博士學位論文

Transversally harmonic maps between foliated Riemannian manifolds

濟州大學校 大學院

數 學 科

鄭　　珉　州

2012年　8月

Transversally harmonic maps between foliated Riemannian manifolds

指導教授 鄭 承 達

鄭 珉 州

이 論文을 理學 博士學位 論文으로 提出함

2012年 6月

鄭珉州의 理學 博士學位 論文을 認准함

審査委員長 ______________________

委　　員 ______________________

委　　員 ______________________

委　　員 ______________________

委　　員 ______________________

濟州大學校 大學院

2012年 6月

Transversally harmonic maps between foliated Riemannian manifolds

Min Joo Jung
(Supervised by professor Seoung Dal Jung)

A thesis submitted in partial fulfillment of the requirement for the degree of Doctor of Science

2012. 6.

This thesis has been examined and approved.

____

____

____

____

____

Date : ________________

Department of Mathematics
GRADUATE SCHOOL
JEJU NATIONAL UNIVERSITY

Contents

⟨**Abstract**⟩

Transversally harmonic maps between foliated Riemannian manifolds

Let $(M,\mathcal{F})$ and $(M',\mathcal{F}')$ be two foliated Riemannian manifolds with M compact. Then we study the first normal variational formula for the transversal energy. Moreover, if we assume that the transversal Ricci curvature of $\mathcal{F}$ is nonnegative and the transversal sectional curvature of $\mathcal{F}'$ is nonpositive, then any transversally harmonic map $\phi:(M,\mathcal{F})\to(M',\mathcal{F}')$ is transversally totally geodesic. In addition, if the transversal Ricci curvature is positive at some point, then ϕ is tansversally constant.

1 Introduction

Harmonic maps are solutions to a natural geometrical variational problem. This notion grew out of essential notions in differential geometry, such as geodesic, minimal surfaces and harmonic functions. Harmonic maps are also closely related to holomorphic maps in several complex variables, to the theory of stochastic processes, to nonlinear field theory in theoretical physics, and to the theory of liquid crystals in materials science.

There are several equivalent definitions for harmonic maps. A map between Riemannian manifolds is harmonic if the divergence of its differential vanishes. Equivalently, harmonic maps are critical points of the energy functional.

On foliated Riemannian manifolds, transversally harmonic maps were introduced by Konderak and Wolak [6] in 2003. Namely, let $(M, g, \mathcal{F})$ and $(M', g', \mathcal{F}')$ be two foliated Riemannian manifolds and let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth *foliated map*, which is a smooth leaf-preserving map. Then ϕ is a *transversally harmonic map* if ϕ is a solution of the equation $\tau_b(\phi) = tr_Q \tilde{\nabla} d_T \phi = 0$, where $\tilde{\nabla}$ be the connection on $Q^* \otimes \phi^{-1} Q'$. Equivalently, ϕ is a critical point of the transversal energy functional on any compact domain of M. That is, transversally harmonic maps are considered as harmonic maps between the leaf spaces ([6, 7]). On a point foliation, transversally harmonic map is just harmonic map. So transversally harmonic maps are generalizations of harmonic maps.

In this thesis, we study transversally harmonic maps and give some interesting facts relating to them. In chapter 2, we review the well-known facts on a foliated Riemannian mannifold. In chapter 3, we review the properties of the transversally harmonic map, which were studied in [7] and give some results. In chapter 4, we give a new proof of the first normal variational formula for the transversal energy $E_B(\phi)$ (Theorem 4.3). In the last chapter, we study the generalized Weitzenböck formula and give some applications (Theorem 5.4 and Theorem 5.5).

2 Riemannian foliation

Let M be a smooth manifold of dimension $p+q$.

Definition 2.1 A family $\mathcal{F} \equiv \{L_\alpha\}_{\alpha \in A}$ of connected subsets of a manifold M^{p+q} is called a p-dimensional(or codimension q) *foliation* if

(1) $\cup_\alpha L_\alpha = M$,

(2) $\alpha \neq \beta \Longrightarrow L_\alpha \cap L_\beta = \varnothing$,

(3) for any point $p \in M$ there exist a C^r−chart(local coordinate system) (φ_p, U_p), such that $p \in U_p$ and if $U_p \cap L_\alpha \neq \varnothing$, then $\varphi_p(U_p \cap L_\alpha) = A_c \cap \varphi(U_p)$, where

$$A_c = \{(x,y) \in \mathbb{R}^p \times \mathbb{R}^q | y = \text{constant}\}.$$

Here (φ_p, U_p) is called a *distinguished*(or *foliated*) chart.

Roughly speaking, a foliation corresponds to a decomposition of a manifold into a union of connected submanifolds of dimension p called it leaves.

Remark. From (3), we know that on $U_i \cap U_j \neq \varnothing$, the coordinate change $\varphi_j^{-1} \circ \varphi_i$: $\varphi_i^{-1}(U_i \cap U_j) \to \varphi_j^{-1}(U_i \cap U_j)$ has the form

$$\varphi_j^{-1} \circ \varphi_i(x,y) = (\varphi_{ij}(x,y), \gamma_{ij}(y)), \tag{2.1}$$

where $\varphi_{ij} : \mathbb{R}^{p+q} \to \mathbb{R}^p$ is a differential map and $\gamma_{ij} : \mathbb{R}^q \to \mathbb{R}^q$ is a diffeomorphism.

Example.

(1) Line foliation. ([17]) Consider a closed 1-form $\omega = adx + bdy, a, b \in \mathbb{R}$ on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then we obtain a family of lines which defines a foliation in T^2. In this case, each leaf

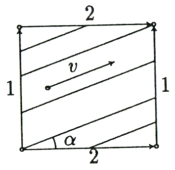

figure 1

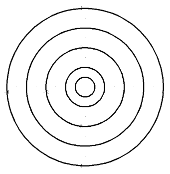

figure 2

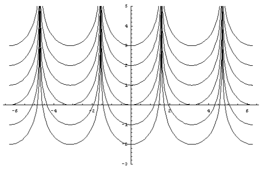

figure 3

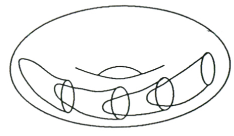

figure 4

is $\mathbb{R}$ (See figure 1).

(2) Circles. In $\mathbb{R}^2$, the differential equation $xdx + ydy = 0$ has $x^2 + y^2 = c^2, c \in \mathbb{R}^+$ as general solution. When c varies, we obtain a family of circles which defines a foliation in $\mathbb{R}^2$. In this case, each leaf is circle (See figure 2).

(3) In the plane $\mathbb{R}^2$, the differential equation $\frac{dy}{dx} = \tan x$ has the solution $y = \log|\sec x| + c, c \in \mathbb{R}$. When c varies, we obtain a foliation as figure 3.

(4) Reeb solid torus. ([17]) On a solid cylinder $D^2 \times \mathbb{R}$, we obtain the Reeb component which is also defined by a submersion $f : D^2 \times \mathbb{R} \to \mathbb{R}$:

$$f(x, y, z) = a(r^2)exp(z),$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$ and $a(r) = exp(-exp(\frac{1}{1-r^2}))$ (See figure 4).

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a Riemannian metric g. Let TM be the tangent bunlde of M, L the tangent bundle of $\mathcal{F}$ and then L is the integrable subbundle of TM. i.e., $X, Y \in \Gamma L \Longrightarrow [X, Y] \in \Gamma L$. Let $Q = TM/L$ the corresponding normal bundle of $\mathcal{F}$, then the metric g_M defines a splitting σ in the exact sequence of vector bundles

$$0 \longrightarrow L \longrightarrow TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \longrightarrow 0\,, \tag{2.2}$$

where $\pi : TM \to Q$ is a projection and $\sigma : Q \to L^\perp$ is a bundle map satisfying $\pi \circ \sigma = id$. Thus $g_M = g_L \oplus g_{L^\perp}$ induces a metric g_Q on Q ; that is,

$$g_Q(s,t) = g_M(\sigma(s), \sigma(t)) \quad \forall s,t \in \Gamma Q. \tag{2.3}$$

So we have an identification $L^\perp$ with Q via an isometric splitting $(Q, g_Q) \cong (L^\perp, g_{L^\perp})$.

Definition 2.2 A Riemannian metric g_Q on Q of a foliation $\mathcal{F}$ is *holonomy invariant* if $\theta(X)g_Q = 0$, for any $X \in \Gamma L$, where $\theta(X)$ is the transverse Lie derivative. i.e.,

$$Xg_Q(s,t) = g_Q(\pi[X, Y_s], t) + g_Q(s, \pi[X, Y_t]), \quad \forall X \in \Gamma L, \quad \forall s,t, \in \Gamma Q, \tag{2.4}$$

where $Y_s = \sigma(s)$ for any $s \in \Gamma Q$.

Definition 2.3 A foliation $\mathcal{F}$ is *Riemannian* if there exists a holonomy invariant metric g_Q on Q. A metric g_M is a *bundle-like* (with respect to $\mathcal{F}$) if the induced metric g_Q is holonomy invariant.

Theorem 2.4 ([14]) *Let $\mathcal{F}$ be a foliation on (M,g). Then the following conditions are equivalent.*

(a) *$\mathcal{F}$ is Riemannian and g is a bundle-like metric.*

(b) *There exists an orthonomal adapted frame $\{E_i, E_a\}$ such that*

$$g(\nabla^M_{E_a} E_i, E_b) + g(\nabla^M_{E_b} E_i, E_a) = 0,$$

where ∇^M be the Levi-Civita connection on M.

(c) *All geodesics orthogonal to a leaf at one point are orthogonal to each leaf at every point.*

Definition 2.5 ([13]) The *transverse Levi-Civita connection* ∇^Q on the normal bundle Q is defined by

$$\nabla^Q_X s = \begin{cases} \pi([X, Y_s]) & \forall X \in \Gamma L, \\ \pi(\nabla^M_X Y_s) & \forall X \in \Gamma L^\perp, \end{cases} \tag{2.5}$$

where ∇^M be the Levi-Civita connection associated to the Riemannian metric g_M and $Y_s = \sigma(s)$.

Then the transverse Levi-Civita connection ∇^Q is metrical and torsion-free with respect to $g_Q = g_{L^\perp}$. That is, $\nabla^Q_X g_Q = 0$ for all $X \in \Gamma TM$ and for any $Y, Z \in \Gamma TM$,

$$T^Q(Y, Z) = \nabla^Q_Y \pi(Z) - \nabla^Q_Z \pi(Y) - \pi[Y, Z] = 0,$$

where T^Q is the transversal torsion tensor field of ∇^Q.

Let the transversal curvature tensor R^Q of $\nabla^Q \equiv \nabla$ is defined by

$$R^Q(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad \forall X, Y \in \Gamma TM. \tag{2.6}$$

It is trivial that $i(X)R^Q = 0$ for any $X \in \Gamma L$, where $i(X)$ is the interior product. In fact, $R^Q(X, Y)s = (\theta(X)\nabla)_Y s = 0$ where $Y \in \Gamma TM$ and $s \in \Gamma Q$ ([14]).

Definition 2.6 The *transversal sectional curvature* K^Q, *transversal Ricci operator* Ric^Q and *transversal scalar curvature* σ^Q with respect to ∇ are defined by

$$K^Q(s, t) = \frac{g_Q(R^Q(s,t)t, s)}{g_Q(s,s)g_Q(t,t) - g_Q(s,t)^2}, \quad \forall s, t, \in \Gamma Q$$

$$Ric^Q(s) = \sum_a R^Q(s, E_a)E_a, \quad \sigma^Q = g_Q(Ric^Q(E_a), E_a),$$

where $\{E_a\}$ is a local orthonomal basic frame of Q.

Definition 2.7 The *mean curvature form* κ of $\mathcal{F}$ is given by

$$\kappa(X) = g_Q(\sum_{i=1}^{p} \pi(\nabla^M_{E_i} E_i), X), \quad \forall X \in \Gamma Q, \tag{2.7}$$

where $\{E_i\}_{i=1,\cdots,p}$ is a local orthonormal basis of L. The foliation $\mathcal{F}$ is said to be *minimal* (or *harmonic*) if $\kappa = 0$.

Definition 2.8 Let $\mathcal{F}$ be an arbitrary foliation on a manifold M. A differential form ω is *basic* if

$$i(X)\omega = 0,\ \theta(X)\omega = 0, \quad \forall X \in \Gamma L, \tag{2.8}$$

where $i(X)$ is an interior product.

Locally, the basic $r-$form ω is expressed by

$$\omega = \sum_{a_1<\cdots<a_r} \omega_{a_1\cdots a_r} dy_{a_1} \wedge \cdots \wedge dy_{a_r} \tag{2.9}$$

where $\frac{\partial \omega_{a_1\cdots a_r}}{\partial x^j} = 0$ for all $j = 1, \cdots, p$.

Let $\Omega^r_B(\mathcal{F})$ be the space of all basic r–forms. Then ([1])

$$\Omega^*_B(M) = \Omega^*_B(\mathcal{F}) \oplus \Omega^*_B(\mathcal{F})^\perp.$$

Notation. Let ω_B is the basic part of the form ω.

Theorem 2.9 ([1]) *For a Riemannian foliation $\mathcal{F}$ on a compact manifold, κ_B is closed, i.e., $d\kappa_B = 0$.*

Definition 2.10 The *basic Laplacian* Δ_B acting on $\Omega^*_B(\mathcal{F})$ by

$$\Delta_B = d_B\delta_B + \delta_B d_B, \tag{2.10}$$

where δ_B is the formal adjoint operator of $d_B = d|_{\Omega_B^*(\mathcal{F})}$, which are locally given by

$$d_B = \sum_a \theta^a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_a i(E_a)\nabla_{E_a} + i(\kappa_B^\sharp), \tag{2.11}$$

where $\kappa_B^\sharp$ is the g_Q–dual vector of κ_B, $\{E_a\}_{a=1,\cdots,q}$ is a local orthonormal basic frame of Q and $\{\theta^a\}$ is its g_Q–dual 1–form.

We define $\nabla_{tr}^*\nabla_{tr} : \Omega_B^r(\mathcal{F}) \to \Omega_B^r(\mathcal{F})$ by

$$\nabla_{tr}^*\nabla_{tr} = -\sum_a \nabla^2_{E_a,E_a} + \nabla_{\kappa_B^\sharp}, \tag{2.12}$$

where $\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla^M_X Y}$ for any $X, Y \in \Gamma TM$.

Proposition 2.11 ([3]) *The operator $\nabla_{tr}^*\nabla_{tr}$ is positive definite and formally self adjoint on the space of basic forms, i.e.,*

$$\int < \nabla_{tr}^*\nabla_{tr}\varphi, \psi > = \int < \nabla_{tr}\varphi, \nabla_{tr}\psi >,$$

where $< \nabla_{tr}\varphi, \nabla_{tr}\psi > = \sum_a < \nabla_{E_a}\varphi, \nabla_{E_a}\psi >$.

Definition 2.12 ([5]) A vector field $Y \in M$ is an *infinitesimal automorphism* of $\mathcal{F}$ if

$$[Y, Z] \in \Gamma L, \quad \forall Z \in \Gamma L.$$

Let $V(\mathcal{F})$ be the space of all infinitesimal automorphaisms and let Let $\bar{V}(\mathcal{F}) = \{\bar{Y} = \pi(Y) | Y \in V(\mathcal{F})\}$. It is trivial that an elements s of $\bar{V}(\mathcal{F})$ satisfies $\nabla_X s = 0$ for all $X \in \Gamma L$. Hence the metric defined by (2.4) induces an identification([10])

$$\bar{V}(\mathcal{F}) \cong \Omega_B^1(\mathcal{F}). \tag{2.13}$$

For the later use, we recall the transversal divergence theorem([16]) on a foliated Riemannian manifold.

Theorem 2.13 (Transversal divergence theorem) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a transversally oriented foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then*

$$\int_M div_\nabla \bar{X} = \int_M g_Q(\bar{X}, \kappa_B^\sharp) \tag{2.14}$$

for all $X \in V(\mathcal{F})$, where $div_\nabla X$ denotes the transversal divergence of X with respect to the connection ∇.

Proof. Let $div_\nabla X$ be the divergence of $X \in \Gamma TM$ with respect to ∇. Then we have

$$\begin{aligned} div_\nabla X &= \sum_i g_M(\nabla^M_{E_i} X, E_i) + \sum_a g_M(\nabla^M_{E_a} X, E_a) \\ &= -\sum_i g_Q(\bar{X}, \pi(\nabla^M_{E_i} E_i)) + \sum_a g_Q(\pi(\nabla^M_{E_a} \bar{X}), E_a) \\ &= -g_Q(\bar{X}, \kappa_B^\sharp) + div_\nabla \bar{X}, \end{aligned}$$

where $\bar{X} = \pi(X)$. By Green's Theorem, we have

$$0 = \int_M div_\nabla X = \int_M div_\nabla \bar{X} - \int_M g_Q(\bar{X}, \kappa_B^\sharp).$$

Hence the proof is completed. $\square$

Now we define the bundle map $A_Y : \Lambda^r Q^* \to \Lambda^r Q^*$ for any $Y \in V(\mathcal{F})$([5]) by

$$A_Y \phi = \theta(Y)\phi - \nabla_Y \phi. \tag{2.15}$$

It is well-known([5]) that for any $s \in \Gamma Q$

$$A_Y s = -\nabla_{Y_s} \bar{Y}, \tag{2.16}$$

where Y_s is the vector field such that $\pi(Y_s) = s$. In fact, $A_Y s = \theta(Y)s - \nabla_Y s = \nabla_Y s - \nabla_{Y_s}\bar{Y} - \nabla_Y s = -\nabla_{Y_s}\bar{Y}$. Since $\theta(X)\phi = \nabla_X\phi$ for any $X \in \Gamma L$, A_Y preserves the basic forms and depends only on $\bar{Y} = \pi(Y)$. Now, we recall the generalized Weitzenböck formula on $\Omega_B^*(\mathcal{F})$.

Theorem 2.14 ([3]) *On a Riemannian foliated manifold $(M, \mathcal{F})$, we have*

$$\Delta_B\phi = \nabla_{tr}^*\nabla_{tr}\phi + F(\phi) + A_{\kappa_B^\sharp}\phi, \quad \phi \in \Omega_B^r(\mathcal{F}), \tag{2.17}$$

where $F(\phi) = \sum_{a,b}\theta^a \wedge i(E_b)R^\nabla(E_b, E_a)\phi$. If ϕ is a basic 1–form, then $F(\phi)^\sharp = Ric^Q(\phi^\sharp)$.

Now we recall a very important lemma for later use. From Proposition 4.1 in [11], it is well-known that $\Delta_B - \kappa_B^\sharp$ on all basic functions is the restriction of $\Delta - \kappa^\sharp$ on all functions. Hence, by maximum and minimum principles, we have the following lemma.

Lemma 2.15 ([4]) *Let $(M, g, \mathcal{F})$ be a closed oriented Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g. If $(\Delta_B - \kappa_B^\sharp)f \geq 0$ (or ≤ 0) for any basic function f, then f is constant.*

3 Transversally harmonic maps between foliated Riemannian manifolds

Let $(M, g, \mathcal{F})$ and $(M', g', \mathcal{F}')$ be two foliated Riemannian manifolds. Let ∇^M and $\nabla^{M'}$ be the Levi-Civita connections of M and M', respectively. Let ∇ and ∇' be the transverse Levi-Civita connections of Q and Q', respectively. Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth *foliated map*, i.e., $d\phi(L) \subset L'$. Then we define $d_T\phi : Q \to Q'$ by

$$d_T\phi := \pi' \circ d\phi \circ \sigma. \tag{3.1}$$

Then $d_T\phi$ is a section in $Q^* \otimes \phi^{-1}Q'$, where $\phi^{-1}Q'$ is the pull-back bundle on M. Let ∇^ϕ and $\tilde{\nabla}$ be the connections on $\phi^{-1}Q'$ and $Q^* \otimes \phi^{-1}Q'$, respectively.

Definition 3.1 Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Then ϕ is called *transversally totally geodesic* if

$$\tilde{\nabla}_{tr} d_T\phi = 0, \tag{3.2}$$

where $(\tilde{\nabla}_{tr} d_T\phi)(X, Y) = (\tilde{\nabla}_X d_T\phi)(Y)$ for any $X, Y \in \Gamma Q$.

Note that if $\phi : M \to M'$ is transversally totally geodesic with $d\phi(Q) \subset Q'$, then for any transversal geodesic γ in M, $\phi \circ \gamma$ is also transversal geodesic.

Definition 3.2 The *transversal tension field* of ϕ is defined by

$$\tau_b(\phi) = tr_Q \tilde{\nabla} d_T\phi = \sum_{a=1}^{q} (\tilde{\nabla}_{E_a} d_T\phi)(E_a), \tag{3.3}$$

where $\{E_a\}$ is a local orthonormal basic frame of Q.

Trivially, the transversal tension field $\tau_b(\phi)$ is a section of $\phi^{-1}Q'$

Definition 3.3 Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Then the map ϕ is said to be *transverally harmonic* if the transversal tension field of ϕ vanishes, i.e., $\tau_b(\phi) = 0$.

Now we recall the O'Neill tensors $\mathcal{A}$ and $\mathcal{T}$([9,14]) on a foliated manifold $(M, \mathcal{F})$, which are defined by

$$\mathcal{A}_X Y = \pi^\perp(\nabla^M_{\pi(X)}\pi(Y)) + \pi(\nabla^M_{\pi(X)}\pi^\perp(Y)) \tag{3.4}$$

$$\mathcal{T}_X Y = \pi^\perp(\nabla^M_{\pi^\perp(X)}\pi(Y)) + \pi(\nabla^M_{\pi^\perp(X)}\pi^\perp(Y)) \tag{3.5}$$

for any $X, Y \in \Gamma TM$, where $\pi^\perp : TM \to L$. It is well known([9]) that

$$\mathcal{A}_{\pi(X)}\pi(Y) = \pi^\perp[\pi(X), \pi(Y)] \tag{3.6}$$

for any vector fields X, Y on M. Then $\mathcal{T} \equiv 0$ is equivalent to the property that all leaves of $\mathcal{F}$ are totally geodesic submanifolds of (M, g) and $\mathcal{A} \equiv 0$ is equivalent to the integrability of Q.

Let $\{E_i\}_{i=1,\cdots,p}$ be a local orthonomal basis of L and $\{E_a\}_{a=1,\cdots,q}$ be a local orthonormal basic frame of Q. Then we have the following.

Theorem 3.4 *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a foliated map. Then*

$$\begin{aligned}
\tau(\phi) &= \tau(\phi|_{\mathcal{F}}) + \tau_b(\phi) - d_T\phi(\kappa^\sharp) + tr_g\phi^*\mathcal{T}' + tr_Q\phi^*\mathcal{A}' \\
&+ \sum_{a=1}^{q} \pi^\perp\{\nabla^{M'}_{\pi^\perp d\phi(E_a)}\pi^\perp d\phi(E_a) + \nabla^{M'}_{\pi d\phi(E_a)}\pi^\perp d\phi(E_a) - d\phi(\nabla_{E_a}E_a)\} \\
&+ \sum_{a=1}^{q} \pi\nabla^{M'}_{\pi^\perp d\phi(E_a)}\pi d\phi(E_a),
\end{aligned}$$

where $\tau(\phi)$ is the tension field of ϕ, $\tau(\phi|_{\mathcal{F}}) = \pi^{\perp}\sum_i(\tilde{\nabla}_{E_i}d\phi)(E_i)$,

$tr_g\phi^*\mathcal{T}' = \sum_{i=1}^{p}\mathcal{T}'_{d\phi(E_i)}d\phi(E_i) + \sum_{a=1}^{q}\mathcal{T}'_{d\phi(E_a)}d\phi(E_a)$ and

$tr_Q\phi^*\mathcal{A}' = \sum_{a=1}^{q}\mathcal{A}'_{d\phi(E_a)}d\phi(E_a)$.

Proof. Let $\{E_i, E_a\}_{i=1,\cdots,p;a=1,\cdots,q}$ be a local orthonormal frame of ΓTM such that $E_i \in \Gamma L, E_a \in \Gamma Q$. By the definition of the tension field, we have

$$\tau(\phi) = \sum_{i=1}^{p}(\tilde{\nabla}_{E_i}d\phi)(E_i) + \sum_{a=1}^{q}(\tilde{\nabla}_{E_a}d\phi)(E_a).$$

Since ϕ is a foliated map, $\pi d\phi(E_i) = 0$ and $\pi^{\perp}d\phi(E_i) = d\phi(E_i)$. Hence we have

$$\begin{aligned}\sum_{i=1}^{p}(\tilde{\nabla}_{E_i}d\phi)(E_i) &= \sum_{i=1}^{p}\{\nabla^{M'}_{d\phi(E_i)}d\phi(E_i) - d\phi(\nabla^{M}_{E_i}E_i)\}\\ &= \tau(\phi|_{\mathcal{F}}) + \sum_{i=1}^{p}\{\pi\nabla^{M'}_{d\phi(E_i)}d\phi(E_i) - \pi d\phi(\nabla^{M}_{E_i}E_i)\}\end{aligned}$$

and

$$\begin{aligned}&\sum_{a=1}^{q}(\tilde{\nabla}_{E_a}d\phi)(E_a)\\ =\ &\tau_b(\phi) + \sum_{a=1}^{q}\{\pi^{\perp}\nabla^{M'}_{\pi d\phi(E_a)}\pi d\phi(E_a) + \nabla^{M'}_{\pi d\phi(E_a)}\pi^{\perp}d\phi(E_a)\}\\ &+ \sum_{a=1}^{q}\{\nabla^{M'}_{\pi^{\perp}d\phi(E_a)}\pi d\phi(E_a) + \nabla^{M'}_{\pi^{\perp}d\phi(E_a)}\pi^{\perp}d\phi(E_a) - \pi^{\perp}d\phi(\nabla^{M}_{E_a}E_a)\}.\end{aligned}$$

From (3.6), we have $\pi^{\perp}\nabla^{M'}_{\pi d\phi(E_a)}\pi d\phi(E_a) = \pi^{\perp}\nabla^{M}_{E_a}E_a = 0$. From (3.4) and (3.5), we have

$$\begin{aligned}\tau(\phi) &= \tau(\phi|_{\mathcal{F}}) + \tau_b(\phi) - \pi d\phi(\sum_{i=1}^{p}\pi(\nabla^{M}_{E_i}E_i)) + \sum_{i=1}^{p}\mathcal{T}'_{d\phi(E_i)}d\phi(E_i)\\ &+ \sum_{a=1}^{q}\{\mathcal{T}'_{d\phi(E_a)}d\phi(E_a) + \mathcal{A}'_{d\phi(E_a)}d\phi(E_a) + \pi\nabla^{M'}_{\pi^{\perp}d\phi(E_a)}\pi d\phi(E_a)\}\\ &+ \sum_{a=1}^{q}\pi^{\perp}\{\nabla^{M'}_{\pi d\phi(E_a)}\pi^{\perp}d\phi(E_a) + \nabla^{M'}_{\pi^{\perp}d\phi(E_a)}\pi^{\perp}d\phi(E_a) - d\phi(\pi\nabla^{M}_{E_a}E_a)\}.\end{aligned}$$

Since $\sum_{i=1}^{p}\pi(\nabla^{M}_{E_i}E_i) = \kappa^{\sharp}$, the proof is completed. □

Corollary 3.5 *If a foliated map $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ satisfies $d\phi(Q) \subset Q'$, then*

$$\tau(\phi) = \tau(\phi|_{\mathcal{F}}) + \tau_b(\phi) - d\phi(\kappa^\sharp) + tr_L \phi^* \mathcal{T}', \tag{3.7}$$

where $tr_L \phi^* \mathcal{T}' = \sum_{i=1}^{p} \mathcal{T}'_{d\phi(E_i)} d\phi(E_i)$.

Proof. Since $d\phi(Q) \subset Q'$, $\pi^\perp d\phi(E_a) = 0$ for all a. Moreover, from (3.5) and (3.6), $\mathcal{A}'_X X = 0$ and $\mathcal{T}'_X Y = 0$ for all $X, Y \in \Gamma Q'$. Hence the proof is completed. □

Corollary 3.6 *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a foliated smooth map. Assume that $\mathcal{F}$ is minimal, $\mathcal{F}'$ is totally geodesic and $d\phi(Q) \subset Q'$. Then ϕ is a harmonic if and only if ϕ is a transversally harmonic and leaf-wise harmonic, i.e., $\tau(\phi|_{\mathcal{F}}) = 0$.*

Proof. Since $\mathcal{F}$ is minimal and $\mathcal{F}'$ is totally geodesic, i.e., $\kappa^\sharp = 0$ and $\mathcal{T}' = 0$, we have from (3.7)

$$\tau(\phi) = \tau(\phi|_{\mathcal{F}}) + \tau_b(\phi).$$

So the proof is completed. □

Corollary 3.7 *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map and $d\phi(Q) \subset Q'$. Then ϕ is a transversally harmonic map if and only if*

$$\pi(\tau(\phi)) = tr_L \phi^* \mathcal{T}' - d\phi(\kappa^\sharp).$$

Now, let $\mathcal{F}$ be a Riemannian flow defined by a unit vector field V on a Riemannian manifold (M^{n+1}, g). Then

$$\kappa^\sharp = \pi(\nabla^M_V V) = \nabla^M_V V. \tag{3.8}$$

In fact, $\nabla^M_V V$ is already orthogonal to the leaves since $g(\nabla^M_V V, V) = 0$. Moreover, it is trivial that $\mathcal{F}$ is totally geodesic if and only if $\mathcal{F}$ is minimal, i.e., $\mathcal{T} = 0$ if and only if $\kappa^\sharp = 0$. Let $\mathcal{F}$ and $\mathcal{F}'$ be two Riemannian flows defined by unit vector fields V and V' on Riemannian manifolds (M, g) and (M', g'), respectively. Let $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a smooth foliated map. Then

$$\tau(\phi|_{\mathcal{F}}) = V(\lambda)V' - \pi^\perp d\phi(\kappa^\sharp), \quad \lambda = (\phi^*\omega')(V), \tag{3.9}$$

where ω' is the dual 1–form of V'. Hence if $d\phi(Q) \subset Q'$, then ϕ is leaf-wise harmonic if and only if λ is basic, i.e., $V(\lambda) = 0$. Hence we have following corollary.

Corollary 3.8 *Let $\mathcal{F}$ and $\mathcal{F}'$ be two Riemannian flows defined by a unit vector fields V and V' on a Riemannian manifolds M and M', respectively. Assume that $\mathcal{F}$ and $\mathcal{F}'$ are minimal. Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map and $d\phi(Q) \subset Q'$. Then ϕ is harmonic if and only if ϕ is transversally harmonic and $(\phi^*\omega')(V)$ is basic.*

Proof. Since $\mathcal{F}$ is minimal, from (3.9)

$$\tau(\phi|_{\mathcal{F}}) = V(\lambda)V', \quad \lambda = (\phi^*\omega')(V).$$

Hence the proof follows from Corollary 3.5. □

Let $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ and $\psi : (M', \mathcal{F}') \to (M'', \mathcal{F}'')$ be smooth foliated maps. Then the composition $\psi \circ \phi : (M, \mathcal{F}) \to (M'', \mathcal{F}'')$ is a smooth foliated map. Moreover, we have

$$d_T(\psi \circ \phi) = d_T\psi \circ d_T\phi \tag{3.10}$$

Hence we have the following proposition.

Proposition 3.9 *Let* $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ *and* $\psi : (M', \mathcal{F}') \to (M'', \mathcal{F}'')$ *be smooth foliated maps. Then*

$$\tilde{\nabla}_{tr} d_T(\psi \circ \phi) = d_T\psi(\tilde{\nabla}_{tr} d_T\phi) + \phi^* \tilde{\nabla}_{tr} d_T\psi, \tag{3.11}$$

where $(\phi^* \tilde{\nabla}_{tr} d_T\psi)(X, Y) = (\tilde{\nabla}_{d_T\phi(X)} d_T\psi)(d_T\phi(Y))$ *for any* $X, Y \in \Gamma Q$.

Proof. From (3.10), we have that, for any $X, Y \in \Gamma Q$,

$$\begin{aligned}(\tilde{\nabla}_{tr} d_T(\psi \circ \phi))(X, Y) &= \nabla_X^{\psi\circ\phi} d_T(\psi \circ \phi)(Y) - d_T(\psi \circ \phi)(\nabla_X Y) \\ &= (\tilde{\nabla}_{d_T\phi(X)} d_T\psi)(d_T\phi(Y)) + d_T\psi((\tilde{\nabla}_X d_T\phi)(Y)) \\ &= (\phi^* \tilde{\nabla}_{tr} d_T\psi)(X, Y) + d_T\psi(\tilde{\nabla}_{tr} d_T\phi)(X, Y),\end{aligned}$$

which proves (3.11). $\square$

Corollary 3.10 *Let* $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ *and* $\psi : (M', \mathcal{F}') \to (M'', \mathcal{F}'')$ *be smooth foliated maps. Then the transversal tension field of the composition is given by*

$$\tau_b(\psi \circ \phi) = d_T\psi(\tau_b(\phi)) + tr_Q \phi^* \tilde{\nabla}_{tr} d_T\psi, \tag{3.12}$$

where $tr_Q \phi^* \tilde{\nabla}_{tr} d_T\psi = \sum_{a=1}^{q} (\tilde{\nabla}_{d_T\phi(E_a)} d_T\psi)(d_T\phi(E_a))$.

Corollary 3.11 *Let* $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ *be a transversally harmonic map and let* $\psi : (M', \mathcal{F}') \to (M'', \mathcal{F}'')$ *be a transversally totally geodesic map. Then* $\psi \circ \phi : (M, \mathcal{F}) \to (M'', \mathcal{F}'')$ *is a transversally harmonic map.*

4 The first normal variational formula

Let $(M, g, \mathcal{F})$ be a foliated Riemannian manifold. Let $vol_L : M \to [0, \infty]$ be the volume function which $vol_L(x)$ is the volume of the leaf passing through $x \in M$. It is trivial that vol_L is a basic function. Then we have the following.

Lemma 4.1 *On a foliated Riemannian manifold* $(M, \mathcal{F})$*, it holds that*

$$d_B vol_L + (vol_L)\kappa_B = 0. \tag{4.1}$$

Proof. Let $\{v_1, \cdots, v_p\}$ be linearly independent vector fields of ΓL such that $vol_L = \chi_{\mathcal{F}}(v_1, \cdots, v_p) = i(v_p)\cdots i(v_1)\chi_{\mathcal{F}}$, where $\chi_{\mathcal{F}}$ is the characteristic form of $\mathcal{F}$. By the Rummler's formula ([13]), $\varphi_0 := d\chi_{\mathcal{F}} + \kappa \wedge \chi_{\mathcal{F}}$ satisfies $i(v_p)\cdots i(v_1)\varphi_0 = 0$. Therefore we have

$$\begin{aligned} i(v_p)\cdots i(v_1) d\chi_{\mathcal{F}} &= -i(v_p)\cdots i(v_1)(\kappa \wedge \chi_{\mathcal{F}}) \\ &= i(v_p)\cdots i(v_2)(\kappa \wedge i(v_1)\chi_{\mathcal{F}}) \\ &= (-1)^{p+1}\kappa \wedge i(v_p)\cdots i(v_1)\chi_{\mathcal{F}} \\ &= (-1)^{p+1}(vol_L)\kappa. \end{aligned}$$

On the other hand, a direct calculation gives

$$d(i(v_p)\cdots i(v_1)\chi_{\mathcal{F}}) = (-1)^p i(v_p)\cdots i(v_1) d\chi_{\mathcal{F}} + \alpha(v_1, \cdots, v_p),$$

where $\alpha(v_1, \cdots, v_p) = \sum_{j=1}^{p} (-1)^{p-j} i(v_p)\cdots i(v_{j+1})\theta(v_j)\{i(v_{j-1})\cdots i(v_1)\chi_{\mathcal{F}}\}$. Since L is integrable, $\alpha(v_1, \cdots, v_p) \in L^*$ and so $\alpha(v_1, \cdots, v_p) = 0$. Since vol_L is a basic function, we have

$$
\begin{aligned}
d_B vol_L &= d_B(i(v_p)\cdots i(v_1)\chi_{\mathcal{F}}) \\
&= (-1)^p i(v_p)\cdots i(v_1) d_B \chi_{\mathcal{F}} \\
&= (-1)^{2p+1}(vol_L)\kappa_B = -(vol_L)\kappa_B.
\end{aligned}
$$

The proof is completed. □

Definition 4.2 Let Ω be a compact domain of M. Then the *transversal energy* of ϕ on $\Omega \subset M$ is defined by

$$E_B(\phi;\Omega) = \frac{1}{2}\int_\Omega |d_T\phi|^2 \frac{1}{vol_L}\mu_M, \tag{4.2}$$

where $|d_T\phi|^2 = \sum_{a=1}^{q} g_{Q'}(d_T\phi(E_a), d_T\phi(E_a))$ and μ_M is the volume element of M.

Let $V \in \phi^{-1}Q'$. Obviously, V may be considered as a vector field on Q' along ϕ. Then there is a 1-parameter family of foliated maps ϕ_t with $\phi_0 = \phi$ and $\frac{d\phi_t}{dt}|_{t=0} = V$. Then the family $\{\phi_t\}$ is said to be a *foliated variation* of ϕ with the *normal variation vector field* V. Then we have the first normal variational formula(cf. [7]).

Theorem 4.3 (The first normal variational formula) *Let $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a smooth foliated map, and all leaves of $\mathcal{F}$ be compact. Let $\{\phi_t\}$ be a smooth foliated variation of ϕ supported in a compact domain Ω. Then*

$$\frac{d}{dt}E_B(\phi_t,\Omega)|_{t=0} = -\int_\Omega < V, \tau_b(\phi) > \frac{1}{vol_L}\mu_M, \tag{4.3}$$

where $V = \frac{d\phi_t}{dt}|_{t=0}$ is the normal variation vector field of $\{\phi_t\}$ and $<\cdot,\cdot>$ is the pull–back metric on $\phi^{-1}Q'$.

Proof. Let Ω be a compact domain of M and let $\{\phi_t\}$ be a foliated variation of ϕ supported in Ω with the normal variation vector field $V \in \phi^{-1}Q'$. Choose a local orthonormal basic frame $\{E_a\}$ on Q such that $(\nabla E_a)_x = 0$, at $x \in M$. Define $\Phi : M \times (-\epsilon, \epsilon) \to M'$ by $\Phi(x,t) = \phi_t(x)$ and set $E = \Phi^{-1}Q'$. Let ∇^Φ denote the pull–back connection on E. Obviously, $d_T\Phi(E_a) = d_T\phi_t(E_a)$ and $d\Phi(\frac{\partial}{\partial t}) = \frac{d\phi_t}{dt}$. Moreover, we have $\nabla^\Phi_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} = \nabla^\Phi_{\frac{\partial}{\partial t}} E_a = \nabla^\Phi_{E_a}\frac{\partial}{\partial t} = 0$. Hence we have

$$\begin{aligned}
\frac{d}{dt}E_B(\phi_t,\Omega) &= \int_\Omega \sum_a < \nabla^\Phi_{\frac{\partial}{\partial t}} d_T\Phi(E_a), d_T\Phi(E_a) > \frac{1}{vol_L}\mu_M \\
&= \int_\Omega \sum_a < \nabla^\Phi_{E_a} d\Phi\left(\frac{\partial}{\partial t}\right), d_T\Phi(E_a) > \frac{1}{vol_L}\mu_M \\
&= \int_\Omega \sum_a \{E_a < \frac{d\phi_t}{dt}, d_T\phi_t(E_a) > - < \frac{d\phi_t}{dt}, \nabla^{\phi_t}_{E_a} d_T\phi_t(E_a) >\} \frac{1}{vol_L}\mu_M \\
&= \int_\Omega \sum_a E_a\{< \frac{d\phi_t}{dt}, d_T\phi_t(E_a) > \frac{1}{vol_L}\}\mu_M \\
&\quad - \int_\Omega \sum_a < \frac{d\phi_t}{dt}, d_T\phi_t(E_a) > E_a\left(\frac{1}{vol_L}\right)\mu_M \\
&\quad - \int_\Omega < \frac{d\phi_t}{dt}, \tau_b(\phi_t) > \frac{1}{vol_L}\mu_M.
\end{aligned}$$

Now we define a normal vector field W_t by

$$W_t = \frac{1}{vol_L}\sum_a < \frac{d\phi_t}{dt}, d_T\phi_t(E_a) > E_a.$$

Then its divergence is

$$div_\nabla W_t = \sum_a E_a\{< \frac{d\phi_t}{dt}, d_T\phi_t(E_a) > \frac{1}{vol_L}\}.$$

By the transversal divergence theorem(Theorem 2.13), we have

$$\begin{aligned}\frac{d}{dt}E_B(\phi_t,\Omega) &= \int_\Omega \{div_\nabla W_t - < \frac{d\phi_t}{dt}, d_T\phi_t(d_B\left(\frac{1}{vol_L}\right)) >\}\mu_M \\ &\quad - \int_\Omega < \frac{d\phi_t}{dt}, \tau_b(\phi_t) > \frac{1}{vol_L}\mu_M \\ &= \int_\Omega < \frac{d\phi_t}{dt}, d_T\phi_t((vol_L)\kappa_B + d_B vol_L) > \frac{1}{vol_L^2}\mu_M \\ &\quad - \int_\Omega < \frac{d\phi_t}{dt}, \tau_b(\phi_t) > \frac{1}{vol_L}\mu_M.\end{aligned}$$

By Lemma 4.1, we have

$$\frac{d}{dt}E_B(\phi_t,\Omega) = -\int_\Omega < \frac{d\phi_t}{dt}, \tau_b(\phi_t) > \frac{1}{vol_L}\mu_M, \tag{4.4}$$

which proves (4.3). □

Corollary 4.4 *Let $\phi : (M,\mathcal{F}) \to (M',\mathcal{F}')$ be a smooth foliated map. Assume that all leaves of $\mathcal{F}$ are compact. Then ϕ is transversally harmonic if and only if ϕ is a critical point of the transversal energy of ϕ supported in a compact domain.*

5 A generalized Weitzenböck type formula and its applications

Let $(M, g, \mathcal{F})$ and $(M', g', \mathcal{F}')$ be two foliated Riemannian manifolds and let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Let $\Omega_B^r(E) = \Omega_B^r(\mathcal{F}) \otimes E$ be the space of E–valued basic r–forms, where $E = \phi^{-1}Q'$. Let $\tilde{\nabla}$ be the transverse Levi-Civita connection on $\Omega_B^r(E)$ and let the transveral curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ is defined by

$$\tilde{R}(X, Y) = [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \tilde{\nabla}_{[X,Y]} \quad X, Y \in \Gamma TM. \tag{5.1}$$

Let $\Phi = \omega \otimes s \in \Omega_B^r(E)$ for any $\omega \in \Omega_B^r(\mathcal{F})$ and $s \in \Gamma E$. Then by a direct calculation, we have

$$\tilde{R}(X, Y)\Phi = R^Q(X, Y)\omega \otimes s + \omega \otimes R^E(X, Y)s. \tag{5.2}$$

Now we define $d_\nabla : \Omega_B^r(E) \to \Omega_B^{r+1}(E)$ by

$$d_\nabla(\omega \otimes s) = d_B\omega \otimes s + (-1)^r \omega \wedge \nabla^\phi s, \tag{5.3}$$

and let δ_∇ be a formal adjoint of d_∇. Then we have the following equations

$$d_\nabla = \sum_a \theta^a \wedge \tilde{\nabla}_{E_a}, \quad \delta_\nabla = -\sum_a i(E_a)\tilde{\nabla}_{E_a} + i(\kappa_B^\sharp), \tag{5.4}$$

where $i(X)(\omega \otimes s) = i(X)\omega \otimes s$ for any $X \in \Gamma TM$, $\{E_a\}$ is an orthonormal basis of Q, $\{\theta^a\}$ its g_Q–dual 1–form and $\kappa = \pi(\sum_a \nabla^M_{E_a} E_a)$ is a mean curvature vector of $\mathcal{F}$. Then the Laplacian Δ on $\Omega_B^*(E)$ is defined by

$$\Delta = \delta_\nabla d_\nabla + d_\nabla \delta_\nabla. \tag{5.5}$$

Moreover, the operators A_X and $\theta(X)$ on $\Omega_B^r(E)$ are extended by

$$A_X(\omega \otimes s) = A_X\omega \otimes s \tag{5.6}$$

$$\theta(X)(\omega \otimes s) = \theta(X)\omega \otimes s + (-1)^r \omega \wedge \nabla_X^{\phi} s \tag{5.7}$$

for any $X \in \Gamma TM$. Then

$$\begin{aligned}
d_\nabla i(X)(\Phi) &= d_\nabla(i(X)\omega \otimes s) = d_B i(X)\omega \otimes s + (-1)^{r-1} i(X)\omega \wedge \nabla^{\phi} s \\
i(X)d_\nabla(\Phi) &= i(X)(d_B\omega \otimes s + (-1)^r \omega \wedge \nabla^{\phi} s) \\
&= i(X)d_B\omega \otimes s + (-1)^r i(X)\omega \wedge \nabla^{\phi} s + (-1)^r \omega \wedge \nabla_X^{\phi} s.
\end{aligned}$$

Hence we can get $\theta(X)(\Phi) = (d_\nabla i(X) + i(X)d_\nabla)(\Phi)$, for any $X \in \Gamma TM$. Hence $\Phi \in \Omega_B^*(E)$ if and only if $i(X)\Phi = 0$ and $\theta(X)\Phi = 0$ for all $X \in \Gamma L$. Then the generalized Weitzenböck type formula (3.17) is extended to $\Omega_B^*(E)$ as follows.

Theorem 5.1 *For any $\Phi \in \Omega_B^*(E)$,*

$$\Delta\Phi = \tilde{\nabla}_{tr}^* \tilde{\nabla}_{tr} \Phi + F(\Phi) + A_{\kappa_B^\sharp} \Phi, \tag{5.8}$$

where $F(\Phi) = \sum_{a,b=1}^{q} \theta^a \wedge i(E_b)\tilde{R}(E_b, E_a)(\Phi)$.

Proof. Let $\Phi = \omega \otimes s \in \Omega_B^r(\mathcal{F}) \otimes \phi^{-1}Q'$. Then by a direct calculation

$$
\begin{aligned}
\delta_\nabla d_\nabla \Phi &= -\sum_{a,b} i(E_b)\tilde\nabla_{E_b}(\theta^a \wedge \tilde\nabla_{E_a}\Phi) + i(\kappa_B^\sharp)(d_B\omega \otimes s + (-1)^r \omega \wedge \nabla^\phi_{E_b} s) \\
&= \sum_{a,b}(-i(E_b)\theta^a \tilde\nabla_{E_b}\tilde\nabla_{E_a}\Phi + \theta^a \wedge i(E_b)\tilde\nabla_{E_b}\tilde\nabla_{E_a}\Phi) \\
&\quad + i(\kappa_B^\sharp) d_B\omega \otimes s + (-1)^r i(\kappa_B^\sharp)\omega \wedge \nabla^\phi_{E_b} s + (-1)^r \omega \wedge \nabla^\phi_{\kappa_B^\sharp} s, \\
d_\nabla \delta_\nabla \Phi &= \sum_{a,b} \theta^a \wedge \tilde\nabla_{E_a}(-i(E_b)\tilde\nabla_{E_b}\Phi + i(\kappa_B^\sharp)\Phi) \\
&= -\sum_{a,b}\theta^a \wedge i(E_b)\tilde\nabla_{E_a}\tilde\nabla_{E_b}\Phi \\
&\quad + \sum_a \theta^a \wedge (\tilde\nabla_{E_a} i(\kappa_B^\sharp)\omega \otimes s + (-1)^{r-1} i(\kappa_B^\sharp)\omega \wedge \nabla^\phi_{E_a} s) \\
&= -\sum_{a,b}\theta^a \wedge i(E_b)\tilde\nabla_{E_a}\tilde\nabla_{E_b}\Phi + (-1)^{r-1} i(\kappa_B^\sharp)\omega \wedge \nabla^\phi_{E_a} s + d_B i(\kappa_B^\sharp)\omega \otimes s.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\Delta\Phi &= \delta_\nabla d_\nabla \Phi + d_\nabla \delta_\nabla \Phi \\
&= -\sum_a \tilde\nabla_{E_a}\tilde\nabla_{E_a}\Phi + \sum_{a,b}\theta^a \wedge i(E_b)\tilde R(E_b, E_a)(\Phi) + (-1)^r \omega \wedge \nabla^\phi_{\kappa_B^\sharp} s \\
&\quad + (i(\kappa_B^\sharp) d_B\omega + d_B i(\kappa_B^\sharp)\omega) \otimes s \\
&= \nabla_{tr}^*\nabla_{tr}\Phi - (-1)^r \omega \wedge \nabla^\phi_{\kappa_B^\sharp} s - \nabla_{\kappa_B^\sharp}\omega \otimes s + F(\Phi) + (-1)^r \omega \wedge \nabla^\phi_{\kappa_B^\sharp} s \\
&\quad + \theta(\kappa_B^\sharp)\omega \otimes s \\
&= \nabla_{tr}^*\nabla_{tr}\Phi + F(\Phi) + A_{\kappa_B^\sharp}\omega \otimes s.
\end{aligned}
$$

Hence the proof is completed. □

Note that $d_T\phi \in \Omega_B^1(E)$ and $|d_T\phi|^2 \in \Omega_B(\mathcal{F})$, then we have the following.

Theorem 5.2 *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Then the* *generalized Weitzenböck formula is given by*

$$\frac{1}{2}\Delta_B|d_T\phi|^2 = <\Delta d_T\phi, d_T\phi> - |\tilde\nabla_{tr} d_T\phi|^2 - <A_{\kappa_B^\sharp}(d_T\phi), d_T\phi> - <F(d_T\phi), d_T\phi>, \tag{5.9}$$

where

$$<F(d_T\phi), d_T\phi> = \sum_a g_{Q'}(d_T\phi(Ric^Q(E_a)), d_T\phi(E_a)) - \sum_{a,b} g_{Q'}(R^{Q'}(d_T\phi(E_b), d_T\phi(E_a))d_T\phi(E_a), d_T\phi(E_b)). \tag{5.10}$$

Proof. Let $\{E_a\}_{a=1,\cdots,q}$ be a local orthonormal basic frame such that at $x \in M$, $(\nabla E_a)_x = 0$. Then at x, we have from (2.10) and (2.11)

$$\frac{1}{2}\Delta_B|d_T\phi|^2 = <\tilde\nabla_{tr}^*\tilde\nabla_{tr} d_T\phi, d_T\phi> - |\tilde\nabla_{tr} d_T\phi|^2. \tag{5.11}$$

From (5.8), we have

$$\frac{1}{2}\Delta_B|d_T\phi|^2 = <\Delta d_T\phi, d_T\phi> - |\tilde\nabla_{tr} d_T\phi|^2 - <A_{\kappa_B^\sharp}(d_T\phi), d_T\phi> - <F(d_T\phi), d_T\phi>.$$

Now, we compute $<F(d_T\phi), d_T\phi>$. Let $\{V_\alpha\}_{\alpha=1,\cdots,q'}$ be a local orthonormal basic frame of Q' and ω^α be its dual coframe field. Let $f^\alpha = \phi^*\omega^\alpha$. Then $d_T\phi$ is expressed by

$$d_T\phi = \sum_{\alpha=0}^{q'} f^\alpha \otimes V_\alpha, \tag{5.12}$$

where $V_\alpha(x) \cong V_\alpha(\phi(x))$. From (5.2)

$$\tilde R(E_a, E_b)d_T\phi = \sum_\alpha R^Q(E_a, E_b)f^\alpha \otimes V_\alpha + \sum_\alpha f^\alpha \otimes R^E(E_a, E_b)V_\alpha, \tag{5.13}$$

where $R^E(E_a,E_b)V_\alpha = R^{Q'}(d_T\phi(E_a), d_T\phi(E_b))V_\alpha$. From (5.11), we have

$$\begin{aligned}
< F(d_T\phi), d_T\phi > &= \sum_{a,b} < \theta^a \wedge i(E_b)\tilde{R}(E_b,E_a)d_T\phi, d_T\phi > \\
&= \sum_{a,b,\alpha,\beta} < \theta^a \wedge i(E_b)R^Q(E_b,E_a)f^\alpha \otimes V_\alpha, f^\beta \otimes V_\beta > \\
&+ \sum_{a,b,\alpha,\beta} g_Q(\theta^a \wedge i(E_b)f^\alpha, f^\beta)g_Q(R^E(E_a,E_b)V_\alpha, V_\beta).
\end{aligned}$$

Note that $d_T\phi(E_a) = \sum_\alpha f^\alpha(E_a)V_\alpha$. Then we have

$$\begin{aligned}
&\sum_{a,b,\alpha,\beta} g_Q(\theta^a \wedge i(E_b)R^Q(E_b,E_a)f^\alpha \otimes V_\alpha, f^\beta \otimes V_\beta) \qquad (5.14)\\
&= \sum_{a,b,\alpha,\beta} (R^Q(E_b,E_a)f^\alpha(E_b) - f^\alpha(R^Q(E_b,E_a)E_b))g_Q(\theta^a, f^\beta) \\
&= \sum_{a,\alpha,\beta} f^\alpha(Ric^Q(E_a))g_Q(\theta^a, f^\beta) \\
&= \sum_a g_{Q'}(d_T\phi(Ric^Q(E_a), d_T\phi(E_a)).
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{a,b,\alpha,\beta} g_Q(\theta^a \wedge i(E_b)f^\alpha, f^\beta)g_Q(R^E(E_a,E_b)V_\alpha, V_\beta) \\
&= \sum_{a,b,\alpha,\beta} g_Q(f^\alpha, E_b)g_Q(\theta^a, f^\beta)g_{Q'}(R^N(d_T\phi(E_b), d_T\phi(E_a))V_\alpha, V_\beta) \\
&= \sum_{a,b} g_{Q'}(R^N(d_T\phi(E_b), d_T\phi(E_a))d_T\phi(E_b), d_T\phi(E_a)).
\end{aligned}$$

Hence we have the equation (5.9). □

Remark. (1) Let $\phi : (M,\mathcal{F}) \to (M',\mathcal{F}')$ be a smooth foliated map. Then,

$$d_\nabla(d_T\phi) = 0, \quad \delta_\nabla(d_T\phi) = -\tau_b(\phi) + i(\kappa_B^\sharp)d_T\phi. \qquad (5.15)$$

(2) If $\phi : (M,\mathcal{F}) \to (M',\mathcal{F}')$ is a transversally harmonic, then

$$\Delta d_T\phi = d_\nabla i(\kappa_B^\sharp)d_T\phi. \qquad (5.16)$$

Corollary 5.3 *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a transversally harmonic map. Then*

$$\frac{1}{2}\Delta_B |d_T\phi|^2 = -|\tilde{\nabla}_{tr} d_T\phi|^2 - < F(d_T\phi), d_T\phi > + \frac{1}{2}\kappa_B^\sharp(|d_T\phi|^2). \tag{5.17}$$

Proof. Since $d_\nabla(d_T\phi) = 0$, we have

$$A_X(d_T\phi) = -\tilde{\nabla}_X d_T\phi + d_\nabla i(X) d_T\phi, \quad \forall X \in \Gamma Q. \tag{5.18}$$

Hence (5.17) follows from (5.16) and (5.18). □

As applications of the generalized Weitzenböck formula, we have the following theorems.

Theorem 5.4 *Let $(M, g, \mathcal{F})$ be a compact foliated Riemannian manifold of nonnegative transversal Ricci curvature Ric^Q and $(M', g', \mathcal{F}')$ be a foliated Riemannian manifold of nonpositive transversal sectional curvature $K^{Q'}$. If $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ is a transversally harmonic, then ϕ is a transversally totally geodesic, i.e., $\tilde{\nabla}_{tr} d_T\phi = 0$. Furthermore,*

(1) If the transversal Ricci curvature Ric^Q of $\mathcal{F}$ is positive somewhere, then ϕ is a transversally constant, i.e., the induced map between leaf spaces is constant.

(2) If the transversal sectional curvature $K^{Q'}$ of $\mathcal{F}'$ is negative, then ϕ is either a transversally constant or $\phi(M)$ is a transversally geodesic closed curve.

Proof. Let $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a transversally harmonic map. Then from (5.17), we have

$$\frac{1}{2}(\Delta_B - \kappa_B^\sharp)|d_T\phi|^2 = -|\tilde{\nabla}_{tr} d_T\phi|^2 - < F(d_T\phi), d_T\phi > . \tag{5.19}$$

Since $Ric^Q \geq 0$ and $K^{Q'} \leq 0$, from (5.8) we have

$$< F(d_T\phi), d_T\phi > \quad \geq 0. \tag{5.20}$$

Hence $(\Delta_B - \kappa_B^\sharp)|d_T\phi|^2 \leq 0$. Then by Lemma 2.15, $|d_T\phi|$ is constant and then we have

$$|\tilde{\nabla}_{tr} d_T\phi|^2 + < F(d_T\phi), d_T\phi >= 0. \tag{5.21}$$

Hence $\tilde{\nabla}_{tr} d_T\phi = 0$, i.e., ϕ is a transversally totally geodesic and

$$\sum_a g_{Q'}(d_T\phi(Ric^Q(E_a)), d_T\phi(E_a)) = 0, \tag{5.22}$$

$$\sum_{a,b} g_{Q'}(R^{Q'}(d_T\phi(E_a), d_T\phi(E_b))d_T\phi(E_b), d_T\phi(E_a)) = 0 \tag{5.23}$$

for any indices a and b. Moreover, if Ric^Q is positive at some point, then $d_T\phi = 0$. i.e., ϕ is a transversally constant, which proves (1). For the proof of (2), if there exists a point $x \in M$ such that at least two vectors in $\{d_T\phi(E_a)\}$ are linearly independent at $\phi(x)$, say $d_T\phi(E_1)$ and $d_T\phi(E_2)$, then our hypothesis contradicts (5.23). Hence the rank of $d_T\phi < 2$, that is the rank of $d_T\phi$ is zero or one everywhere. If rank$(d_T\phi) = 0$, then ϕ is a transversally constant and if rank$(d_T\phi) = 1$, then $\phi(M)$ is closed transversally geodesic. □

Theorem 5.5 *Let $(M, g, \mathcal{F})$ be a compact foliated Riemannian manifold and let $(M', g', \mathcal{F}')$ be a foliated Riemannian manifold. Assume that λ and μ are two positive constants such that $Ric^Q \geq \lambda$ id and $K^{Q'} \leq \mu$, where Ric^Q denotes the transversal Ricci curvature of M and $K^{Q'}$ denotes the transversal sectional curvature of M'. Let $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a transversally harmonic map with $max\{rank_T\phi\} \leq C$, where $C \geq 2$ is constant. If*

$|d_T\phi|^2 \le \frac{\lambda C}{\mu(C-1)}$, *then* ϕ *is transversally constant or* ϕ *is transversally totally geodesic. In particular, if* $|d_T\phi|^2 \le \frac{\lambda}{\mu}$, *then* ϕ *is transversally constant.*

Proof. Let $\{E_a\}_{a=1,\cdots,q}$ be a local orthonormal basic frame of Q such that

$$d_T\phi = \left(\begin{array}{ccc|c} \lambda_1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & * \\ 0 & \cdots & \lambda_q & \\ \hline & * & & 0 \end{array}\right) \text{ and } rank_T\phi = q \le C$$

and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_q > 0$.

Since $g_{Q'}(d_T\phi(E_a), d_T\phi(E_b))|_x = \lambda_a\delta_{ab}$, $|d_T\phi|^2 = \sum_{a=1}^{q} g_{Q'}(d_T\phi(E_a), d_T\phi(E_a)) = \sum_{a=1}^{q}\lambda_a$.

Then we have

$$\begin{aligned} \frac{1}{2}\Delta_B|d_T\phi|^2 &= \frac{1}{2}\kappa_B^\sharp(|d_T\phi|^2) - |\tilde\nabla_{tr} d_T\phi|^2 - \sum_{a=1}^{q} g_{Q'}(d_T\phi(Ric^Q(E_a)), d_T\phi(E_a)) \\ &\quad + \sum_{a,b=1}^{q}\{|d_T\phi(E_a)|^2|d_T\phi(E_b)|^2 - g_{Q'}(d_T\phi(E_a), d_T\phi(E_b))^2\}K_{ab}^{Q'} \\ &\le \frac{1}{2}\kappa_B^\sharp(|d_T\phi|^2) - |\tilde\nabla_{tr} d_T\phi|^2 - \lambda|d_T\phi|^2 + \mu\left(|d_T\phi|^4 - \sum_{a=1}^{q}\lambda_a^2\right), \end{aligned}$$

where $K_{ab}^{Q'} = g_{Q'}(R^{Q'}(d_T\phi(E_a), d_T\phi(E_b))d_T\phi(E_b), d_T\phi(E_a))$ is the transversal sectional curvature spanned by $d_T\phi(E_a)$ and $d_T\phi(E_b)$.

Using the Scwarz's inequality, we have

$$\begin{aligned} |d_T\phi|^4 &= \left(\sum_{a=1}^{q}\lambda_a\right)\left(\sum_{b=1}^{q}\lambda_b\right) = \sum_{a,b=1}^{q}\lambda_a\lambda_b \le \frac{1}{2}\sum_{a,b=1}^{q}(\lambda_a^2+\lambda_b^2) \\ &= \frac{1}{2}\sum_{a=1}^{q}\left(q\lambda_a^2 + \sum_{b=1}^{q}\lambda_b^2\right) = \frac{1}{2}\left(q\sum_{a=1}^{q}\lambda_a^2 + q\sum_{b=1}^{q}\lambda_b^2\right) = q\sum_{a=1}^{q}\lambda_a^2 \le C\sum_{a=1}^{q}\lambda_a^2. \end{aligned}$$

Therefore $|d_T\phi|^4 - \sum_{a=1}^{q}\lambda_a^2 \le |d_T\phi|^4 - \frac{1}{C}|d_T\phi|^4 = \frac{(C-1)}{C}|d_T\phi|^4$

and then by hypothesis,

$$\begin{aligned}\frac{1}{2}\Delta_B|d_T\phi|^2 &\le \frac{1}{2}\kappa_B^\sharp(|d_T\phi|^2) - |\tilde\nabla_{tr}d_T\phi|^2 - \lambda|d_T\phi|^2 + \frac{\mu(C-1)}{C}|d_T\phi|^4 \\ &= \frac{1}{2}\kappa_B^\sharp(|d_T\phi|^2) - |\tilde\nabla_{tr}d_T\phi|^2 - |d_T\phi|^2\left(\lambda - \frac{\mu(C-1)}{C}|d_T\phi|^2\right) \\ &\le \frac{1}{2}\kappa_B^\sharp(|d_T\phi|^2). \qquad (5.24)\end{aligned}$$

Hence from Lemma 2.15, $|d_T\phi|$ is constant and then

$$|\tilde\nabla_{tr}d_T\phi|^2 + |d_T\phi|^2\left(\lambda - \frac{\mu(C-1)}{C}|d_T\phi|^2\right) = 0. \qquad (5.25)$$

Therefore $\tilde\nabla_{tr}d_T\phi = 0$(i.e., ϕ is transversally totally geodesic) and $|d_T\phi|^2\left(\lambda - \frac{\mu(C-1)}{C}|d_T\phi|^2\right) = 0$. If $|d_T\phi| = 0$, then ϕ is a transversally constant and if $|d_T\phi| \ne 0$, then $|d_T\phi|^2 = \frac{\lambda C}{\mu(C-1)}$. Particularly, if $|d_T\phi|^2 \le \frac{\lambda}{\mu}\left(< \frac{\lambda C}{\mu(C-1)}\right)$, then ϕ is transversally constant. □

Remark. For the point foliation, Theorem 5.4 and Theorem 5.5 are found in [2] and [12], respectively.

Example. Let T^2 be the flat 2−torus paramerized by the angles (u,v) with $0 \le u,v < 2\pi$. Let $\bar\phi : T^2 \to S^3$ be defined by

$$\bar\phi(u,v) = (cosu, sinu, cosv, sinv)/\sqrt{2},$$

considered as a point in $\mathbb{R}^4$. Then $\bar\phi$ is harmonic but not totally geodesic[2]. Now let (F,h) and (F',h') be Riemannian manifolds. Consider the foliations on $T^2\times F$ and $S^3\times F'$ given by the projections on the first component $\pi_1 : T^2\times F \to T^2, \pi_2 : S^3\times F' \to S^3$, respectively. Then the projections $\pi_i (i = 1,2)$ are Riemannian fibrations, and so the foliations are Riemannian. Let $\phi : T^2\times F \to S^3\times F'$ be a foliated smooth map, which is given by

$$\phi((u,v),x) = (\bar{\phi}(u,v), f(u,v,x))$$

for any $x \in F$, where $f : T^2 \times F \to F'$ is smooth. Then ϕ is transversally harmonic because $\bar{\phi}$ is harmonic. But ϕ is not totally geodesic because $\bar{\phi}$ is not totally geodesic[6].

References

[1] J. A. Alvarez López, *The basic component of the mean curvature of Riemannian foliations*, Ann. Global Anal. Geom. 10(1992), 179-194.

[2] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86(1964), 106-160.

[3] S. D. Jung, *The first eigenvalue of the transversal Dirac operator*, J. Geom. Phys. 39(2001), 253-264.

[4] S. D. Jung, K. R. Lee and K. Richardson, *Generalized Obata theorem and its applications on foliations*, J. Math. Anal. Appl. 376(2011), 129-135.

[5] F. W. Kamber and Ph. Tondeur, *Infinitesimal automorphisms and second variation of the energy for harmonic foliations*, Tôhoku Math. J. 34(1982), 525-538.

[6] J. Konderak and R. Wolak, *On transversally harmonic maps between manifolds with Riemannian foliations*, Quart. J. Math. Oxford Ser.(2) 54(2003), 335-354.

[7] J. Konderak and R. Wolak, *Some remarks on transversally harmonic maps*, Glasgow Math. J. 50(2008), 1-16.

[8] P. Molino, *Riemannian foliations*, translated from the French by Grant Cairns, Boston: Birkhäser, 1988.

[9] B. O'Neill, *The fundamental equations of a submersion*, Mich. Math. J. 13(1966), 459-469.

[10] H. K. Pak and J. H. Park, *Transversal harmonic transformations for Riemannian foliations*, Ann. Global Anal. Geom. 30(2006), 97-105.

[11] E. Park and K. Richardson, *The basic Laplacian of a Riemannian foliation*, Amer. J. Math. 118(1996), 1249-1275.

[12] H. C. J. Sealey, *Harmonic maps of small energy*, Bull. London Math. Soc. V13(5)(1981), 405-408.

[13] Ph. Tondeur, *Foliations on Riemannian manifolds*, New-York: Springer-Verlag, 1988.

[14] Ph. Tondeur, *Geometry of foliations*, Basel: Birkhäuser Verlag, 1997.

[15] Y. L. Xin, *Geometry of harmonic maps*, Birkhäuser, Boston, 1996.

[16] S. Yorozu and T. Tanemura, *Green's theorem on a foliated Riemannian manifold and its applications*, Acta Math. Hungar. 56(1990), 239-245.

[17] Vladimir Y. Rovenskii, *Foliations on Riemannian Manifolds and Submanifolds*, Birkhäuser, Boston, 1998.

<국문 초록>

엽층적 리만다양체상에서의 횡단선의 조화사상

엽층적 리만다양체 $(M,\mathcal{F})$와 $(M',\mathcal{F}')$에 대하여 M이 컴팩트인 경우에 대하여, 우선 횡단하는 힘에 대한 첫 번째 변분공식에 대하여 연구하였다. 엽층구조 $\mathcal{F}$의 횡단하는 Ricci 곡률이 음(-)이 아니고 $\mathcal{F}$‘의 횡단하는 절단선의 곡률이 양(+)이 아닐 때, 횡단적으로 조화사상 $\phi : (M,\mathcal{F}) \rightarrow (M',\mathcal{F}')$는 횡단적으로 완전히 측지선이 된다. 특히, 횡단하는 Ricci 곡률이 양(+)인 점이 존재할 때, ϕ는 횡단적으로 상수가 된다.

감사의 글

아무도 그에게 수심을 일러 준 일이 없기에
흰 나비는 도무지 바다가 무섭지 않다.

청무우 밭인가 해서 내려갔다가는
어린 날개가 물결에 절어서
공주처럼 지쳐서 돌아온다.

- 바다와 나비(작가 김기림) 중에서

논문심사를 마치고 얼마 지나지 않아 갑자기 떠오른 시 '바다와 나비'를 인용해 보았습니다. 저는 저 흰 나비와 같이 아무것도 모른 채 박사과정을 시작한 것 같다는 생각이 들었습니다. 석사 때는 느껴보지 못했던 책임감까지 더해지니 정말 힘들어서 포기하고 싶은 순간도 더러 있었습니다. 하지만 세상만사 지나가면 별 일 아니었다고, 이제와 생각해보니 무엇이 그리 힘들었나 싶습니다. 그래도 저 나비와 같은 무모함이 있었기 때문에 제가 지금 이 글을 쓸 수 있는 게 아닌가 싶네요. ^^

물결에 절은 날개를 말리는데 도움을 주신 많은 분들이 떠오릅니다.
우선, 말리는 일 뿐만 아니라 다시 날 수 있게 큰 용기와 지식을 주신 우리 정승달 교수님! 정말 감사합니다! 예전이나 지금이나 교수님이 안계셨으면 지금의 저는 없는 거나 마찬가지이지요. "무엇을 모르고 있는지 아는 것이 제일 중요하다!"라는 말씀 항상 잊지 않고 어떤 일을 하던지 항상 본질을 먼저 생각하겠습니다. 그리고 조금 더 멀리 날 수 있게 도와주신 제주대 수학과 양영오 교수님, 송석준 교수님, 방은숙 교수님, 윤용식 교수님, 유상욱 교수님 그리고 진현성 교수님 감사합니다. 제 논문 심사를 위해 이 곳 제주까지 오신 울산대 강태호 교수님, 경희대 김병학 교수님 그리고 제주대 초등교육과 최근배 교수님 감사합니다.

대학원에 들어와서 동고동락한 금란, 항상 대견스러워 하시는 고연순 선생님, 그리고 우리 미분기하학팀!!! 논문 열심히 쓰라고 힘을 준 친구들(선영, 안나, 소현, 영심, 희정, 은이, 엠티빙자, 가뭄에단비, compact set, 새빛 등) 모두 고마움을 느낍니다. 힘들다고 칭얼대면 힘내서 열심히 하라고 격려해주셔서 많은 위로가 되었습니다. 졸업하기까지 시간이 오래 걸려서 그런지 고마운 사람이 많네요. ^^

마지막으로 존재만으로도 큰 힘이 되는 우리 가족들에게도 고마움을 전합니다.

2012년 6월