^2+5a-1=0$

**Proof,** Consider the matrix $B \in \Omega(A_4)$ obtained from A, by replacing columns 2 and 3 by their average. *i, e*



$$B = \begin{pmatrix} a & \frac{1-a}{2} & \frac{1-a}{2} & 0 \\ a_{21} & \frac{a_{22}+a_{23}}{2} & \frac{a_{22}+a_{23}}{2} & a_{24} \\ a_{31} & \frac{a_{32}+a_{33}}{2} & \frac{a_{32}+a_{33}}{2} & a_{34} \\ 0 & \frac{1-a}{2} & \frac{1-a}{2} & a \end{pmatrix}$$

Then, per(B)=per(A). Again, consider the matrix $C \in \Omega(A_4)$ obtain from B by replacing rows 2 and 3 by their average. *i, e*

$$C = \begin{pmatrix} a & \frac{1-a}{2} & \frac{1-a}{2} & 0 \\ \frac{1-a}{2} & \frac{a}{2} & \frac{a}{2} & \frac{1-a}{2} \\ \frac{1-a}{2} & \frac{a}{2} & \frac{a}{2} & \frac{1-a}{2} \\ 0 & \frac{1-a}{2} & \frac{1-a}{2} & 0 \end{pmatrix}$$

Then, per(C) = per(B) = per(A) by Lemma 2.3. Evaluating the permanent of C

$$\mathrm{per}(C) = a \ \mathrm{per} \begin{pmatrix} \frac{a}{2} & \frac{a}{2} & \frac{1-a}{2} \\ \frac{a}{2} & \frac{a}{2} & \frac{1-a}{2} \\ \frac{1-a}{2} & \frac{1-a}{2} & a \end{pmatrix} + 2 \cdot \frac{1-a}{2} \ \mathrm{per} \begin{pmatrix} \frac{1-a}{2} & \frac{1-a}{a} & 0 \\ \frac{a}{2} & \frac{a}{2} & \frac{1-a}{2} \\ \frac{1-a}{2} & \frac{1-a}{2} & a \end{pmatrix}$$

$$= a \left\{ \frac{a}{2} \left( \frac{a^2}{2} + \left(\frac{1-a}{2}\right)^2 \right) + \frac{a}{2} \left( \frac{a^2}{2} + \left(\frac{1-a}{2}\right)^2 \right) + \frac{1-a}{2} \left( \frac{a(1-a)}{4} \times 2 \right) \right\}$$

$$+ (1-a) \left\{ \frac{1-a}{2} \left( \frac{a^2}{2} + \left(\frac{1-a}{2}\right)^2 \right) \times 2 \right\}$$



$$= a \left\{ \frac{a}{2} \left( \frac{a^2}{2} + \frac{1-2a+a^2}{4} \right) \times 2 + \frac{a(1-2a+a^2)}{4} \right\} + (1-a)^2 \left( \frac{a^2}{2} + \frac{1-2a+a^2}{4} \right)$$

$$= a^2 \left\{ \frac{a^2+1-2a+a^2}{2} \right\} + (1-a)^2 \left( \frac{2a^2+1-2a+a^2}{4} \right)$$

$$= \frac{a^2}{2} (2a^2-2a+1) + \frac{1}{4} (1-2a+a^2)(1-2a+3a^2)$$

$$= a^4 - a^3 + \frac{a^2}{2} + \frac{1}{4} (3a^4-6a^3+3a^2-2a^3+4a^2-2a+a^2-2a+1)$$

$$= \frac{7}{4} a^4 - 3a^3 + \frac{5}{2} a^2 - a + \frac{1}{4}$$

Put,

$$f(a) = \text{per}(C) = \frac{7}{4}a^4 - 3a^3 + \frac{5}{2}a^2 - a + \frac{1}{4}$$

Let us,

$$f'(a) = 7a^3 - 9a^2 + 5a - 1 = 0$$

Put,

$$a = y + \frac{3}{7}$$

Then,

$$y^3 + \frac{8}{49}y + \frac{2}{7^3} = 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (3.3)$$

Put,

$$P = \frac{8}{49 \times 3} = \frac{8}{148}, \quad q = \frac{2}{7^3}$$

Then,

$$U^3,\ V^3 \text{ are roots of } \quad t^2 + \frac{2}{7^3}t - \left(\frac{8}{147}\right)^3 = 0$$

and,



$$U^3 = \frac{1}{2}\left(-\frac{2}{7^3} + \sqrt{\left(\frac{2}{343}\right)^2 + 4\left(\frac{8}{147}\right)^3}\right) = \frac{-9+7\sqrt{33}}{7^3 \times 3^2}$$

$$V^3 = \frac{1}{2}\left(-\frac{2}{7^3} - \sqrt{\left(\frac{2}{343}\right)^2 - 4\left(\frac{8}{147}\right)^3}\right) = \frac{-9-7\sqrt{33}}{7^3 \times 3^2}$$

Hence,

$$a = \sqrt[3]{\frac{-9+7\sqrt{33}}{7^3 \times 3^2}} - \sqrt[3]{\frac{9+7\sqrt{33}}{7^3 \times 3^2}} + \frac{3}{7} \fallingdotseq 0.3920314 \qquad \blacksquare$$

**Proposition 3.4.** On the face $\Omega(A_4)$, we have the minimum permanent as Lemma 3.3 which occurs uniquely at

$$A = \begin{bmatrix} a & \frac{1-a}{2} & \frac{1-a}{2} & 0 \\ \frac{1-a}{2} & \frac{a}{2} & \frac{a}{2} & \frac{1-a}{2} \\ \frac{1-a}{2} & \frac{a}{2} & \frac{a}{2} & \frac{1-a}{2} \\ 0 & \frac{1-a}{2} & \frac{1-a}{2} & a \end{bmatrix} \qquad \cdots\cdots (3.4)$$

where $a$ is the value of Lemma 3.3.

**Proof,** Assume that

$$\mathrm{per}(\bar{A}) = \min\{\mathrm{per}(Y) : Y \in \Omega(A_4)\}$$

and put,

$$\bar{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad \cdots\cdots (3.5)$$



Since, the columns 2 and 3 has the same zero pattern. We may have a matrix

$$B = [b_{ij}] \in \Omega(A_4)$$

such that

$$b_{i1} = a_{i1},\ b_{i4} = a_{i4},\quad b_{i2} = b_{i3} = \frac{a_{i2} + a_{i3}}{2}\ (i = 1, 2, 3, 4)$$

but, since B has the entries $b_{12} = b_{13}$, $b_{42} = b_{43}$. We may write B as follows:

$$B = \begin{pmatrix} b_{11} & \frac{1-b_{11}}{2} & \frac{1-b_{11}}{2} & 0 \\ b_{21} & b_{22} & b_{22} & b_{24} \\ b_{31} & b_{32} & b_{32} & b_{34} \\ 0 & \frac{1-b_{44}}{2} & \frac{1-b_{44}}{2} & b_{44} \end{pmatrix} \quad \cdots\cdots\cdots\cdots\cdots (3.6)$$

Then, $\mathrm{per}(B)=\mathrm{per}(\bar{A})$ and hence B is a minimizing matrix in $\Omega(A_4)$ by Lemma 2.3. Again, since the rows 2 and 3 has the same zero pattern. We may have a matrix

$$C=[C_{ij}]\in\Omega(A_4)$$

such that

$$C_{1i}=b_{1i},\ C_{4i}=B_{4i},\ C_{2i}=C_{3i}=\frac{b_{2i}+b_{3i}}{2}(i=1,2,3,4)$$

but, since C has the entries $C_{12}=C_{13}=C_{21}=C_{31}$, $C_{24}=C_{34}=C_{42}=C_{43}$, $C_{22}=C_{23}=C_{32}=C_{33}$

We may write C as follows :



$$C = \begin{pmatrix} a & \frac{1-a}{2} & \frac{1-a}{2} & 0 \\ \frac{1-a}{2} & \frac{a+b}{4} & \frac{a+b}{4} & \frac{1-b}{2} \\ \frac{1-a}{2} & \frac{a+b}{4} & \frac{a+b}{4} & \frac{1-b}{2} \\ 0 & \frac{1-b}{2} & \frac{1-b}{2} & b \end{pmatrix} \quad \cdots\cdots\cdots\cdots\cdots (3.7)$$

where, $a=C_{11}$, $b=C_{44}$, then $\mathrm{per}(C)=\mathrm{per}(B)=\mathrm{per}(\bar{A})$ by Lemma 2.3. And hence, C is a minimizing matrix in $\Omega(A_4)$. Then, C is fully indecomposable, and hence

$\mathrm{per}\ c(1\mid 1)=\mathrm{per}\ c(4\mid 4)=\mathrm{per}(C)$. Since

$$\text{per }(1|1) = \frac{a+b}{4}\left\{\frac{a+b}{4}\cdot b+\left(\frac{1-b}{2}\right)^2\right\}2+\frac{1-b}{2}\left\{\frac{a+b}{4}\cdot\frac{1-b}{2}\cdot 2\right\}$$

$$= \frac{a+b}{8}\left(3b^2+(a-4)b+2\right) \quad \cdots\cdots (3.8)$$

And similarly

$$\text{per } C\ (4|4) = \frac{1-a}{2}\left\{\frac{(a+b)(1-a)}{8}\cdot 2\right\}+2\cdot\frac{a+b}{4}\left\{\frac{a(a+b)}{4}+\left(\frac{1-a}{2}\right)^2\right\}$$

$$=\frac{a+b}{8}\left(1-2a+a^2+a^2+ab+1-2a+a^2\right)$$

$$=\frac{a+b}{8}\left(3a^2+(b-4)a+2\right) \quad \cdots\cdots (3.9)$$

Therefore, from (3.8) and (3.9), we have

$$0 = \text{per } C\ (1|1)-\text{per}C\,(4|4)$$

$$= \frac{a+b}{8}\left\{3\,(b^2-a^2)-4\,(b-a)\right\} = \frac{a+b}{8}(b-a)\left\{3\,(b+a)-4\right\}$$



Then, $a+b=0$ or $3(b+a)-4=0$ or $b-a=0$ $\cdots\cdots$ (3.10)

We compute the per(C) for case 1) $a=b$ for case 2) $a+b=0$ for case 3) $a+b=4/3$ respectively.

Case 1) $a=b$, then the minimum permanent obtained in Lemma 3.3. That is

$$\text{per}(C) = \text{per}\begin{pmatrix} a & \frac{1-a}{2} & \frac{1-a}{2} & 0 \\ \frac{1-a}{2} & \frac{a}{2} & \frac{a}{2} & \frac{1-a}{2} \\ \frac{1-a}{2} & \frac{a}{2} & \frac{a}{2} & \frac{1-a}{2} \\ 0 & \frac{1-a}{2} & \frac{1-a}{2} & a \end{pmatrix} \quad \cdots\cdots (3.11)$$

has minimum value at which is the given value

$$a = \frac{1}{7}\left( \sqrt[3]{\frac{-9+7\sqrt{33}}{9}} - \sqrt[3]{\frac{9+7\sqrt{33}}{9}} + 3 \right) \text{ in Lemma 3.3.}$$

Case 2) $a+b=0$, since $a \geqq 0$, and $b \geqq 0$, hence $a=b=0$.

Then,

$$C = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \cdots\cdots\cdots\cdots (3.12)$$

per $(C) = \frac{1}{4}$ : not minimum in $\Omega(A_4)$ by case 1) and Lemma 3.3.

Case 3) $a+b=\frac{4}{3}$, we replace $b=\frac{4}{3}-a$ to C in (3.7).

Then,



$$C = \begin{pmatrix} a & \frac{1-a}{2} & \frac{1-a}{2} & 0 \\ \frac{1-a}{2} & \frac{1}{3} & \frac{1}{3} & \frac{3a-1}{6} \\ \frac{1-a}{2} & \frac{1}{3} & \frac{1}{3} & \frac{3a-1}{6} \\ 0 & \frac{3a-1}{6} & \frac{3a-1}{6} & \frac{4-3a}{3} \end{pmatrix} \cdots\cdots\cdots\cdots (3.13)$$

And

$$\text{per}(C) = a \cdot \text{per}\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{3a-1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{3a-1}{6} \\ \frac{3a-1}{6} & \frac{3a-1}{6} & \frac{4-3a}{3} \end{pmatrix} + \frac{1-a}{2} \text{ per}\begin{pmatrix} \frac{1-a}{2} & \frac{1}{3} & \frac{3a-1}{6} \\ \frac{1-a}{2} & \frac{1}{3} & \frac{3a-1}{6} \\ 0 & \frac{3a-1}{6} & \frac{4-3a}{3} \end{pmatrix} \times 2$$

$$= a\left\{\frac{1}{3}\left\{\frac{4-3a}{9}+\frac{(3a-1)^2}{36}\right\}\times 2+\frac{3a-1}{6}\left(\frac{3a-1}{18}\right)\times 2\right\}$$

$$+(1-a)\left\{\frac{1-a}{2}\left\{\frac{4-3a}{9}+\frac{(3a-1)^2}{36}\right\}\times 2\right\}$$

$$=\frac{1}{2^2\cdot 3^3}\left(27a^4-72a^3+138a^2-120a+51\right) \quad\cdots\cdots\cdots\cdots \quad (3.14)$$

Let, $f(a)=27a^4-72a^3+138a^2-120a+51$.

In order to find the minimum value of $2^2\cdot 3^3\cdot f(a)$.

We use the derivative function with respect to $a$, then,

$$f'(a)=108a^3-216a^2+276a-120=0$$

Put, $a=y+\frac{2}{3}$

Then, $y^3+\left(-\frac{4}{3}+\frac{23}{9}\right)y=0$

hence, $y=0$ and hence $a=\frac{2}{3}$



Per(C) has minimum (and least) value at $a=\frac{2}{3}=b$. then, it becomes the case 1) and this is not minimum in $\Omega(A_4)$. In fact,

$$C=\begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix}$$

And by (3.14)

$$\text{per}(C)=\frac{1}{2^2\cdot 3^3}\left\{27\left(\frac{2}{3}\right)^4-72\left(\frac{2}{3}\right)^3+138\left(\frac{2}{3}\right)^2-120\cdot\frac{2}{3}+51\right\}\fallingdotseq 0.1512345$$

Therefore, comparing the case 1)~case 3), we know that per $(\bar{A})$ has the minimum value at the given matrix (3.4), where $a = \frac{1}{7}\left( \sqrt[3]{\frac{-9+7\sqrt{33}}{9}} - \sqrt[3]{\frac{9+7\sqrt{33}}{9}} + 3 \right)$ ◘

**Theorem 3.5.** Let $D_n = [d_{ij}]$ be n-square (0.1)-matrix such that $d_{ij}=1$ if $|\, i-j \,| \leqq 1$ and $d_{ij}=0$ Otherwise.

Then, for any $A \epsilon\ \Omega(D_n)$

$$\mathrm{per}(A) \geqq \frac{1}{2^{n-1}}$$

with equality if and only if $a_{11} = a_{nn} = a_{k,k+1} = a_{k+1,k} = \frac{1}{2}$ for $k=1, \cdots\cdots, n-1$

**Proof,** We prove the theorem by induction on n, when $n=2$, it is easily verified. Assume that the theorem holds for $n-1$. Then

Let,

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots\cdots & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \ \cdots\cdots & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} \ \ 0 \ \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots\cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \cdots\cdots & 0 & a_{n,n-1} & a_{n,n} \end{pmatrix}_{n \times n}$$

be the minimizing matrix in $\Omega(D_n)$.

Then,

$$C(A) = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots\cdots & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \ \cdots\cdots & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} \ \ 0 \ \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & \cdots & \cdots\cdots & a_{n-1,n-2} & a_{n-1,n-1}+a_{n,n-1} \end{pmatrix}$$

is the contraction (on the $n$th column) of A. Since $\overline{C(A)}$ is the minimizing matrix in $\Omega(C(A))$, we have per $\overline{C(A)} \leq$ per $C(A) = 2$ per $(A)$ by Lemma 2.2.

But,

$$\mathrm{per}(\overline{C(A)}) = \frac{1}{2^{n-2}}$$

by induction assumption, and has the form

$$\overline{C(A)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & & & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \cdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (n-1)\times(n-1)$$

Hence, $\mathrm{per}(A) \geq \frac{1}{2}\mathrm{per}(\overline{C(A)}) \geq \frac{1}{2^{n-1}}$

and, equality holds for $\overline{C(A)} = C(A)$.



*i, e,* A has the form

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & & & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & & & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \cdots & & 0 \\ \cdots & & & & & & \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & k & \frac{1}{2}-k \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2}-k & k+\frac{1}{2} \end{pmatrix}$$

since,

$$\mathrm{per}(A) = \frac{1}{2^{n-2}}\left(2k^2 + \frac{1}{2}\right) \geqq \frac{1}{2^{n-1}}$$

The minimizing property of A shows that $k=0$

Hence, we have the required form A, such that

$$\mathrm{per}(A) = \min\{\mathrm{per}(X) \mid X \in \Omega(D_n)\}$$ ▣

# IV. PARTIAL SOLUTION FOR A WANG'S CONJECTURE ON PERMANENT FUNCTION

E. T. Wang's conjecture :

If S is a doubly stochastic $n \times n$ matrix, $n \geqq 2$ then,

$$per(s) \geqq per\left(\frac{nJ_n + S}{n+1}\right) \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (4.1)$$

If $n \geqq 3$, equality can hold in (4.1) if and only if $S = J_n$

**Proposition 4.1.** The above conjecture implies the Van der Waerden's conjecture

**Proof,** Let $f : \Omega_n \rightarrow \Omega_n$ be defined by

$$f(s) = J_n - \frac{J_n - S}{n+1} = \frac{nJ_n + S}{n+1} = \frac{(n+1)J_n + S - J_n}{n+1} = J_n - \frac{J_n - S}{n+1}$$

Moreover,



$$f^k(s) = J_n - \frac{J_n - S}{(n+1)^k}$$

Then, if $S \neq J_n$, the inequality (4.1) becomes

$$per(s) > per(f(s))$$

and therefore $per(s) > per(f(s)) > per(f^k(s))$

for $k = 2, 3, \cdots\cdots$

Hence,

$$per(s) = \lim_{k \to \infty} per(f^k(s)) = per\left[\lim_{k \to \infty}\left(J_n - \frac{J_n - S}{(n+1)^k}\right)\right]$$

$$= \text{per}(J_n) = \frac{n!}{n^n}$$

**Lemma 4.2[6].** Let $A=(a_{ij})$ be an $n \times n$ real matrix all of whose row sums and column sums are equal to 0. Then the sum of all subpermanents of A of order 2 is positive unless $A=0$.

**Theorem 4.3.** If $S=(s_{ij})$ is doubly stochastic matrix in a sufficiently small neighborhood of $J_n$, then,

$$\text{per}(s) \geqq \text{per}\left(\frac{nJ_n + S}{n+1}\right) \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (4.2)$$

and the equality holds in (4.2) if and only if $S=J_n$

**Proof,** If $n=2$, then the theorem is trivial. Let $n \geqq 3$, and let $A=(a_{ij})=S-J_n$, and $\sigma_k$ denote the sum of all subpermanents of A of order K. Then A satisfies the hypothesis of Lemma 4.2.

Hence $\sigma_2 > 0$ unless $A=0$, that is, $S=J_n$. Now $\sigma_1=0$, and therefore



$$\begin{aligned}
\text{per}(s) &= \text{per}(J_n + A) \\
&= \frac{n!}{n^n} + \frac{(n-1)!}{n^{n-1}}\sigma_1 + \frac{(n-2)!}{n^{n-2}}\sigma_2 + \cdots + \frac{2!}{n^2}\sigma_{n-2} + \frac{1!}{n}\sigma_{n-1} + \sigma_n \\
&= \frac{n!}{n^n} + \frac{(n-2)!}{n^{n-2}}\sigma_2 + \sum_{t=3}^{n} \frac{(n-t)!}{n^{n-t}}\sigma_t
\end{aligned}$$

On the other hand

$$\text{per}\left(\frac{nJ_n + S}{n+1}\right) = \text{per}\left(J_n + \frac{A}{n+1}\right)$$

$$= \frac{n!}{n^n} + \frac{(n-2)!}{n^{n-2}} \frac{1}{(n+1)^2}\sigma_2 + \cdots + \sum_{t=3}^{n} \frac{(n-t)!}{n^{n-t}} \frac{1}{(n+1)^t}\sigma_t$$

Hence, if $A \neq 0$, and if all the entries of A are sufficiently small in absolute value so that

$$\frac{(n-2)!}{n^{n-2}}\left(1-\frac{1}{(n+1)^2}\right)\sigma_2+\sum_{t=3}^{n}\left(1-\frac{1}{(n+1)^t}\right)\frac{(n-t)!}{n^{n-t}}\sigma_t>0 \quad \cdots\cdots (4.3)$$

Then

$$\mathrm{per}(s)-\mathrm{per}\left(\frac{nJ_n+S}{n+1}\right)>0$$

If $S \neq J_n$ is a doubly stochastic matrix in a sufficiently small neighborhood of $J_n$ so that (4.3) holds, then (4.2) cannot be an equality. Of course, if $S=J_n$, then actually $S=\frac{nJ+S}{n+1}$ and equality trivially holds in (4.2). ◘

**Lemma 4.4.**[6]. If A is positive semi-definite hermitian with eigenvalues $\lambda_1, \cdots\cdots, \lambda_n$, then we may write $A=U^*DU$, where U is unitary and

$D=\mathrm{diag}(\lambda_1, \cdots\cdots, \lambda_n)$, and

$$\mathrm{per}(A)=\sum_{w\in G_{n,n}}\frac{1}{\mu(w)}|\mathrm{per}(U[w|1,\cdots,n])|^2\prod_{t=1}^{n}\lambda_t^{m_t(w)} \quad \cdots\cdots (4.4)$$

where $\mu(w)=\prod_{t=1}^{n} m_t(w)!$ and $m_t(w)$ denotes the number of times the integer $t$ occurs in w.

**Theorem 4.5.** If S is a positive semi-definite symmetric doubly stochastic matrix, then

$$\mathrm{per}(s)\geq \mathrm{per}\left(\frac{nJ_n+S}{n+1}\right) \quad \cdots\cdots (4.5)$$

and the equality can hold in (4.5) if and only if $S=J_n$

**Proof,** Let $1=\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n = 0$ be the eigenvalues of S. Since S and $J_n$ commute, the eigenvalues of $\dfrac{nJ_n+S}{n+1}$ are

$$1, \frac{\lambda_2}{n+1}, \cdots\cdots\cdots, \frac{\lambda_n}{n+1}$$

Let $V_1 = \dfrac{(1,\cdots,1)}{\sqrt{n}}$, $v_2,\cdots,v_n$ be an orthonormal set of eigenvectors common to S and $\dfrac{nJ_n+S}{n+1}$, and let U be the unitary matrix whose $i$-th row is $V_i$, $i=1, \cdots, n$. by (4.4).

$$\mathrm{per}(s) = \sum_{\gamma \in G_{n,n}} \frac{C\gamma}{\mu(\gamma)} \prod_{t=2}^{n} (\lambda_t)^{m_t(\gamma)}$$

and

$$\mathrm{per}\left(\frac{nJ_n+S}{n+1}\right) = \sum_{\gamma \in G_{n,n}} \frac{C\gamma}{\mu(\gamma)} \prod_{t=2}^{n} \left(\frac{\lambda_t}{n+1}\right)^{m_t(\gamma)}$$

where,

$$C_\gamma = |\,\mathrm{per}\,(U[\gamma|1, \cdots, n])\,|^2,$$

$$\mu(\gamma) = \prod_{t=1}^{n} m_t(\gamma)!$$



and $m_t(\gamma)$ denotes the number of times the integer $t$ occurs in $\gamma$.

Clearly

$$\frac{C_\gamma}{\mu(\gamma)} \prod_{t=2}^{n} \lambda_t^{m_t(\gamma)} \geqq \frac{C_\gamma}{\mu(\gamma)} \prod_{t=2}^{n} \left(\frac{\lambda_t}{n+1}\right)^{m_t(\gamma)} \quad \cdots\cdots\cdots\cdots \ (4.6)$$

for any $\gamma$, and hence the inequality (4.5) follows. If equality holds in (4.5), then (4.6) is equality for every $\gamma$. We show by an appropriate choice of $\gamma$ that this implies that $\lambda_2 = \cdots\cdots = \lambda_n = 0$.

let, $V_2 = (X_1, \cdots, X_n)$, and suppose that

$X_{i1}, \cdots, X_{ik}$ are nonzero and $X_j=0$ for $j \notin \{i_1, \cdots\cdots, i_k\}$.

Let, $\gamma = (\gamma_1, \cdots, \gamma_n)$ where $\gamma_1 = \cdots = \gamma_{n-k} = 1$ and

$\gamma_{n-k+1} = \cdots = \gamma_n = 2$

Then

$$C_\gamma = |\operatorname{per}(u[1, \cdots, 1, 2, \cdots, 2 \mid 1, \cdots, n])|^2$$

$$= \frac{1}{n^{n-k}} \operatorname{per} \left| \begin{pmatrix} 1 & \cdots\cdots\cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots\cdots\cdots & 1 \\ X_1 & \cdots\cdots\cdots & X_n \\ \vdots & & \vdots \\ X_1 & \cdots\cdots\cdots & X_n \end{pmatrix} \right|^2 \quad \cdots\cdots\cdots (4.7)$$

We evaluate the permanent in (4.7) using the Laplace expansion on the last K rows ;

$$C_\gamma = \frac{1}{n^{n-k}} \left| K!(n-k)! \left( \prod_{s=1}^{k} X_{i_s} \right) \right|^2$$

Hence $C_\gamma \neq 0$, and therefore equality in (4.6) implies that



$$\lambda_2^k = \left( \frac{\lambda_2}{n+1} \right)^k$$

and thus $\gamma_2 = 0$. But $\gamma_2 = \cdots = \gamma_n = 0$, the doubly stochastic matrix S has rank 1, and therefore $S = J_n$.

The converse is obvious. ◘

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〈 國文抄錄 〉

# 行列上에서 permanent 函數의 極小값

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본 논문에서는 doubly stochastic 行列들에서 permanent 函數의 極小값을 찾는 문제를 硏究하여, 몇 가지의 주어진 面에서 極小값을 구하고, 이 極小값을 결정하는 극소 行列들을 규명하였다.

또한 permanent 函數에 관하여 E. T. Wang 이 제시한 미 해결 문제의 하나를 硏究하여 두가지 부분적인 해답을 구하였다.