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Extreme preservers of integer matrix pairs derived from column rank inequalities

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CONTENTS

<**Abstract**>

Extreme preservers of integer matrix pairs derived from column rank inequalities

In this thesis, we investigate the surjective linear operators that preserve the sets of integer matrix pairs. These sets are naturally occurred at the extreme cases for the column rank inequalities relative to the sums and multiplications of integer matrices. These sets were constructed with the nonnegative integer matrix pairs which are related with the ranks of the sums and difference of two integer matrices or compared between their column ranks and their real ranks.

That is, we construct the following sets ;

$$\mathcal{C}_{AN}(\mathcal{Z}^+) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{Z}^+)^2 \mid c(X+Y) = n\};$$

$$\mathcal{C}_{AR}(\mathcal{Z}^+) = \{(X,Y) \in \mathcal{M}_{m,n}(\mathcal{Z}^+)^2 \mid c(X+Y) = |\rho(X) - \rho(Y)|\};$$

$$\mathcal{C}_{M2}(\mathcal{Z}^+) = \{(X,Y) \in \mathcal{M}_{n}(\mathcal{Z}^+)^2 \mid c(XY) = c(X)\};$$

$$\mathcal{C}_{MR}(\mathcal{Z}^+) = \{(X,Y) \in \mathcal{M}_{n}(\mathcal{Z}^+)^2 \mid c(X+Y) = \rho(X) + \rho(Y) - n\};$$

For these sets, we consider the linear operators that preserve their properties. We characterize those linear operators as $T(X) = PXQ$ or $T(X) = PXP^t$ with appropriate permutation matrices P and Q. We also give some examples of non-surjective linear maps that preserve these sets.

Extreme preservers of integer matrix pairs derived from column rank inequalities

1 Introduction

The linear preserver problems are one of the most active research subjects in matrix theory during last one hundred years, which concern the characterizations of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. For survey of these types of problems, we refer to the article of Song([12]) and the papers in [11]. The specified frame of problems is of interest both for matrices with entries from a field and for matrices with entries from an arbitrary semiring such as Boolean algebra, nonnegative integers, and fuzzy semiring. It is necessary to note that there are several rank functions over a semiring that are analogues of the classical function of the matrix rank over a field. Detailed research and self-contained information about rank functions over semirings can be found in [2] and [12].

There are some results on the inequalities for the rank function of matrices(see [2] - [6]). Beasley and Guterman ([2]) investigated the rank inequalities of matrices over semirings. And they characterized the equality cases for some rank inequalities in [5].

In this thesis, we construct the sets of nonnegative integer matrix pairs. These sets are naturally occurred at the extreme cases for the column rank inequalities derived from the addition and multiplication of nonnegative integer matrix pairs. We characterize the linear operators that preserve these extreme sets of nonnegative integer matrix pairs.

Definition 1.1. A *semiring* $\mathcal{S}$ consists of a set $\mathcal{S}$ and two binary operations, addition and multiplication, such that:

- $\mathcal{S}$ is an Abelian monoid under addition (identity denoted by 0);
- $\mathcal{S}$ is an Abelian monoid under multiplication (identity denoted by 1);
- multiplication is distributive over addition;
- $s0 = 0s = 0$ for all $s \in \mathcal{S}$.

Let $\mathcal{Z}^+$ be the set of nonnegative integers. Then $\mathcal{Z}^+$ becomes a semiring under the usual addition and multiplication. In this thesis we will study the matrices over the semiring $\mathcal{Z}^+$.

Definition 1.2. A semiring is called *antinegative* if the zero element is the only element with an additive inverse.

It is straightforward to see that the nonnegative integer semiring $\mathcal{Z}^+$ is antinegative.

Let $\mathcal{M}_{m,n}(\mathcal{S})$ denote the set of $m \times n$ matrices with entries from a semiring $\mathcal{S}$. If $m = n$, we use the notation $\mathcal{M}_n(\mathcal{S})$ instead of $\mathcal{M}_{n,n}(\mathcal{S})$.

A vector space is usually only defined over fields or division rings, and modules are generalizations of vector spaces defined over rings. We generalize the concept of vector spaces to *semiring vector spaces* defined over arbitrary semirings.

Definition 1.3. Given a semiring $\mathcal{S}$, we define a *semiring vector space*, $V(\mathcal{S})$, to be a nonempty set with two operations, addition and scalar multiplication such that $V(\mathcal{S})$ is closed under addition and scalar multiplication, addition is associative and commutative, and such that for all $\mathbf{u}$ and $\mathbf{v}$ in $V(\mathcal{S})$ and $r, s \in \mathcal{S}$:

1. There exists a $\mathbf{0}$ such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$,
2. $1\mathbf{v} = \mathbf{v} = \mathbf{v}1$,
3. $rs\mathbf{v} = r(s\mathbf{v})$,
4. $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$, and
5. $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$.

Definition 1.4. A set of vectors with entries from a semiring is called *linearly independent* if there is no vector in this set that can be expressed as a nontrivial linear combination of the others.

Definition 1.5. A collection of linearly independent vectors is said to be a *basis* of the vector space V over a semiring if its linear span is V. The *dimension* of V is a minimal number of vectors in any basis of V.

The following rank functions are usual in the semiring context.

Definition 1.6. The matrix $A \in \mathcal{M}_{m,n}(\mathcal{S})$ is said to be of *factor rank* k ($rank(A) = k$) if there exist matrices $B \in \mathcal{M}_{m,k}(\mathcal{S})$ and $C \in \mathcal{M}_{k,n}(\mathcal{S})$ such that $A = BC$ and k is the smallest positive integer for which such factorization exists. By definition, the only matrix with factor rank 0 is the zero matrix, O.

Definition 1.7. The matrix $A \in \mathcal{M}_{m,n}(\mathcal{S})$ is said to be of *column rank* k ($c(A) = k$) if the dimension of the linear span of the columns of A is equal to k.

Definition 1.8. The matrix $A \in \mathcal{M}_{m,n}(\mathcal{S})$ is said to be of *row rank* k ($r(A) = k$) if the dimension of the linear span of the rows of A is equal to k.

Definition 1.9. The matrix $A \in \mathcal{M}_{m,n}(\mathcal{S})$ is said to be of *term-rank* k ($t(A) = k$) if the least number of lines needed to include all nonzero elements of A is equal to k.

Example 1.10. It follows that

$$1 \leq rank(A) \leq c(A) \leq n$$

for all nonzero matrix $A \in \mathcal{M}_{m,n}(\mathcal{Z}^+)$. These inequalities may be strict: let

$$A = \begin{bmatrix} 3 & 4 \end{bmatrix} \in \mathcal{M}_{1,2}(\mathcal{Z}^+).$$

Then $rank(A) = 1 \ < \ 2 = c(A)$.

Definition 1.11. A *line* of a matrix A is a row or a column of the matrix A.

If $\mathcal{S}$ is a subsemiring of a field then there is a usual rank function $\rho(A)$ for any matrix $A \in \mathcal{M}_{m,n}(\mathcal{S})$. Easy examples show that over semirings these functions are not equal in general. However, the inequalities $r(A) \geq \rho(A)$ and $c(A) \geq \rho(A)$ always hold.

It is well-known that the behavior of the function ρ on the matrices over a field with respect to matrix addition and multiplication is given by the following inequalities([5]):

- the rank-sum inequalities:

$$|\rho(A) - \rho(B)| \leq \rho(A+B) \leq \rho(A) + \rho(B),$$

- Sylvester's laws:

$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min\{\rho(A), \rho(B)\},$$

- and the Frobenius inequality:

$$\rho(AB) + \rho(BC) \leq \rho(ABC) + \rho(B),$$

where A, B are conformal matrices with entries from a field.

Arithmetic properties of column rank (or row rank, factor rank) depend on the structure of semiring of entries. It is restricted by the following list of inequalities established in [2]:

Let $\mathcal{S}$ be an antinegative semiring without zero divisors.

Then for $0 \neq A, B \in \mathcal{M}_{m,n}(\mathcal{S})$,

1. $1 \leq c(A+B), r(A+B)$;
2. $c(A+B) \leq n$;
3. $r(A+B) \leq m$;

If $0 \neq A \in \mathcal{M}_{m,n}(\mathcal{S})$, $0 \neq B \in \mathcal{M}_{n,k}(\mathcal{S})$

4. if $c(A) + r(B) > n$ then $c(AB), r(AB) \geq 1$;
5. $c(AB) \leq c(B)$;

6. $r(AB) \le r(A)$;

Let $\mathcal{S}$ be a subsemiring of $\Re^+$, the nonnegative reals. Then for $A, B \in \mathcal{M}_{m,n}(\mathcal{S})$ one has that

7. $c(A+B), r(A+B) \ge |\rho(A) - \rho(B)|$.

For $A \in \mathcal{M}_{m,n}(\mathcal{S})$, $B \in \mathcal{M}_{n,k}(\mathcal{S})$ one has that

8. if $\rho(A) + \rho(B) \le n$ then $c(AB), r(AB) \ge 0$;

9. if $\rho(A) + \rho(B) > n$ then $c(AB), r(AB) \ge \rho(A) + \rho(B) - n$.

2 Preliminaries

Let us construct the set of matrix pairs that arise as extremal cases in the inequalities listed above section on matrices over $\mathcal{Z}^+$:

$$\mathcal{C}_{AN}(\mathcal{Z}^+) = \{(X, Y) \in \mathcal{M}_{m,n}(\mathcal{Z}^+)^2 \mid c(X+Y) = n\};$$

$$\mathcal{C}_{A1}(\mathcal{Z}^+) = \{(X, Y) \in \mathcal{M}_{m,n}(\mathcal{Z}^+)^2 \mid c(X+Y) = 1\};$$

$$\mathcal{C}_{AR}(\mathcal{Z}^+) = \{(X, Y) \in \mathcal{M}_{m,n}(\mathcal{Z}^+)^2 \mid c(X+Y) = |\rho(X) - \rho(Y)|\};$$

$$\mathcal{C}_{M2}(\mathcal{Z}^+) = \{(X, Y) \in \mathcal{M}_{n}(\mathcal{Z}^+)^2 \mid c(XY) = c(Y)\};$$

$$\mathcal{C}_{M0}(\mathcal{Z}^+) = \{(X, Y) \in \mathcal{M}_{n}(\mathcal{Z}^+)^2 \mid c(XY) = 0\};$$

$$\mathcal{C}_{M1}(\mathcal{Z}^+) = \{(X, Y) \in \mathcal{M}_{n}(\mathcal{Z}^+)^2 \mid c(X) + c(Y) > n \text{ and } c(XY) = 1\};$$

$$\mathcal{C}_{MR}(\mathcal{Z}^+) = \{(X, Y) \in \mathcal{M}_{n}(\mathcal{Z}^+)^2 \mid c(XY) = \rho(X) + \rho(Y) - n\}.$$

As it was proved in [2] the inequalities 1 — 9 in section 1 are sharp and the best possible. The natural question is to characterize the equality cases in the above inequalities. Even over fields this is an open problem, see [9, 10, 14, 15] for more details. The structure of matrix varieties which arise as extremal cases in these inequalities is far from being understood over fields, as well as over semirings. A usual way to generate elements of such a variety is to find a tuple of matrices which belongs to it and to act on this tuple by various linear operators that preserve this variety. The investigation of the corresponding problems over semirings for the factor rank function, term and zero term rank functions was done in [3, 4]. This paper is a continuation of [3, 4] and is devoted to study linear operators that preserve extremal cases of rank inequalities with respect to row and column ranks. The complete classification of linear operators that preserve cases of equalities in various matrix inequalities over fields was obtained in [1, 5, 6, 8]. For the details on linear operators preserving matrix invariants one can see [11] and references therein.

Definition 2.1. An operator $T : \mathcal{M}_{m,n}(\mathcal{Z}^+) \to \mathcal{M}_{m,n}(\mathcal{Z}^+)$ is called *linear* if $T(\alpha X) = \alpha T(X)$ and $T(X+Y) = T(X) + T(Y)$ for all $X, Y \in \mathcal{M}_{m,n}(\mathcal{Z}^+)$, $\alpha \in \mathcal{Z}^+$.

Definition 2.2. We say that an operator $T : \mathcal{M}_{m,n}(\mathcal{Z}^+) \to \mathcal{M}_{m,n}(\mathcal{Z}^+)$ *preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ implies that $T(X) \in \mathcal{P}$, or, if $\mathcal{P}$ is a set of ordered pairs [triples], that $(X, Y) \in \mathcal{P}$ $[(X, Y, Z) \in \mathcal{P}]$ implies $(T(X), T(Y)) \in \mathcal{P}$ $[(T(X), T(Y), T(Z)) \in \mathcal{P}]$.

Definition 2.3. The matrix $X \circ Y$ denotes the *Hadamard* or *Schur product*, i.e., the (i, j) entry of $X \circ Y$ is $x_{i,j}y_{i,j}$.

Definition 2.4. An operator $T : \mathcal{M}_{m,n}(\mathcal{Z}^+) \to \mathcal{M}_{m,n}(\mathcal{Z}^+)$ is called a (P, Q, B)-*operator* if there exist permutation matrices $P \in \mathcal{M}_m(\mathcal{Z}^+)$ and $Q \in \mathcal{M}_n(\mathcal{Z}^+)$, and a matrix $B = [b_{i,j}] \in \mathcal{M}_{m,n}(\mathcal{Z}^+)$, $b_{i,j}$ are nonzero elements from $\mathcal{Z}^+$ for all i, j, $1 \le i \le m$, $1 \le j \le n$, such that $T(X) = P(X \circ B)Q$ for all $X \in \mathcal{M}_{m,n}(\mathcal{Z}^+)$ or when $m = n$ $T(X) = P(X \circ B)^t Q$ for all $X \in \mathcal{M}_n(\mathcal{Z}^+)$ where X^t denotes the transpose of X. An operator T is called a *nontransposing* (P, Q, B)-*operator* if there exist permutation matrices $P \in \mathcal{M}_n(\mathcal{Z}^+)$ and $Q \in \mathcal{M}_n(\mathcal{Z}^+)$, and a matrix $B = [b_{i,j}] \in \mathcal{M}_{m,n}(\mathcal{Z}^+)$, $b_{i,j}$ are nonzero elements from $\mathcal{Z}^+$ for all i, j, $1 \le i \le m$, $1 \le j \le n$, such that $T(X) = P(X \circ B)Q$ for all $X \in \mathcal{M}_{m,n}(\mathcal{Z}^+)$. For the case of $B = J$, (P, Q, B)-operator is called (P, Q)-operator.

Definition 2.5. We say that the matrix A *dominates* the matrix B if and only if $b_{i,j} \ne 0$ implies that $a_{i,j} \ne 0$, and we write $A \ge B$ or $B \le A$ in this case.

Definition 2.6. If A and B are matrices and $A \geq B$ we let $A \backslash B$ denote the matrix C where

$$c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} \neq 0 \\ a_{i,j} & \text{otherwise} \end{cases} .$$

The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J, and O, respectively. The matrix $E_{i,j}$, called a *cell*, denotes the matrix with 1 in (i, j) position and zero elsewhere. A weighted cell is any nonzero scalar multiple of a cell, i.e., $\alpha E_{i,j}$ is a weighted cell for any $0 \neq \alpha \in \mathcal{S}$. Let R_i denote the matrix whose i^{th} row is all ones and all other rows are zero, and C_j denote the matrix whose j^{th} column is all ones and all other columns are zero. We let $|A|$ denote the number of nonzero entries in the matrix A. We denote by $A[i_1, \ldots, i_k | j_1, \ldots, j_l]$ the $k \times l$-submatrix of A which lies in the intersection of the $i_1, \ldots, i_k$ rows and $j_1, \ldots, j_l$ columns.

We obtain some basic results on the linear operators on $\mathcal{M}_{m,n}(\mathcal{Z}^+)$ for later use.

Theorem 2.7. *Let $T : \mathcal{M}_{m,n}(\mathcal{Z}^+) \to \mathcal{M}_{m,n}(\mathcal{Z}^+)$ be a linear operator. Then the following are equivalent:*

1. *T is bijective.*
2. *T is surjective.*
3. *There exists a permutation σ on $\{(i, j) \mid i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$.*

Proof. It is trivial that (1) implies (2) and (3) implies (1).

We now show that 2) implies 3).

We assume that T is surjective. Then, for any pair (i, j), there exists some X such that $T(X) = E_{i,j}$. Clearly $X \neq O$ by the linearity of T. Thus there is a pair of indices (r, s) such that $X = x_{r,s}E_{r,s} + X'$ where (r, s) entry of X' is zero and the following two conditions are satisfied: $x_{r,s} \neq 0$ and $T(E_{r,s}) \neq O$. Indeed, if in the contrary for all pairs (r, s) either $x_{r,s} = 0$ or $T(E_{r,s}) = O$

then $T(X) = 0$ which contradicts with the assumption $T(X) = E_{i,j}$. Since $\mathcal{Z}^+$ is an antinegative semiring without zero divisors it follows that

$$T(x_{r,s}E_{r,s}) \leq T(x_{r,s}E_{r,s}) + T(X \setminus (x_{r,s}E_{r,s})) = T(X) = E_{i,j}.$$

Hence, $x_{r,s}T(E_{r,s}) = T(x_{r,s}E_{r,s}) \leq E_{i,j}$ and $T(E_{r,s}) \neq O$ by the above. Therefore, $T(E_{r,s}) \leq E_{i,j}$. Indeed, if on the contrary, $T(E_{r,s})$ is a sum of certain multiples of cells then so is $x_{r,s}T(E_{r,s})$, since $\mathcal{Z}^+$ is an antinegative semiring without zero divisors.

Let $P_{i,j} = \{E_{r,s} \mid T(E_{r,s}) \leq E_{i,j}\}$. By the above $P_{i,j} \neq \emptyset$ for all (i,j). By its definition $P_{i,j} \cap P_{u,v} = \emptyset$ whenever $(i,j) \neq (u,v)$. That is $\{P_{i,j}\}$ is a set of mn nonempty sets which partition the set of cells. By the pigeonhole principle, we must have that $\mid P_{i,j} \mid = 1$ for all (i,j). Necessarily, for each pair (r,s) there is a unique pair (i,j) such that $T(E_{r,s}) = b_{r,s}E_{i,j}$. That is there is some permutation σ on $\{(i,j) \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\}$ such that for some scalars $b_{i,j}$, $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$. We now only need to show that the $b_{i,j} = 1$. Since T is surjective and $T(E_{r,s}) \not\leq E_{\sigma(i,j)}$ for $(r,s) \neq (i,j)$, there is some α such that $T(\alpha E_{i,j}) = E_{\sigma(i,j)}$. But then, since T is linear,

$$E_{\sigma(i,j)} = T(\alpha E_{i,j}) = \alpha T(E_{i,j}) = \alpha b_{i,j}E_{\sigma(i,j)}.$$

That is, $\alpha b_{i,j} = 1$, or $b_{i,j} = 1$ in $\mathcal{Z}^+$. ∎

Lemma 2.8. *Let $T : \mathcal{M}_{m,n}(\mathcal{Z}^+) \to \mathcal{M}_{m,n}(\mathcal{Z}^+)$ be an operator which maps lines to lines and is defined by $T(E_{i,j}) = E_{\sigma(i,j)}$, where σ is a permutation on the set $\{(i,j) \mid i = 1,2,\ldots,m; j = 1,2,\ldots,n\}$. Then T is a (P,Q)-operator.*

Proof. Since no combination of a rows and b columns can dominate J where $a + b = m$ unless $b = 0$ (or if $m = n$, if $a = 0$) we have that either the image of each row is a row and the image of each column is a column, or $m = n$ and the image of each row is a column and the image of each column is a row. Thus, there are permutation matrices P and Q such that $T(R_i) \leq PR_iQ$ and $T(C_j) \leq PC_jQ$ or, if $m = n$, $T(R_i) \leq P(R_i)^tQ$ and $T(C_j) \leq P(C_j)^tQ$. Since each cell lies in the intersection of a row and a column and T maps nonzero cells to nonzero cells, it follows that $T(E_{i,j}) = PE_{i,j}Q$, or, if $m = n$, $T(E_{i,j}) = PE_{j,i}Q = P(E_{i,j})^tQ$. ∎

Remark 2.9. One can easily check that if $m = 1$ or $n = 1$ then all operators under consideration are (P, Q)-operators, if $m = n = 1$ then all operators under consideration are (P, P^t)-operators.

Henceforth we will always assume that $m, n \geq 2$.

3 Linear Operators Preserving Extreme Set of Integer Matrix Pairs

In this section we investigate the linear operators that preserve the sets $\mathcal{C}_{**}(\mathcal{Z}^+)$ defined above section.

Definition 3.1. Let $\mathbf{E}$ be the matrix $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \in \mathcal{M}_{3,4}(\mathcal{Z}^+)$.

Lemma 3.2. *For the matrix* $\mathbf{E} \in \mathcal{M}_{3,4}(\mathcal{Z}^+)$ *in the above Definition 3.1, we have* $c(\mathbf{E}) = 4$ *and* $c(\mathbf{E}^t) = 3$ *over* $\mathcal{Z}^+$.

Proof. It is straightforward to check that three rows of $\mathbf{E}$ are linearly independent. Thus $c(\mathbf{E}^t) = 3$. In order to prove the other equality we consider the column space

$$V = \left\{ \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \delta \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \middle| \alpha, \beta, \gamma, \delta \in \mathcal{S} \right\}$$

of the matrix $\mathbf{E}$. One can easily check that V is not a space of all 3-element column vectors with the entries from $\mathcal{Z}^+$ since $[1,0,0]^t \notin V$. Suppose that X is a basis for V and that X has only three elements, $X = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, where $\mathbf{x}_i = \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ x_{i,3} \end{bmatrix}$.

Since $[0,1,0]^t \in V$, we have that there are $a_1, a_2, a_3 \in \mathcal{Z}^+$ such that

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 = [0,1,0]^t .$$

But then,

$$a_1x_{1,1} + a_2x_{2,1} + a_3x_{3,1} = 0,$$
$$a_1x_{1,2} + a_2x_{2,2} + a_3x_{3,2} = 1,$$

and

$$a_1 x_{1,3} + a_2 x_{2,3} + a_3 x_{3,3} = 0.$$

By reordering, if necessary, we can assume that $a_1 \neq 0$, so that $x_{1,1} = x_{1,3} = 0$, and since the zero vector is never an element of a basis, $x_{1,2} \neq 0$. Thus we have $\mathbf{x}_1 = [0, x_{1,2}, 0]^t$.

Further, since $[1,1,0]^t \in V$, there are $b_1, b_2, b_3 \in \mathcal{Z}^+$ such that

$$b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + b_3\mathbf{x}_3 = [1,1,0]^t.$$

Since $x_{1,1} = 0$ we must have that the first coordinate in either $\mathbf{x}_2$ or $\mathbf{x}_3$ is nonzero since

$$b_1 x_{1,1} + b_2 x_{2,1} + b_3 x_{3,1} = 1.$$

By renumbering we can assume that $x_{2,1} \neq 0$ and that $b_2 \neq 0$. But then, the third coordinate in $\mathbf{x}_2$ must be zero, since

$$b_1 x_{1,3} + b_2 x_{2,3} + b_3 x_{3,3} = 0.$$

Thus we must have that $\mathbf{x}_2 = [x_{2,1}, x_{2,2}, 0]^t$ where $x_{2,1} \neq 0$.

Now, $[1,0,1]^t \in V$, so that there are $c_1, c_2, c_3 \in \mathcal{Z}^+$ such that

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = [1,0,1]^t.$$

Since $x_{1,2} \neq 0$ we must have that $c_1 = 0$. Since $x_{1,3} = x_{2,3} = 0$, we have

$$c_1 x_{1,3} + c_2 x_{2,3} + c_3 x_{3,3} = c_3 x_{3,3} = 1.$$

Thus, $c_3 = 1 \neq 0$ and $x_{3,3} = 1 \neq 0$.

Since

$$a_1 x_{1,1} + a_2 x_{2,1} + a_3 x_{3,1} = 0,$$

$$a_1 x_{1,3} + a_2 x_{2,3} + a_3 x_{3,3} = 0,$$

$x_{2,1} \neq 0$ and $x_{3,3} \neq 0$, we have that $a_2 = a_3 = 0$. Further $b_3 = 0$ since $x_{3,3} = 1$ and

$$b_1 x_{1,3} + b_2 x_{2,3} + b_3 x_{3,3} = 0.$$

Now consider that since $[0,0,1]^t \in V$, we must have that there are $d_1, d_2, d_3 \in \mathcal{Z}^+$ such that

$$d_1\mathbf{x}_1 + d_2\mathbf{x}_2 + d_3\mathbf{x}_3 = [0,0,1]^t.$$

Since $x_{1,2} \neq 0$ and $x_{2,1} \neq 0$ we must have that $d_1 = d_2 = 0$ and hence $[0,0,1]^t = d_3\mathbf{x}_3$. It follows that $\mathbf{x}_3 = [0,0,x_{3,3}]^t$.

Also, since

$$c_1x_{1,1} + c_2x_{2,1} + c_3x_{3,1} = c_2x_{2,1} = 1$$

we have that $c_2 \neq 0$. However,

$$c_1x_{1,2} + c_2x_{2,2} + c_3x_{3,2} = c_2x_{2,2} + c_3x_{3,2} = 0.$$

Since $c_2 \neq 0$ and

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = [1,0,1]^t,$$

we must have $x_{2,2} = 0$. That is $\mathbf{x}_2 = [x_{2,1},0,0]^t \in V$, a contradiction since $c_2\mathbf{x}_2 = [1,0,0]^t \notin V$.

Therefore we have that V must have dimension 4. Thus $c(\mathbf{E}) = 4$. ■

3.1 Linear Operators that Preserve $\mathcal{C}_{AN}(\mathcal{Z}^+)$

Lemma 3.3. *If $T : \mathcal{M}_{m,n}(\mathcal{Z}^+) \to \mathcal{M}_{m,n}(\mathcal{Z}^+)$ is a surjective linear operator which preserves $\mathcal{C}_{AN}(\mathcal{Z}^+)$, then T maps lines to lines.*

Proof. Suppose that T^{-1} does not map lines to lines. Then, there are two non collinear cells which are mapped to a line. There are two cases, they are mapped to two cells in a column or two cells in a row by Theorem 2.7.

If two non-collinear cells are mapped to two cells in a column, we may assume without loss of generality that $T(E_{1,1}+E_{2,2}) = E_{1,1}+E_{2,1}$. If $n \leq m$ it suffices to consider $A = E_{1,1}+E_{2,2}+\ldots+E_{n,n}$. In this case, $T(A)$ has column rank at most $n-1$, i.e., $(0,A) \in \mathcal{C}_{AN}(\mathcal{Z}^+)$, $(0,T(A)) \notin \mathcal{C}_{AN}(\mathcal{Z}^+)$, a contradiction. Let us consider the case $m < n$. Then we choose a matrix $A' \in \mathcal{M}_{m-2,n-2}(\mathcal{Z}^+)$ such that $c(A') = n-2$. Let us choose A' with the minimal number of non-zero entries. Let $A = O_2 \oplus A' \in \mathcal{M}_{m,n}(\mathcal{Z}^+)$. Thus $c(A) = c(A') = n-2$. Hence $(E_{1,1}+E_{2,2},A) \in \mathcal{C}_{AN}(\mathcal{Z}^+)$. Since T preserves $\mathcal{C}_{AN}(\mathcal{Z}^+)$, it follows that $(E_{1,1}+E_{2,1},T(A)) \in \mathcal{C}_{AN}(\mathcal{Z}^+)$, i.e., $c(E_{1,1}+E_{2,1}+T(A)) = n$. Therefore $c(T(A)[1,\ldots,m;3,\ldots,n]) \geq n-2$. Since the column rank of any matrix cannot exceed the number of columns, $c(T(A)[1,\ldots,m;3,\ldots,n]) = n-2$. Further, $|T(A)[1,\ldots,m;3,\ldots,n]| < |A| = |A'|$ since T transforms cells to cells and at least one cell has to be mapped into the 2^{nd} column. Thus we can have an $(m-2)\times(n-2)$ submatrix of $T(A)[1,\ldots,m;3,\ldots,n]$ whose column rank is $n-2$ and the number of whose nonzero entries are less than that of A'. This contradicts the choice of A' with the minimal number of non-zero entries.

If two non-collinear cells are mapped to two cells in a row, we may assume without loss of generality that $T(E_{1,1}+E_{2,2}) = E_{1,1}+E_{1,2}$. In this case, by considering the matrices $E_{1,1}+E_{2,2}$ and A chosen above, the result follows.

Thus, T maps lines to lines. ■

Theorem 3.4. *Let $m \neq n$ or $m = n \geq 4$ and $T : \mathcal{M}_{m,n}(\mathcal{Z}^+) \to \mathcal{M}_{m,n}(\mathcal{Z}^+)$ be a surjective linear operator. Then T preserves $\mathcal{C}_{AN}(\mathcal{Z}^+)$ if and only if T is a nontransposing (P,Q)-operator.*

Proof. It is easily checked that all nontransposing (P,Q)-operators preserve $\mathcal{C}_{AN}(\mathcal{Z}^+)$.

Suppose that T preserves $\mathcal{C}_{AN}(\mathcal{Z}^+)$. By Lemma 3.3 we have that T preserves lines and by applying Theorem 2.7 to Lemma 2.8 we have that T is a (P,Q)-operator. Since all nontransposing (P,Q)-operators preserve $\mathcal{C}_{AN}(\mathcal{Z}^+)$ it only remains to show that if $m = n$ then the transposition does not preserve $\mathcal{C}_{AN}(\mathcal{Z}^+)$. Let

$$A = \begin{bmatrix} \mathbf{E} & O \\ O & I_{n-4} \\ O & O \end{bmatrix} \in \mathcal{M}_n(\mathcal{Z}^+).$$

Then by Lemma 3.2 we have that $c(A) = n$ and $c(A^t) = n-1$, so that $(A,O) \in \mathcal{C}_{AN}(\mathcal{Z}^+)$ while $(A^t,O) \notin \mathcal{C}_{AN}(\mathcal{Z}^+)$. Thus T is a nontransposing (P,Q)-operator. ■

3.2 Linear Operators that Preserve $\mathcal{C}_{A1}(\mathcal{Z}^+)$

Lemma 3.5. *If $T : \mathcal{M}_{m,n}(\mathcal{Z}^+) \to \mathcal{M}_{m,n}(\mathcal{Z}^+)$ is a surjective linear operator which preserves $\mathcal{C}_{A1}(\mathcal{Z}^+)$, then T maps lines to lines.*

Proof. Suppose that T does not map lines to lines. Then, without loss of generality, we may assume that either $T(E_{1,1}+E_{1,2}) = E_{1,1}+E_{2,2}$ or $T(E_{1,1}+E_{2,1}) = E_{1,1}+E_{2,2}$ by Theorem 2.7. In either case, let $Y = O$ and X be either $E_{1,1}+E_{1,2}$ or $E_{1,1}+E_{2,1}$, so that $(X,Y) \in \mathcal{C}_{A1}$ while $(T(X),T(Y)) \notin \mathcal{C}_{A1}$, a contradiction. Thus T maps lines to lines. ■

Theorem 3.6. *Let $T : \mathcal{M}_{m,n}(\mathcal{Z}^+) \to \mathcal{M}_{m,n}(\mathcal{Z}^+)$ be a surjective linear operator. Then T preserves $\mathcal{C}_{A1}(\mathcal{Z}^+)$ if and only if T is a nontransposing (P,Q)-operator.*

Proof. It is easily checked that all nontransposing (P,Q)-operators preserve $\mathcal{C}_{A1}(\mathcal{Z}^+)$.

Suppose that T preserves $\mathcal{C}_{A1}(\mathcal{Z}^+)$. By applying Lemma 3.5 and Theorem 2.7 to Lemma 2.8 we have that if T preserves $\mathcal{C}_{A1}(\mathcal{Z}^+)$ then T is a (P,Q)-operator. Since all nontransposing (P,Q)-operators preserve $\mathcal{C}_{A1}(\mathcal{Z}^+)$ it only remains to show that if $m = n$ then the transposition does not preserve $\mathcal{C}_{A1}(\mathcal{Z}^+)$. Let

$$X = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \bigoplus O_{n-2,n-1} \in \mathcal{M}_n(\mathcal{Z}^+)$$

and $Y = O$, $(X,Y) \in \mathcal{C}_{A1}(\mathcal{Z}^+)$ but $(X^t,Y^t) \notin \mathcal{C}_{A1}(\mathcal{Z}^+)$. So, transposition operator does not preserve the set $\mathcal{C}_{A1}(\mathcal{Z}^+)$. Thus T is a nontransposing (P,Q)-operator. ■

3.3 Linear Operators that Preserve $\mathcal{C}_{AR}(\mathcal{Z}^+)$

Lemma 3.7. *If $T : \mathcal{M}_{m,n}(\mathcal{Z}^+) \to \mathcal{M}_{m,n}(\mathcal{Z}^+)$ is a surjective linear operator which preserves $\mathcal{C}_{AR}(\mathcal{Z}^+)$, $\min\{m,n\} \geq 3$, then T maps lines to lines.*

Proof. The sum of three distinct weighted cells has column rank at most 3. Thus $T(E_{1,1}+E_{1,2}+E_{2,1})$ is either a sum of 3 collinear cells, and hence has column rank 1, or is contained in two lines, and hence has real rank 2, or is the sum of three cells of column rank 3 and hence of real rank 3. Now, for $X = E_{1,1}+E_{1,2}+E_{2,1}$ and $Y = E_{2,2}$, we have that $(X,Y) \in \mathcal{C}_{AR}(\mathcal{Z}^+)$, and the image of Y is a single cell, and hence $\rho(T(Y)) = 1$. Now, if $\rho(T(X)) = 3$, then $T(X+Y)$ must have column rank 3 or 4, and hence $(T(X), T(Y)) \notin \mathcal{C}_{AR}(\mathcal{Z}^+)$, a contradiction. If $\rho(T(X)) = 1$, clearly $(T(X), T(Y)) \notin \mathcal{C}_{AR}(\mathcal{Z}^+)$ since $T(X+Y) \neq O$. Thus $\rho(T(X)) = 2$, and $c(T(X+Y)) = 1$. However it is straightforward to see that the sum of four cells has the column rank 1 if and only if they lie either in a line or in the intersection of two rows and two columns. The matrix $T(X+Y)$ is a sum of four cells. These cells do not lie in a line since $\rho(T(X)) = 2$. Thus $T(X+Y)$ must be the sum of four cells which lie in the intersection of two rows and two columns. Similarly, for any i, j, k, l, $T(E_{i,j} + E_{i,k} + E_{l,j} + E_{l,k})$ must lie in the intersection of two rows and two columns. It follows that any two rows must be mapped into two lines. By the bijectivity of T, if some pair of two rows is mapped into two rows (resp. columns), any pair of two rows is mapped into two rows (resp. columns). Similarly, if some pair of two columns is mapped into two rows (resp. columns), any pair of two columns is mapped into two rows (resp. columns).

Now, the image of three rows is contained in three lines, two of which are the image of two rows, thus, every row is mapped into a line. Similarly for columns. Thus, T maps lines to lines. ∎

Theorem 3.8. *Let $m \neq n$ or $m = n \geq 4$, and $T : \mathcal{M}_{m,n}(\mathcal{Z}^+) \to \mathcal{M}_{m,n}(\mathcal{Z}^+)$ be a surjective linear operator. Then T preserves $\mathcal{C}_{AR}(\mathcal{Z}^+)$ if and only if T is a nontransposing (P, Q)-operator.*

Proof. It is easily checked that all nontransposing (P, Q)-operators preserve $\mathcal{C}_{AR}(\mathcal{Z}^+)$.

By applying Lemma 3.7 and Theorem 2.7 to Lemma 2.8 we have that if T preserves $\mathcal{C}_{AR}(\mathcal{Z}^+)$ then T is a (P, Q)-operator. Since all nontransposing (P, Q)-operators preserve $\mathcal{C}_{AR}(\mathcal{Z}^+)$ it only remains to show that in the case $m = n$ the operator $X \to X^t$ does not preserve $\mathcal{C}_{AR}(\mathcal{Z}^+)$. Let

$$X = \begin{bmatrix} \mathbf{E}^t & O \\ O & O \end{bmatrix} \in \mathcal{M}_n(\mathcal{Z}^+)$$

and $Y = O$. Then $(X, Y) \in \mathcal{C}_{AR}(\mathcal{Z}^+)$ while $(X^t, Y^t) \notin \mathcal{C}_{AR}(\mathcal{Z}^+)$. So, transposition operator does not preserve the set $\mathcal{C}_{AR}(\mathcal{Z}^+)$. Thus T is a nontransposing (P, Q)-operator. ∎

3.4 Linear Operators that Preserve $\mathcal{C}_{M2}(\mathcal{Z}^+)$

Lemma 3.9. *If $T : \mathcal{M}_n(\mathcal{Z}^+) \to \mathcal{M}_n(\mathcal{Z}^+)$ is a surjective linear operator which preserves $\mathcal{C}_{M2}(\mathcal{Z}^+)$, then T maps lines to lines.*

Proof. Suppose that T^{-1} does not map columns to lines, say, without loss of generality, that $T^{-1}(E_{1,1}+E_{2,1}) \geq E_{1,1}+E_{2,2}$. Then $T(I)$ has nonzero entries in at most $n-1$ columns. Suppose $T(I)$ has all zero entries in column j. Then for $X = I$ and $Y = T^{-1}(E_{j,1})$, we have $XY = Y$ however, $T(X)T(Y) = O$. This contradicts the fact that T preserves $\mathcal{C}_{M2}(\mathcal{Z}^+)$. Suppose that T^{-1} does not map rows to lines. Say, without loss of generality, that $T^{-1}(E_{1,1}+E_{1,2}) \geq E_{1,1}+E_{2,2}$. That is $T(E_{1,1}+E_{2,2}) = E_{1,1}+E_{1,2}$. Then for $X = E_{1,1}+E_{2,2}+[O_2 \oplus I_{n-2}]$, $T(X)$ has column rank at most $n-1$ since either the first two columns of $T(X)$ are equal or at least one of the columns from the 3$^{\text{rd}}$ through the n^{th} is zero. Let $Y = T^{-1}(I)$, then we have that $(X,Y) \in \mathcal{C}_{M2}(\mathcal{Z}^+)$, since $c(XZ) = c(Z)$ for any Z, while $c(T(X)I) = c(T(X)) = n-1 < c(I) = c(T(Y))$ so that $(T(X),T(Y)) \notin \mathcal{C}_{M2}(\mathcal{Z}^+)$, a contradiction.

Similarly, if $T^{-1}(E_{1,1}+E_{2,1}) \geq E_{1,1}+E_{2,2}$ then the second column of $T(X)$ is zero and the same pair $(X,Y) \in \mathcal{C}_{M2}(\mathcal{Z}^+)$ gives the contradiction.

Thus T^{-1} and hence T map lines to lines. ∎

Theorem 3.10. *Let $n \geq 4$, and $T : \mathcal{M}_n(\mathcal{Z}^+) \to \mathcal{M}_n(\mathcal{Z}^+)$ be a surjective linear operator. Then T preserves $\mathcal{C}_{M2}(\mathcal{Z}^+)$ if and only if T is a nontransposing (P,P^t)-operator.*

Proof. It is easily checked that all nontransposing (P,Q)-operators preserve $\mathcal{C}_{M2}(\mathcal{Z}^+)$.

By applying Lemma 3.9 and Theorem 2.7 to Lemma 2.8 we have that if T preserves $\mathcal{C}_{M2}(\mathcal{Z}^+)$ then T is a (P,Q)-operator.

To see that the operator $T(X) = PX^tQ$ does not preserve $\mathcal{C}_{M2}(\mathcal{Z}^+)$, it suffices to consider $T_0(X) = X^t$, since row and column permutations preserve $\mathcal{C}_{M2}(\mathcal{Z}^+)$. Let

$$X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \bigoplus I_{n-4}$$

and

$$Y=\begin{bmatrix}0&1&0&0\\0&0&0&0\\1&0&0&0\\1&1&0&0\end{bmatrix}\bigoplus I_{n-4}.$$

Then $(X,Y)\in\mathcal{C}_{M2}(\mathcal{Z}^+)$ while $(X^t,Y^t)\notin\mathcal{C}_{M2}(\mathcal{Z}^+)$. So, transposition operator does not preserve the set $\mathcal{C}_{M2}(\mathcal{Z}^+)$. Thus T is a nontransposing (P,Q)-operator.

It remains to prove that $Q=P^t$. Assume in the contrary that $QP\neq I$. Suppose that $T_1(X)=(QP)X$ transforms the r^{th} row into the t^{th} row for some $r\neq t$. We consider the matrix

$$X=\sum_{i\neq t}E_{i,i},\quad Y=E_{r,r}.$$

Then $XY=E_{r,r}=Y$, which implies $(X,Y)\in\mathcal{C}_{M2}(\mathcal{Z}^+)$, but

$$T(X)T(Y)=PXQPYQ=P(\sum_{i\neq t}E_{i,i})E_{t,r}Q=P(I\backslash E_{t,t})E_{t,r}Q=P0Q=0,$$

which implies that $(T(X),T(Y))\notin\mathcal{C}_{M2}(\mathcal{Z}^+)$. Therefore T does not preserve the set $\mathcal{C}_{M2}(\mathcal{Z}^+)$, a contradiction. Thus $Q=P^t$ and T is a nontransposing (P,P^t)-operator. ∎

3.5 Linear Operators that Preserve $\mathcal{C}_{M1}(\mathcal{Z}^+)$

Lemma 3.11. *If $T:\mathcal{M}_n(\mathcal{Z}^+)\to\mathcal{M}_n(\mathcal{Z}^+)$ is a surjective linear operator which preserves $\mathcal{C}_{M1}(\mathcal{Z}^+)$, then T maps lines to lines.*

Proof. Recall that if $(X,Y)\in\mathcal{C}_{M1}(\mathcal{Z}^+)$ then $c(X)+c(Y)>n$. We assume that there exist indices i,j,k,l, $i\neq k$, $j\neq l$ such that nonzero entries of $T(E_{i,j})$ and $T(E_{k,l})$ lie in a line. Let $T(E_{i,j})=E_{s,t}$. Then either $T(E_{k,l})=E_{s,t'}$ or $T(E_{k,l})=E_{s',t}$. In both cases $c(T(E_{i,j}+E_{k,l}))=1$. Let

$Y' \in \mathcal{M}_n(\mathcal{Z}^+)$ be a matrix such that $Y'+E_{j,i}+E_{l,k}$ is a permutational matrix. We consider $X = E_{i,j} + E_{k,l}$, $Y = Y' + E_{l,k}$. Then $XY = E_{k,k}$ and $(X, Y) \in \mathcal{C}_{M1}(\mathcal{Z}^+)$. However, since $c(T(X)) = 1$ in either case, and $c(T(Y)) \leq n-1$, $c(T(X)) + c(T(Y)) \leq n$. Finally, we have that $(T(X), T(Y)) \notin \mathcal{C}_{M1}(\mathcal{Z}^+)$, a contradiction. ■

Theorem 3.12. *Let $n \geq 3$, and $T : \mathcal{M}_n(\mathcal{Z}^+) \to \mathcal{M}_n(\mathcal{Z}^+)$ be a surjective linear operator. Then T preserves $\mathcal{C}_{M1}(\mathcal{Z}^+)$ if and only if T is a nontransposing (P, P^t)-operator.*

Proof. It is straightforward that operators under consideration preserve the set $\mathcal{C}_{M1}(\mathcal{Z}^+)$.

By applying Lemma 3.11 and Theorem 2.7 to Lemma 2.8 we have that if T preserves $\mathcal{C}_{M1}(\mathcal{Z}^+)$ then T is a (P, Q)-operator. Similar to the proof of Theorem 3.10 we consider $T_0(X) = X^t$. Let

$$X = \begin{bmatrix} O & I_2 \\ O & O \end{bmatrix},$$

and

$$Y = \begin{bmatrix} I_{n-1} & O \\ O & O \end{bmatrix}.$$

Then $c(XY) = c(E_{1,n-1}) = 1$ and hence $(X, Y) \in \mathcal{C}_{M1}(\mathcal{Z}^+)$ while $c(X^tY^t) = c(\begin{bmatrix} O & O \\ I_2 & O \end{bmatrix}) = 2$ and hence $(X^t, Y^t) \notin \mathcal{C}_{M1}(\mathcal{Z}^+)$. This proves that T is a non-transposing (P, Q)-operator.

Let us check that $Q = P^t$. Assume in the contrary that $QP \neq I$. Suppose that $T_1(X) = (QP)X$ transforms the p^{th} row into the s^{th} and the r^{th} row into t^{th} with $r \neq s, t$ since $n \geq 3$. We consider the matrix $X = \sum_{i \neq r} E_{i,i}$, $Y = E_{p,p} + E_{r,r}$. Then $XY = E_{p,p}$ and hence $(X, Y) \in \mathcal{C}_{M1}(\mathcal{Z}^+)$. And we have that

$$c(T(X)) + c(T(Y)) = n + 1 > n$$

and

$$T(X)T(Y) = PXQPYQ = P(\sum_{i \neq r} E_{i,i})(E_{s,p} + E_{t,r})Q.$$

Thus $c(T(X)T(Y)) = 2$, that is, $(T(X), T(Y)) \notin \mathcal{C}_{M1}(\mathcal{Z}^+)$, a contradiction. Hence $Q = P^t$ and T is a nontransposing (P, P^t)-operator. ■

3.6 Linear Operators that Preserve $\mathcal{C}_{MR}(\mathcal{Z}^+)$

Lemma 3.13. *If $T : \mathcal{M}_n(\mathcal{Z}^+) \to \mathcal{M}_n(\mathcal{Z}^+)$ is a surjective linear operator which preserves $\mathcal{C}_{MR}(\mathcal{Z}^+)$, then T maps lines to lines.*

Proof. If T does not preserve lines, then, as in the proof of Lemma 3.11, there exist indices i, j, k, l, $i \neq k$, $j \neq l$ such that nonzero entries of $T(E_{i,j})$ and $T(E_{k,l})$ lie in a line. Let $X' \in \mathcal{M}_n(\mathcal{Z}^+)$ be a matrix such that $X' + E_{i,j} + E_{k,l}$ is a permutational matrix, $X = X' + E_{i,j} + E_{k,l}$. Then $(X, O) \in \mathcal{C}_{MR}$. However, $c(T(X)) \leq n - 1$, and hence $\rho(T(X)) \leq n - 1$. Thus $(T(X), O) \notin \mathcal{C}_{MR}(\mathcal{Z}^+)$, a contradiction. ■

Theorem 3.14. *Let $T : \mathcal{M}_n(\mathcal{Z}^+) \to \mathcal{M}_n(\mathcal{Z}^+)$ be a surjective linear operator. Then T preserves $\mathcal{C}_{MR}(\mathcal{Z}^+)$ if and only if T is a nontransposing (P, P^t)-operator.*

Proof. It is straightforward that operators under consideration preserve the set $\mathcal{C}_{MR}(\mathcal{Z}^+)$.

By applying Lemma 3.13 and Theorem 2.7 to Lemma 2.8 we have that if T preserves $\mathcal{C}_{MR}(\mathcal{Z}^+)$ then T is a (P, Q)-operator.

Let

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \bigoplus I_{n-2},$$

and

$$Y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \bigoplus I_{n-2}.$$

Then

$$XY = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \bigoplus I_{n-2},$$

and hence $(X, Y) \in \mathcal{C}_{MR}(\mathcal{Z}^+)$. But

$$X^t Y^t = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \bigoplus I_{n-2},$$

and hence $(X^t, Y^t) \notin \mathcal{C}_{MR}(\mathcal{Z}^+)$. This proves that T is a non-transposing (P, Q)-operator.

Let us check that $Q = P^t$. Assume in the contrary that $QP \neq I$. Let QP transforms r^{th} row to t^{th} with $r \neq t$. We consider the matrix $X = \sum_{i \neq r} E_{i,i}$, $Y = E_{r,r}$. Thus $(X, Y) \in \mathcal{C}_{MR}(\mathcal{Z}^+)$. But

$$T(X)T(Y) = PXQPYQ = P(\sum_{i \neq r} E_{i,i})(E_{t,r})Q \neq 0,$$

which implies that $(T(X), T(Y)) \notin \mathcal{C}_{MR}(\mathcal{Z}^+)$, a contradiction. Hence $Q = P^t$ and T is a nontransposing (P, P^t)-operator. ∎

3.7 Linear Operators that Preserve $\mathcal{C}_{M0}(\mathcal{Z}^+)$

Lemma 3.15. *If $T : \mathcal{M}_n(\mathcal{Z}^+) \to \mathcal{M}_n(\mathcal{Z}^+)$ is a surjective linear operator which preserves $\mathcal{C}_{M0}(\mathcal{Z}^+)$, then T maps columns to columns and rows to rows.*

Proof. Suppose that T does not map columns to columns. Say $T(C_j)$ is not a column. Then $T(J \backslash C_j)$ has no zero column. Then $(J \backslash C_j, E_{j,j}) \in \mathcal{C}_{M0}(\mathcal{Z}^+)$, while $(T(J \setminus C_j), T(E_{j,j})) \notin \mathcal{C}_{M0}(\mathcal{Z}^+)$, a contradiction.

Suppose that T does not preserve rows, then, say, $T(R_i)$ is not a row. It follows that $T(J \setminus R_i)$ has no zero row. Then $(E_{i,i}, J \setminus R_i) \in \mathcal{C}_{M0}(\mathcal{Z}^+)$, while $(T(E_{i,i}), T(J \setminus R_i)) \notin \mathcal{C}_{M0}(\mathcal{Z}^+)$, a contradiction. ∎

Theorem 3.16. *Let $T : \mathcal{M}_n(\mathcal{Z}^+) \to \mathcal{M}_n(\mathcal{Z}^+)$ be a surjective linear operator. Then T preserves $\mathcal{C}_{M0}(\mathcal{Z}^+)$ if and only if T is a nontransposing (P, P^t)-operator.*

Proof. It is straightforward that operators under consideration preserve the set $\mathcal{C}_{M0}(\mathcal{Z}^+)$.

Since, by Lemma 3.15 T preserves columns and rows, it preserves lines and hence, by Lemma 2.8, T is a (P, Q)-operator. Since T maps columns to columns, T is clearly a nontransposing (P, Q)-operator.

We now only need show that $Q = P^t$. If not, say $QPE_{r,s} = E_{t,s}$ with $t \neq r$. Then $(E_{t,t}, E_{r,s}) \in \mathcal{C}_{M0}(\mathcal{Z}^+)$. However,

$$T(E_{t,t})T(E_{r,s}) = PE_{t,t}QPE_{r,s}Q = P(E_{t,t}E_{t,s})Q \neq O$$

so that $(T(E_{t,t}), T(E_{r,s})) \notin \mathcal{C}_{M0}(\mathcal{Z}^+)$, a contradiction. Hence $Q = P^t$ and T is a nontransposing (P, P^t)-operator. ■

3.8 Examples of non-surjective Linear Operators that Preserve $\mathcal{C}_{**}(\mathcal{Z}^+)$

Let us see that there exists non-surjective linear preservers of the sets $\mathcal{C}_{**}(\mathcal{Z}^+)$.

Example 3.17.

1. A linear operator T_3 defined on the basis by $T_3(E_{i,j}) = E_{j,j}$ is a non-surjective $\mathcal{C}_{AN}(\mathcal{Z}^+)$-preserver.

2. A linear operator T_4 defined on the basis by $T_4(E_{i,j}) = E_{1,1}$ is a non-surjective $\mathcal{C}_{A1}(\mathcal{Z}^+)$-preserver.

Proof.

1. By its definition T_3 is not surjective. To see that T_3 preserves $\mathcal{C}_{AN}(\mathcal{Z}^+)$ we note that for any $A, B \in \mathcal{M}_{m,n}(\mathcal{Z}^+)$ if $c(T_3(A + B)) < n$ then $T_3(A+B)$ has a zero column (since $T_3(X)$ is a diagonal matrix for any $X \in \mathcal{M}_{m,n}(\mathcal{Z}^+)$). Thus the sum of all entries of a certain column of $A+B$ is zero. By the antinegativity of $\mathcal{Z}^+$ it follows that there is a zero column in $A + B$, i.e., $c(A + B) < n$. Therefore if $(A, B) \in \mathcal{C}_{AN}(\mathcal{Z}^+)$, then $(T_3(A), T_3(B)) \in \mathcal{C}_{AN}(\mathcal{Z}^+)$. Hence, T_3 preserves $\mathcal{C}_{AN}(\mathcal{Z}^+)$. Thus T_3 is a non-surjective $\mathcal{C}_{AN}(\mathcal{Z}^+)$-preserver.

2. If $(A, B) \in \mathcal{C}_{A1}(\mathcal{Z}^+)$, then $(T_4(A), T_4(B)) \in \mathcal{C}_{A1}(\mathcal{Z}^+)$ since all the image of $C_{A1}(\mathcal{Z}^+)$ under T_4 is the pair $(E_{1,1}, E_{1,1})$, which is in $\mathcal{C}_{A1}(\mathcal{Z}^+)$. Thus T_4 is a non-surjective $\mathcal{C}_{A1}(\mathcal{Z}^+)$-preserver.

■

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(국문초록)

본 논문에서는 음이 아닌 정수에서 원소를 가지는 행렬의 짝들로 구성되는 집합들을 보존하는 전사 선형연산자의 형태를 규명하는 연구를 하였다. 이것은 선형보존자 문제의 일환으로서 세계적으로 선형대수학자들이 100여 년 동안 연구해오는 중요한 연구과제의 하나이다.

본 논문의 연구에 앞선 연구들에서 행렬의 계수를 보존하는 선형연산자가 규명되었고, 또 행렬의 계수 부등식과 관련된 행렬 짝들의 집합을 보존하는 선형연산자의 연구가 있었다. 본 논문은 비음의 정수 상에서 원소를 갖는 행렬들에 대하여 열 계수가 일반 계수와 다르다는 것에 착안하여, 일반계수에 대한 연구를 열 계수에 관한 연구로 옮겨서 연구하였다. 본 연구에서 구성한 행렬 짝들의 집합들은 두 정수 행렬들의 합과 곱의 열 계수와 관련된 부등식의 극치인 경우들에서 자연스럽게 나타나는 행렬 짝들의 집합들이다. 이 행렬 짝들의 집합들은 두 정수 행렬의 열 계수들의 합과 차 또는 이 정수 행렬을 실수 행렬로 간주할 때 나타나는 실수 행렬 계수의 차와 관련된 부등식들에서 극치인 경우들로 구성하였다.

곧, 그 중 몇 가지를 열거하면 다음과 같다.

$$C_{AN}(Z^+) = \{(X, Y) \in M_{m,n}(Z^+)^2 \mid c(X+Y) = n\};$$

$$C_{AR}(Z^+) = \{(X, Y) \in M_{m,n}(Z^+)^2 \mid c(X+Y) = |\rho(X) - \rho(Y)|\};$$

$$C_{M2}(Z^+) = \{(X, Y) \in M_n(Z^+)^2 \mid c(XY) = c(X)\};$$

$$C_{MR}(Z^+) = \{(X, Y) \in M_n(Z^+)^2 \mid c(X+Y) = \rho(X) + \rho(Y) - n\};$$

이상의 행렬 짝들의 집합을 선형연산자 T로 보내어 그 집합의 특성들을 보존하는 전사 선형연산자를 연구하여 그 형태를 규명하였다. 곧, 이러한 행렬 짝들의 집합을 보존하는 전사 선형연산자의 형태는 $T(X) = PXQ$ 또는 $T(X) = PXP^t$로 나타남을 보이고, 이들을 증명하였다. 여기서 P와 Q는 각 행과 열에 1이 하나만 있는 순열 행렬이다. 또한 본 논문에서 규명한 전사 선형연산자 외에도 위의 집합들을 보존하는 전사가 아닌 선형연산자들의 예를 보였다.