FUNCTION SPACES IN SEQUENTIAL CONVERGENCE SPACES OVER VARIABLE BASE SPACES

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1. Introduction

Since I. M. James has been promoting the fibrewise viewpoint systematically in topology [5-7], fibrewise topology has been emerged as a subject in its own right. As a matter of fact in many directions interests in research on fibrewise theory are growing now. Many of the familiar definitions and theorems of ordinary topology can be generalized, in a natural way, so that one can develop a theory of topology over a base space. On the other hand, the theory of fibration have been developed in the situation of having variable base spaces. In particular, given fibrations $p : X \to A$ and $q : Y \to B$, the construction and properties of a function space $C_{AB}(X,Y)$ and an associated fibration $p \cdot q : C_{AB}(X,Y) \to A \times B$ are mainly concerned. In this case, fibrewise exponential laws play crucial role.

And sequential language is useful as an alternative in first countable spaces, so there seems to be a reason for direct study of sequential convergence. From these points of view, it is considered to be meaningful to study the exponential laws in sequential convergence spaces over variable base spaces.

In this paper, we introduce sequential convergence spaces over variable base spaces and construct a function space structure which will allow us some exponential laws.

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2. Function spaces in sequential convergence spaces over variable base spaces

For any set X, let $X^{\mathbf{N}}$ be the set of all sequences on X. A *sequential convergence space* is an ordered pair (X, ξ) of sets, where $\xi \subseteq X^{\mathbf{N}} \times X$ is a specified relation between sequences $(u_n) \in X^{\mathbf{N}}$ and points $x \in X$ subject to the following three axioms:

(1) If $u_n = x$ for all n, then $((u_n), x) \in \xi$.
(2) If $((u_n), x) \in \xi$, then for any subsequence $(u_{s(n)})$ of (u_n), $(u_{s(n)}, x) \in \xi$.
(3) If $(u_n) \in X^{\mathbf{N}}$ is such that every subsequence $(u_{s(n)})$ has a further subsequence $(u_{ts(n)})$ with $(u_{ts(n)}, x) \in \xi$, then $((u_n), x) \in \xi$.

In what follows we will express the statement $((u_n), x) \in \xi$ by writing (u_n) converges to x in (X, ξ).

Let (X, ξ) and (Y, η) be sequential convergence spaces and $f : X \to Y$ be a map. Then f is called a ***sequentially continuous map*** if $(f(u_n))$ converges to $f(x)$ in (Y, η) whenever (u_n) converges to x in (X, ξ).

The class of all sequential convergence spaces and sequentially continuous maps form a category which is denoted by **Seq**. Then the followings are well known results.

Proposition 2.1. **Seq** *has initial structures over* **Set**.

Proposition 2.2. **Seq** *has final structures over* **Set**.

Let $p : X \to A$ be a sequentially continuous map. In this case, we say that X is a *sequential convergence space over* A and p is a *projection.* Let $p : X \to A$ and $q : Y \to B$ be sequentially continuous maps where X, Y, A and B are sequential convergence spaces.

Let

$$C_{AB}(X,Y) = \bigcup_{a \in A, b \in B} C(X_a, Y_b)$$

as a set, where $C(X_a, Y_b)$ is the set of all sequentially continuous maps from X_a to Y_b. On $C_{AB}(X,Y)$, define $((f_n), f) \in \xi$, where $\xi \subseteq C_{AB}(X,Y)^{\mathbb{N}} \times C_{AB}(X,Y)$ and $f \in C(X_a, Y_b)$, if

(1) for any subsequence $(f_{s(n)})$ of (f_n) and any sequence (x_n) in X which converges to $x \in X_a$, the sequence $(\mathbf{f}_{s(n)}(x_n))$ which is defined by

$$\mathbf{f}_{s(n)}(x_n) = \begin{cases} f_{s(n)}(x_n) & \text{if it is defined} \\ f(x) & \text{otherwise} \end{cases}$$

converges to $f(x)$ in Y and

(2) the sequence $((p\cdot q)(f_n))$ converges to $(p\cdot q)(f)$, where $p\cdot q : C_{AB}(X,Y) \to A \times B$ is the map defined by $(p \cdot q)(g) = (a, b)$ for $g \in C(X_a, Y_b)$.

From now on, the sequence $(\mathbf{f}_{s(n)}(x_n))$ is called the sequence induced by $(f_{s(n)})$ and (x_n).

Proposition 2.3. *$(C_{AB}(X,Y), \xi)$ is a sequential convergence space over $A \times B$.*

Proof. Let $f_n = f$ for all $n \in \mathbb{N}$, where $f \in C(X_a, Y_b)$. Then if (x_n) converges to $x \in X_a$, for any subsequence $(f_{s(n)})$ of (f_n), the sequence $(\mathbf{f}_{s(n)}(x_n))$ induced by $(f_{s(n)})$ and (x_n) which is defined by

$$\mathbf{f}_{s(n)}(x_n) = \begin{cases} f_{s(n)}(x_n) & \text{if it is defined} \\ f(x) & \text{otherwise} \end{cases}$$

is the image of some mixed sequence of some subsequence of (x_n) and a constant sequence (x) under f. Hence $(\mathbf{f}_{s(n)}(x_n))$ converges to $f(x)$ in Y, since f is sequentially continuous, . Moreover, $(p \cdot q)(f_n) = (a, b)$ for all $n \in \mathbb{N}$ and hence $((p \cdot q)(f_n))$ is a constant sequence in $A \times B$. So $((p \cdot q)(f_n))$ converges to $(p \cdot q)(f)$. Therefore $((f_n), f) \in \xi$.

Suppose (f_n) converges to f in $C_{AB}(X,Y)$ and $f \in C(X_a, Y_b)$. Let $(f_{s(n)})$ be a subsequence of (f_n). We have to show that for any subsequence $(f_{ts(n)})$ of $(f_{s(n)})$ and a sequence (x_n) which converges to $x \in X_a$ in X, the sequence induced by $(f_{ts(n)})$ and (x_n) converges to $f(x)$. Note that $(f_{ts(n)})$ is also a subsequence of (f_n). So, since (f_n) converges to f in

$C_{AB}(X,Y)$, the sequence induced by $(f_{ts(n)})$ and (x_n) converges to $f(x)$. And, since $((p\cdot q)(f_{s(n)}))$ is a subsequence of $((p\cdot q)(f_n))$, $((p\cdot q)(f_{s(n)}))$ converges to $(p\cdot q)(f)$. Therefore, $((f_{s(n)}), f) \in \xi$.

Let (f_n) be a sequence in $C_{AB}(X,Y)$ such that any subsequence of (f_n) contains a further subsequence which converges to $f \in C(X_a, Y_b)$. We have to show that for any subsequence $(f_{s(n)})$ of (f_n) and a sequence (x_n) which converges to $x \in X_a$ in X, the sequence $(\mathbf{f}_{s(n)}(x_n))$ induced by $(f_{s(n)})$ and (x_n) converges to $f(x)$ in Y. Since Y is a sequential convergence space, it is enough to show that each subsequence of $(\mathbf{f}_{s(n)}(x_n))$ contains a further subsequence which converges to $f(x)$. Let $(\mathbf{f}_{ts(n)}(x_{t(n)}))$ be a subsequence of $(\mathbf{f}_{s(n)}(x_n))$. Note that this is a sequence induced by $(f_{ts(n)})$ and $(x_{t(n)})$. Since $(f_{ts(n)})$ is a subsequence of (f_n), $(f_{ts(n)})$ has a further subsequence $(f_{wts(n)})$ which converges to f. Hence the sequence $(\mathbf{f}_{wts(n)}(x_{wt(n)}))$ induced by $(f_{wts(n)})$ and $(x_{wt(n)})$ converges to $f(x)$ in Y. But this sequence is a subsequence of $(\mathbf{f}_{ts(n)}(x_{t(n)}))$, i.e., $(\mathbf{f}_{ts(n)}(x_{t(n)}))$ contains a further subsequence which converges to $f(x)$. So (f_n) converges to f. By the similar argument, $((p\cdot q)(f_n))$ converges to $(p\cdot q)(f)$, since $A\times B$ is a sequential convergence space. Therefore, $((f_n), f) \in \xi$.

Note that $p\cdot q$ is sequentially continuous by definition. In all, $(C_{AB}(X, Y), \xi)$ is a sequential convergence space over $A\times B$.

If $A = B = \{*\}$, then this structure is equal to the function space structure in sequential convergence spaces defined in [9]. And, the subspace structure on $C(X_a, Y_b)$ with respect to $C_{AB}(X,Y)$ is the sequential convergence structure defined in [9].

3. Exponential laws

In this section, we introduce some exponential laws in sequential convergence spaces over variable base spaces.

The function $p\cdot q : C_{AB}(X,Y) \to A\times B$ and the function $p\cdot_1 q : C_{AB}(X,Y) \to A$, where $p\cdot q = pr_1 \circ (p\cdot q)$, are sequentially continuous.

So we can take $C_{AB}(X,Y)$ to be either an object over $A \times B$ or an object over A.

Proposition 3.1. *The evaluation map* $ev : X \times_A C_{AB}(X,Y) \to Y$ *defined by* $ev(x,f) = f(x)$ *is sequentially continuous.*

Proof. Let (x_n, f_n) be a sequence in $X \times_A C_{AB}(X,Y)$ which converges to (x,f), where $x \in X_a$ and $f \in C(X_a, Y_b)$. Then (x_n) converges to x in X and (f_n) converges to f in $C_{AB}(X,Y)$. So, the sequence $(\mathbf{f}_n(x_n))$ induced by (f_n) and (x_n) converges to $f(x)$ in Y. But, since $f_n : X_a \to Y_b$ and $x_n \in X_a$ for all $n \in \mathbb{N}$, this sequence is equal to $(ev(x_n, f_n))$. Hence $(ev(x_n, f_n))$ converges to $f(x) = ev(x,f)$ in Y. Therefore, ev is sequentially continuous.

Theorem 3.2. *Let* $p : X \to A, q : Y \to B$ *and* $r : Z \to D$ *be sequentially continuous maps. Then the map*

$$\phi : C_{ABD}(X \times Y, Z) \to C_{ABD}(X, C_{BD}(Y,Z))$$

which is defined by $\phi(f)(x)(y) = f(x,y)$ *is an isomorphism, where* $f : X_a \times Y_b \to Z_d, x \in X_a$ *and* $y \in Y_b$.

Proof. For $f : X_a \times Y_b \to Z_d$, $\phi(f)$ is a function from X_a to $C(Y_b, Z_d)$. Thus ϕ is well defined and it is easy to show that ϕ is a bijection. First, we want to show that ϕ is continuous. Let (f_n) converges to f in $C_{ABD}(X \times Y, Z)$. We need to show that $(\phi(f_n))$ converges to $\phi(f)$ in $C_{ABD}(X, C_{BD}(Y,Z))$, i.e., for any subsequence $(\phi(f_{s(n)}))$ of $(\phi(f_n))$ and a sequence (x_n) in X which converges to $x \in X_a$, the sequence $(\phi(\mathbf{f}_{s(n)})(x_n))$ induced by $(\phi(f_{s(n)}))$ and (x_n) which is defined by

$$\phi(\mathbf{f}_{s(n)})(x_n) = \begin{cases} \phi(f_{s(n)})(x_n) & \text{if it is defined} \\ \phi(f)(x) & \text{otherwise} \end{cases}$$

converges to $\phi(f)(x)$ in $C_{BD}(Y,Z)$. Hence we have to show that for any subsequence $(\phi(\mathbf{f}_{ts(n)})(x_{t(n)}))$ of the sequence $(\phi(\mathbf{f}_{s(n)})(x_n))$ and a sequence (y_n) in Y which converges to $y \in Y_b$, the sequence induced by $(\phi(\mathbf{f}_{ts(n)})(x_{t(n)}))$ and (y_n) converges to $\phi(f)(x)(y)$ in Z. This sequence is given by

$$\phi(\mathbf{f}_{ts(n)})(x_{t(n)})(y_n) = \begin{cases} \phi(f_{ts(n)})(x_{t(n)})(y_n) & \text{if it is defined} \\ \phi(f)(x)(y) & \text{otherwise} \end{cases}$$

Note that $(x_{t(n)}, y_n)$ converges to $(x, y) \in X_a \times Y_b$. Hence, since (f_n) converges to f in $C_{ABD}(X \times Y, Z)$, the sequence induced by a subsequence $(f_{ts(n)})$ of (f_n) and $(x_{t(n)}, y_n)$ converges to $f(x,y)$. That is, the sequence

$$\mathbf{f}_{ts(n)}(x_{t(n)}, y_n) = \begin{cases} f_{ts(n)}(x_{t(n)}, y_n) & \text{if it is defined} \\ f(x,y) & \text{otherwise} \end{cases}$$

converges to $f(x,y)$. But, this sequence is the same as above induced sequence. Hence the sequence induced by $(\phi(\mathbf{f}_{ts(n)})(x_{t(n)}))$ and (y_n) converges to $f(x,y) = \phi(f)(x)(y)$ in Z. Therefore, ϕ is sequentially continuous. It is easy to show that $(((p \cdot q) \cdot r)(\phi(f_n)))$ converges to $(((p \cdot q) \cdot r)(\phi(f)))$.

For the converse, let φ be the inverse of ϕ. Then $\varphi(f)(x,y) = f(x)(y)$ for $f : X_a \to C(Y_b, Z_d)$ and $(x,y) \in X_a \times Y_b$. Note that φ is well defined. Suppose (f_n) converges to f in $C_{ABD}(X, C_{BD}(Y, Z))$. We need to show that $(\varphi(f_n))$ converges to $\varphi(f)$ in $C_{ABD}(X \times Y, Z)$. Let $(\varphi(f_{s(n)}))$ be a subsequence of $(\varphi(f_n))$ and (x_n, y_n) be a sequence in $X \times Y$ which converges to $(x,y) \in X_a \times Y_b$. Note that (x_n) converges to $x \in X_a$ and (y_n) converges to $y \in Y_b$. Since (f_n) converges to f in $C_{ABD}(X, C_{BD}(Y, Z))$, the sequence $(\mathbf{f}_{s(n)}(x_n))$ induced by $(f_{s(n)})$ and (x_n) converges to $f(x)$ in $C_{BD}(Y, Z)$. And hence, $(\mathbf{f}_{s(n)}(x_n)(y_n))$ converges to $f(x)(y)$ in Z. This sequence is given by

$$\mathbf{f}_{s(n)}(x_n)(y_n) = \begin{cases} f_{s(n)}(x_n)(y_n) & \text{if it is defined} \\ f(x)(y) & \text{otherwise} \end{cases}$$

But, this sequence is the same as the sequence induced by $(\varphi(f_{s(n)}))$ and (x_n, y_n). So the sequence induced by $(\varphi(f_{s(n)}))$ and (x_n, y_n) converges to $f(x)(y) = \varphi(f)(x,y)$ in Z. Hence φ is sequentially continuous. Moreover, the fact that $(((p \times q) \cdot r)(\varphi(f_n)))$ converges to $(((p \times q) \cdot r)(\varphi(f)))$ is easily proved.

In all, ϕ is an isomorphism.

For given $p : X \to B$ and $q : Y \to B$, let

$$C_B(X,Y) = \bigcup_{b \in B} C(X_b, Y_b)$$

as a set, where $C(X_b, Y_b)$ is the set of all sequentially continuous maps from X_b to Y_b. Define $((f_n), f) \in \xi$, where $\xi \subseteq C_B(X,Y)^{\mathbb{N}} \times C_B(X,Y)$ and $f \in C(X_b, Y_b)$, if

(1) for any subsequence $(f_{s(n)})$ of (f_n) and any sequence (x_n) in X which converges to $x \in X_b$, the sequence

$$\mathbf{f}_{s(n)}(x_n) = \begin{cases} f_{s(n)}(x_n) & \text{if it is defined} \\ f(x) & \text{otherwise} \end{cases}$$

converges to $f(x)$ in Y,

(2) the sequence $((p \cdot q)(f_n))$ converges to $(p \cdot q)(f)$, where $p \cdot q : C_B(X,Y) \to B$ is the map defined by $(p \cdot q)(g) = b$ for $g \in C(X_b, Y_b)$.

Then it can be proved that $(C_B(X,Y), \xi)$ is a sequential convergence space and we can get the following isomorphism.

Corollary 3.3. *Let $p : X \to B, q : Y \to B$ and $r : Z \to B$ be sequentially continuous maps. Then there is an isomorphism*

$$\overline{\phi} : C_B(X \times_B Y, Z) \to C_B(X, C_B(Y,Z))$$

Proof. It is enough to consider the following commutative diagram

$$\begin{array}{ccc} C_B(X \times_B Y, Z) & \xrightarrow{\overline{\phi}} & C_B(X, C_B(Y,Z)) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ C_{BBB}(X \times Y, Z) & \xrightarrow[\phi]{} & C_{BBB}(X, C_{BB}(Y,Z)) \end{array}$$

where $\overline{\phi}$ is the restriction and corestriction of ϕ.

Now, we will consider another type of exponential law. Let $p : X \to A$ and $q : Y \to B$ be sequentially continuous maps. A ***fibre preserving map*** from X to Y is a pair of sequentially continuous maps $g : X \to Y$ and $h : A \to B$ such that $q \circ g = h \circ p$, i.e., the following diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[h]{} & B \end{array}$$

commutes. We write this map by $(g, h) : p \to q$. For the case of $A = B$, the sequentially continuous map $f : X \to Y$ such that $q \circ f = p$ is called a ***sequentially continuous map over*** B.

Let $p : X \to B, q : Y \to B$ and $r : Z \to D$ be sequentially continuous maps. Let $M_{BD}(Y, Z) = \{(g, h) | (g, h) : q \to r\}$. We consider $M_{BD}(Y, Z)$ as a subspace of $C(Y, Z) \times C(B, D)$. And let $M_B(X, Y) = \{f : X \to Y | f$ is sequentially continuous map over $D\}$. We give $M_B(X, Y)$ the subspace structure of $C(X, Y)$. $M_B(X, Y)$ can be considered as a subspace of $M_{BB}(X, Y)$. In fact, $M_B(X, Y)$ is isomorphic to a subspace of $M_{BB}(X, Y)$ in which f_0 is fixed as 1_B.

Consider $M_{XD}(X \times_B Y, Z)$ and $M_B(X, C_{BD}(Y, Z))$. In this case, $X \times_B Y$ is considered as a space over X with natural projection and $C_{BD}(Y, Z)$ as a space over B with projection $pr_1 \circ (q \cdot r)$. Define a function $\psi : M_{XD}(X \times_B Y, Z) \to M_B(X, C_{BD}(Y, Z))$ as follows. For $(g, h) \in M_{XD}(X \times_B Y, Z)$, the rule $\psi(g, h)(x)(y) = g(x, y)$ defines $\psi(g, h)(x)$ as a function from Y_b to Z_d where $p(x) = q(y) = b, f_0(x) = d$. For such $x \in X_b$, $\psi(g, h)(x)$ is the composite morphism $Y_b \cong \{x\} \times Y_b \xrightarrow{j} X_b \times Y_b \xrightarrow{g_b} Z_d$, where $(f_1)_b$ is the appropriate restriction and corestriction of f. Thus $\psi(g, h)$ is a function, from X to $C_{BD}(Y, Z)$, that is clearly over B.

Proposition 3.4. *The map*

$$\psi : M_{XD}(X \times_B Y, Z) \to M_B(X, C_{BD}(Y, Z))$$

is sequentially continuous.

Proof. Let (f_n, g_n) converge to (f, g) in $M_{XD}(X \times_B Y, Z)$. Then (f_n) converges to f in $C(X \times_B Y, Z)$. We want to show that $(\psi(f_n, g_n))$ converges to $\psi(f, g)$ in $M_B(X, C_{BD}(Y, Z))$. Let (x_n) converge to $x \in X_b$ and (y_n) converges to $y \in Y_b$. We have to show that for any subsequence $(\psi(f_{s(n)}, g_{s(n)}))$ of $(\psi(f_n, g_n))$, the sequence $(\psi(\mathbf{f}_{s(n)}, \mathbf{g}_{s(n)})\ (x_n))$ induced by $(\psi(f_{s(n)}, g_{s(n)}))$ and (x_n) converges to $\psi(f, g)(x)$ in $C_{BD}(Y, Z)$. So we have to show that for any subsequence $(\psi(\mathbf{f}_{ts(n)}, \mathbf{g}_{ts(n)})(x_{t(n)}))$, the sequence $(\psi(\mathbf{f}_{ts(n)}, \mathbf{g}_{ts(n)})(x_{t(n)})(y_n))$ induced by $(\psi(\mathbf{f}_{ts(n)}, \mathbf{g}_{ts(n)})(x_{t(n)}))$ and (y_n) converges to $\psi(f, g)(x)(y)$. Note that this sequence is given by

$$\psi(\mathbf{f}_{ts(n)}, \mathbf{g}_{ts(n)})(x_{t(n)})(y_n) = \begin{cases} \psi(f_{ts(n)}, g_{ts(n)})(x_{t(n)})(y_n) \\ \text{if it is defined} \\ \psi(f, g)(x)(y) \quad \text{otherwise} \end{cases}$$

Consider the sequence $(x'_{t(n)})$ defined by $x'_{t(n)} = x_{t(n)}$ if $x_{t(n)} \in X_b$ and $x'_{t(n)} = x$ if $x_{t(n)} \notin X_b$ and similarly for (y'_n). Since (f_n) converges to f in $C(X \times_B Y, Z)$, $(x'_{t(n)})$ converges to $x \in X_b$ and (y'_n) converges to $y \in Y_b$, $(f_{ts(n)}(x'_{t(n)}, y'_n))$ converges to $f(x, y)$ in Z. But this sequence is the same as the above sequence. Hence $(\psi(\mathbf{f}_{ts(n)}, \mathbf{g}_{ts(n)})(x_{t(n)})(y_n))$ converges to $f(x, y) = \psi(f, g)(x)(y)$. Therefore, ψ is sequentially continuous.

Next consider the function $\varphi : M_B(X, C_{BD}(Y, Z)) \to C(X \times_B Y, Z) \times C(X, D)$ defined by $\varphi(f) = (g, h)$ for $f \in M_B(X, C_{BD}(Y, Z))$, where $g(x, y) = f(x)(y)$ and $h(x) = (pr_2 \circ (p \cdot r))(f(x))$.

Proposition 3.5. *The map*

$$\varphi : M_B(X, C_{BD}(Y, Z)) \to C(X \times_B Y, Z) \times C(X, D)$$

is sequentially continuous.

Proof. Let (f_n) converge to f in $M_B(X, C_{BD}(Y,Z))$. Let $\varphi(f_n) = (g_n, h_n)$ and $\varphi(f) = (g,h)$. We want to show that (g_n) converges to g in $C(X \times_B Y, Z)$ and (h_n) converges to h in $C(X,D)$. Let (x_n, y_n) converges to (x,y) in $X \times_B Y$. Note that $g_n(x,y) = f_n(x)(y)$ and $h_n(x) = (pr_2 \circ (p \cdot r))(f_n(x))$. First, we have to show that for any subsequence $(g_{s(n)})$ of (g_n), $(g_{s(n)}(x_n, y_n))$ converges to $g(x,y)$ in Z. Since (f_n) converges to f in $M_B(X, C_{BD}(Y,Z))$ and (x_n) converges to $x \in X_b$, $(f_{s(n)}(x_n))$ converges to $f(x)$ in $C_{BD}(Y,Z)$. So, since (y_n) converges to $y \in Y_b$, $(f_{s(n)}(x_n)(y_n))$ converges to $f(x)(y) = g(x,y)$ in Z. But, this sequence is the same as $g_{s(n)}(x_{s(n)}, y_n)$. Hence (g_n) converges to g in $C(X \times_B Y, Z)$.

And, since (f_n) converges to f in $M_B(X, C_{BD}(Y,Z))$ and $pr_2 \circ (q \cdot r)$ is sequentially continuous, (h_n) converges to h in $C(X,D)$.

In all, φ is sequentially continuous.

Using the above propositions, we can prove the following theorem.

Theorem 3.6. *Let $p : X \to B$, $q : Y \to B$ and $r : Z \to D$ be sequentially continuous maps. Then*

$$\psi : M_{XD}(X \times_B Y, Z) \to M_B(X, C_{BD}(Y,Z))$$

which is defined by $\psi(f_1, f_0)(x)(y) = f_1(x,y)$ is an isomorphism.

Proof. Note that the image of φ is contained in $M_{XD}(X \times_B Y, Z)$, i.e., $\varphi : M_B(X, C_{BD}(Y,Z)) \to M_{XD}(X \times_B Y, Z)$ is well defined. This function is the inverse of ψ. Hence the result follows.

The following is a special case of the above theorem.

Corollary 3.7. *Let $p : X \to B$, $q : Y \to B$ and $r : Z \to B$ be sequentially continuous maps. Then there is an isomorphism*

$$\overline{\psi} : M_B(X \times_B Y, Z) \to M_B(X, C_B(Y,Z))$$

Proof. It is enough to consider the following commutative diagram

$$\begin{array}{ccc} M_B(X \times_B Y, Z) & \xrightarrow{\overline{\psi}} & M_B(X, C_B(Y, Z)) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ M_{XB}(X \times_B Y, Z) & \xrightarrow[\psi]{} & M_B(X, C_{BB}(Y, Z)) \end{array}$$

where $\overline{\psi}$ is the restriction and corestriction of ψ. In this case, we consider $M_B(X \times_B Y, Z)$ as a subspace of $M_{XB}(X \times_B Y, Z)$ in which f_0 is fixed as p.

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