碩士學位論文

COLUMN RANK PRESERVERS BETWEEN DIFFERENT BINARY BOOLEAN MATRIX SPACES.

濟州大學校　大學院

數　學　科

吳　奈　映

2013年　12月

COLUMN RANK PRESERVERS BETWEEN DIFFERENT BINARY BOOLEAN MATRIX SPACES.

指導敎授 宋 錫 準

吳 奈 映

이 論文을 理學 碩士學位 論文으로 提出함

2013年 12月

吳奈映의 理學 碩士學位 論文을 認准함

審査委員長 ________________ 印

委 員 ________________ 印

委 員 ________________ 印

濟州大學校 大學院

2013年 12月

COLUMN RANK PRESERVERS BETWEEN DIFFERENT BINARY BOOLEAN MATRIX SPACES.

Na Yeong Oh
(Supervised by professor Seok Zun Song)

A thesis submitted in partial fulfillment of the requirement for the degree of Master of Science

2013. 12.

Department of Mathematics
GRADUATE SCHOOL
JEJU NATIONAL UNIVERSITY

CONTENTS

<Abstract>

COLUMN RANK PRESERVERS BETWEEN DIFFERENT BINARY BOOLEAN MATRIX SPACES.

Matrix theory on the semirings has been developed by many linear algebraists containing Beasley and Pullman since 1981. Moreover there are many papers on linear operators on a matrix space that preserve matrix functions over various algebraic structures. But there are few papers of linear transformations from one matrix space into another matrix space that preserve matrix functions over an algebraic structure.

Let $\mathbb{F}$ be a field and $\mathbb{M}_{m,n}(\mathbb{F})$ denote the vector space of all $m \times n$ matrices over $\mathbb{F}$. Over the last century, a great deal of effort has been devoted to the following problem.:

Characterize those linear operators $T : \mathbb{M}_{m,n}(\mathbb{F}) \to \mathbb{M}_{m,n}(\mathbb{F})$ which leave a function or set invariant. We call this a Linear Preserver Problem.

The study of these operators began in 1897 when Frobenius characterized the linear operators that preserve the determinant over complex matrices and over real symmetric matrices.

In this thesis we consider linear transformations from $m \times n$ Boolean matrices into $p \times q$ Boolean matrices that preserve column rank. We characterize linear transformations that preserve column rank between different Boolean matrix spaces. This results extend the results on the linear operators from $m \times n$ Boolean matrices into itself that preserve column rank. The main theorem is the following:

Theorem: Let $1 < k < l \leq m \leq n$ and $k + 1 < m$. Assume $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation that preserves column rank k and column rank l, or if T strongly preserves column rank k, then T has the form $T(A) = P[A \oplus O]Q$ with permutation matrices P and Q of orders p and q, respectively. And the converse holds.

1 Introduction

Matrix theory on the semirings has been developed by many linear algebraists containing Beasley and Pullman since 1981 ([9]). Moreover there are many papers on linear operators on a matrix space that preserve matrix functions over various algebraic structures ([9]). But there are few papers of linear transformations from one matrix space into another matrix space that preserve matrix functions over an algebraic structure([12]).

Let $\mathbb{F}$ be a field and $\mathbb{M}_{m,n}(\mathbb{F})$ denote the vector space of all $m \times n$ matrices over $\mathbb{F}$. Over the last century, a great deal of effort has been devoted to the following problem. Characterize those linear operators $T : \mathbb{M}_{m,n}(\mathbb{F}) \to \mathbb{M}_{m,n}(\mathbb{F})$ which leave a function or set invariant. We call this a Linear Preserver Problem ([8], [9]). The most typical and oldest type of linear preserver problem is as follows :

Let f be a function on $\mathbb{M}_{m,n}(\mathbb{F})$. Characterize those T on $\mathbb{M}_{m,n}(\mathbb{F})$ such that $f(T(A)) = f(A)$ for all $A \in \mathbb{M}_{m,n}(\mathbb{F})$.

The study of these operators began in 1897 ([9]) when Frobenius characterized the linear operators that preserve the determinant over complex matrices and over real symmetric matrices. Frobenius proved that the linear operators that preserve the determinant consists of linear transformations of the form

$$T(A) = UAV \text{ or } T(A) = UA^tV$$

for all $A \in \mathbb{M}_{m,n}(\mathbb{C})$, where C is complex field and $det(UV) = 1$.

We now turn our attention to matrices over semirings, in particular Boolean algebra.

Boolean algebra is named after the British Mathematician George Boole (1813 - 1864). The Boolean algebra of two elements is most frequently used in combinatorial applications, and all other finite Boolean algebras are direct sums of copies of it ([7]).

Applications of the theory of Boolean matrices are of fundamental importance in the formation and analysis of many classes of discrete structural models which arise in the physical, biological, and social sciences. The theory is also intimately related to many branches of mathematics, including relation theory, logic, graph theory, lattice theory and algebraic semigroup theory ([5], [7]).

Boolean matrices have different properties from matrices over a field, due to the fact that addition in a Boolean algebra does not make it a group. Boolean matrices may arise from graphs or from nonegative real matrices by replaceing all positive entries by 1, but their most frequent occurence is in the representation of binary relation.

The study of the characterization of linear operators that preserve invariants of matrices over semirings is a counterpart for the study of preservers over fields, and it has its own importance.

In [2], Beasley and Pullman established analogous results over Boolean algebra to many preserver problems for matrices over field.

In this paper we consider linear transformations from $m \times n$ Boolean matrices into $p \times q$ Boolean matrices that preserve column rank. We study linear transformation that preserve column rank between different Boolean matrix spaces. This results extend the results on the linear operators from $m \times n$ Boolean matrices into itself that preserve column rank.

2 Preliminaries and Definitions

Definition 2.1. *[3] A semiring is a set* $\mathbb{S}$ *equipped with two binary operations* $+$ *and* $\cdot$ *such that* $(\mathbb{S}, +)$ *is a commutative monoid with identity element* 0 *and* $(\mathbb{S}, \cdot)$ *is a monoid with identity element* 1*. In addition, the operations* $+$ *and* $\cdot$ *are connected by distributivity of* $\cdot$ *over* $+$*, and* 0 *annihilates* $\mathbb{S}$*.*

Definition 2.2. *[3] A semiring* $\mathbb{S}$ *is called antinegative if* 0 *is the only element to have an additive inverse.*

The following are some examples of antinegative semirings which occur in combinatorics. Let $\mathbb{B} = \{0, 1\}$. Then $(\mathbb{B}, +, \cdot)$ is an antinegative semiring (the *binary Boolean semiring*) if arithmetic in $\mathbb{B}$ follows the usual rules except that $1 + 1 = 1$. If $\mathbb{P}$ is any subring of $\mathbb{R}$ with identity, the reals (under real addition and multiplication), and $\mathbb{P}^+$ denotes the nonnegative part of $\mathbb{P}$, then $\mathbb{P}^+$ is an antinegative semiring. In particular $\mathbb{Z}^+$, the nonnegative integers, is an antinegative semiring.

Hereafter, $\mathbb{S}$ will denote an arbitrary commutative and antinegative semiring.

Definition 2.3. *Let* $\mathbb{M}_{m,n}(\mathbb{S})$ *and* $\mathbb{M}_{p,q}(\mathbb{S})$ *be the set of all* $m \times n$ *and* $p \times q$ *matrices respectively with entries in a semiring* $\mathbb{S}$*. Algebraic operations on* $\mathbb{M}_{m,n}(\mathbb{S})$ *and* $\mathbb{M}_{p,q}(\mathbb{S})$ *are defined as if the underlying scalars were in a field.*

Definition 2.4. *[3] The column rank,* $c(A)$*, of* $A \in \mathbb{M}_{m,n}(\mathbb{B})$ *is the dimension of the column space of* A*.*

From now on we will assume that $2 \le m \le n$. It follows that $1 \le c(A) \le n$ for all nonzero $A \in \mathbb{M}_{m,n}(\mathbb{B})$.

Definition 2.5. *Let* $C_k^{(r,s)}$ *denote the set of all matrices in* $\mathbb{M}_{r,s}(\mathbb{B})$ *whose column rank is* k*.*

Let $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{p,q}(\mathbb{B})$ be a linear transformation. If f is a function defined on $\mathbb{M}_{m,n}(\mathbb{B})$ and on $\mathbb{M}_{p,q}(\mathbb{B})$, then T *preserves* the function f if $f(T(A)) = f(A)$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$.

If $\mathbb{X}$ is a subset of $\mathbb{M}_{m,n}(\mathbb{B})$ and $\mathbb{Y}$ is a subset of $\mathbb{M}_{p,q}(\mathbb{B})$, then T *preserves* the pair $(\mathbb{X}, \mathbb{Y})$ if $A \in \mathbb{X}$ implies $T(A) \in \mathbb{Y}$. T *strongly preserves* the pair $(\mathbb{X}, \mathbb{Y})$ if $A \in \mathbb{X}$ implies $T(A) \in \mathbb{Y}$. T *strongly preserves column rank* k if T strongly preserves the pair $(C_k^{(m,n)}, C_k^{(p,q)})$.

Song [11] has characterized linear operators on $\mathbb{M}_n(\mathbb{B})$ that preserve column rank as follows:

T is a column rank preserver if and only if T preserves column ranks 1, 2 and 3. (1.1)

T is a column rank preserver if and only if T has the form of $T(X) = PXQ$ where P, Q are permutation matrices. (1.2)

In this paper, we study linear transformations that preserve column rank between different matrix spaces.

Definition 2.6. *The matrix $A^{(m,n)}$ denotes a matrix in $\mathbb{M}_{m,n}(\mathbb{B})$, $O^{(m,n)}$ is the $m \times n$ zero matrix, I_n is the $n \times n$ identity matrix, $I_k^{(m,n)} = I_k \oplus O_{m-k,n-k}$, and $J^{(m,n)}$ is the $m \times n$ matrix all of whose entries are 1. Let $E_{i,j}^{(m,n)}$ be the $m \times n$ matrix whose (i,j)th entry is 1 and whose other entries are all 0, and we call $E_{i,j}^{(m,n)}$ a cell. An $m \times n$ matrix $L^{(m,n)}$ is called a full line matrix if*

$$L^{(m,n)} = \sum_{l=1}^{n} E_{i,l}^{(m,n)} \quad \text{or} \quad L^{(m,n)} = \sum_{k=1}^{m} E_{k,j}^{(m,n)}$$

for some $i \in \{1, \ldots, m\}$ or for some $j \in \{1, \ldots, n\}$; $R_i^{(m,n)} = \sum_{l=1}^{n} E_{i,l}^{(m,n)}$ is the ith full row matrix and $C_j^{(m,n)} = \sum_{k=1}^{m} E_{k,j}^{(m,n)}$ is the jth full column matrix. We will suppress the subscripts or superscripts on these matrices when the orders are evident from the context and we write A, O, I, I_k, J, $E_{i,j}$, L, R_i and C_j respectively.

The following is obvious by the definition of column rank of matrices over antinegative semirings.

Lemma 2.7. *For matrices A and B in $\mathbb{M}_{m,n}(\mathbb{B})$, we have*

$$c(A+B) \leq c(A) + c(B).$$

Example 2.8. *If A and B are matrices in $M_{3,3}(\mathbb{B})$,*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

we have

$$A+B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

but $c(A) > c(A+B)$.

Definition 2.9. *If A and B are matrices in $\mathbb{M}_{m,n}(\mathbb{S})$, we say that B dominates A (written $A \sqsubseteq B$ or $B \sqsupseteq A$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i and j. This provides a reflexive and transitive relation on $\mathbb{M}_{m,n}(\mathbb{S})$.*

Example 2.10. *If A and B are matrices in $M_{3,3}(\mathbb{B})$,*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then $A \sqsubseteq B$, but $c(B) \leq c(A)$.

Definition 2.11. *As usual, for any matrix A and lists L_1 and L_2 of row and column indices respectively, $A(L_1 \mid L_2)$ denotes the submatrix formed by omitting the rows L_1 and columns L_2 from A and $A[L_1 \mid L_2]$ denotes the submatrix formed by choosing the rows L_1 and columns L_2 from A.*

Definition 2.12. *[12] If $1 \leq m, n$ and $1 \leq p, q$ and $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation, then T is a (P, Q)-*block-transformation *if there are permutation matrices $P \in \mathbb{M}_p(\mathbb{B})$ and $Q \in \mathbb{M}_q(\mathbb{B})$ such that*

- *$m \leq p$ and $n \leq q$, and $T(A) = P[A \oplus O]Q$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$.*

3 A characterization of column rank preservers of Boolean matrices.

Definition 3.1. *[4] For a linear transformation $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{p,q}(\mathbb{B})$, we say that T*

(1) preserves column rank k if $c(T(X)) = k$ whenever $c(X) = k$ for all $X \in \mathbb{M}_{m,n}(\mathbb{B})$, or equivalently if T preserves the pair $(C_k^{(m,n)}, C_k^{(p,q)})$;

(2) strongly preserves column rank k if $c(T(X)) = k$ if and only if $c(X) = k$ for all $X \in \mathbb{M}_{m,n}(\mathbb{B})$, or equivalently if T strongly preserves the pair $(C_k^{(m,n)}, C_k^{(p,q)})$;

(3) preserves column rank if it preserves column rank k for every $k(\leq n)$.

In this section we provide characterizations of linear transformations $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{p,q}(\mathbb{B})$ that preserve column ranks k and l, where $1 \leq k < l \leq m \leq n$.

Example 3.2. *Let*

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

be a matrix in $\mathbb{M}_4(\mathbb{B})$. And $T(A) = A^t$. Then the columns of A are linearly independent. Hence $c(A) = 4$. But $c(A^t) = 3$ since the first three columns of A^t compose a basis of the column space of A^t. Therefore, T does not preserves column rank.

Theorem 3.3. *Let $1 \leq m, n$ and $1 \leq p, q$ and $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{p,q}(\mathbb{B})$ be a linear transformation. Then T strongly preserves column rank 1 if and only if T is a (P, Q)-block-transformation.*

Proof. If T is a (P, Q)-block transformation, then

$$c(T(A)) = c(P[A \oplus O]Q) = c(A \oplus O) = c(A).$$

Thus T strongly preserves column rank 1.

Assume that T strongly preserves column rank 1. Then, the image of each line in $\mathbb{M}_{m,n}(\mathbb{B})$ is a line in $\mathbb{M}_{p,q}(\mathbb{B})$. For since if not, T does not preserves column rank 1.

Case 1. $T(R_1^{(m,n)}) \sqsubseteq R_1^{(p,q)}$.
Suppose that $T(C_j^{(m,n)}) \sqsubseteq R_i^{(p,q)}$. Then, since $E_{1,j}^{(m,n)}$ is in both $R_1^{(m,n)}$ and $C_j^{(m,n)}$ and since $T(E_{1,j}^{(m,n)}) \neq O$, we must have $i = 1$. But then, for $j \neq k$

$$T(E_{2,j}^{(m,n)} + E_{1,k}^{(m,n)}) \sqsubseteq R_1^{(p,q)}$$

and hence, has column rank 1. But

$$c(E_{2,j}^{(m,n)} + E_{1,k}^{(m,n)}) = 2,$$

a contradiction. Thus the image of any column is dominated by a column. Further, since the sum of two columns may has column rank 2, the image of distinct columns must be dominated by distinct columns. Let

$$\phi : \{1, \cdots m\} \rightarrow \{1, \cdots, p\}$$

be a mapping defined by $\phi(i) = j$ if $T(R_i^{(m,n)}) \sqsubseteq R_j^{(p,q)}$ and define

$$\theta : \{1, \cdots n\} \rightarrow \{1, \cdots, p\}$$

by $\theta(i) = j$ if $T(C_i^{(m,n)}) \sqsubseteq C_j^{(p,q)}$. Then, it is easily seen that ϕ and θ are injective mappings, and hence, $m \leq p$ and $n \leq q$. Let $\phi' : \{1, \cdots, p\} \rightarrow \{1, \cdots, p\}$ and $\theta' : \{1, \cdots, q\} \rightarrow \{1, \cdots, q\}$ be injective mappings such that $\phi' \mid_{\{1, \cdots m\}} = \phi$ and $\theta' \mid_{\{1, \cdots n\}} = \theta$. Let $P_{\phi'}$ and $Q_{\theta'}$ denote the permutation matrices corresponding to the permutations ϕ' and θ'.

In this case we have that $m \leq p$ and $n \leq q$, and

$$T(A) = P_{\phi'}[A \oplus O]Q_{\theta'}$$

for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$, that is T is a (P, Q)-block-transformation.

Case 2. $T(R_1^{(m,n)}) \sqsubseteq C_1^{(p,q)}$.
The transposing transformation T does not preserve column rank by Example 3.1. So we have not this case. ■

Lemma 3.4. *Let $3 \leq k \leq m \leq n$. If $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation that preserves column rank k and column rank 1, then T strongly preserves column rank 1.*

Proof. If $k = 2$ then clearly T strongly preserves column rank 1. Assume that $k \geq 3$. Suppose a column rank 2 matrix is mapped to a column rank 1 matrix. Without loss of generality, $c(T(E_{1,1} + E_{2,2})) = 1$. But then, since T preserves term rank 1,

$$\begin{aligned} & c(T(E_{1,1} + E_{2,2} + E_{3,3} + \cdots + E_{k,k})) \\ &= c(T(E_{1,1} + E_{2,2}) + T(E_{3,3}) + \cdots + T(E_{k,k})) \\ &\leq c(T(E_{1,1} + E_{2,2})) + c(T(E_{3,3})) + \cdots + c(T(E_{k,k})) \\ &= 1 + (k-2) < k, \end{aligned}$$

a contradiction. Thus, T strongly preserves column rank 1. ∎

Corollary 3.5. *Let $2 < k \leq m, n$ and $1 \leq p, q$ and $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ be a linear transformation. Then T preserves column rank 1,2 and column rank k if and only if T is a (P, Q)-block-transformation.*

Proof. By Lemma 3.4, T strongly preserves column rank 1. By Theorem 3.3, the corollary follows. ∎

We now come to the main theorem of this section:

Theorem 3.6. *Let $1 < k < l \leq m \leq n$ and $k + 1 < m$. Assume $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation that preserves column rank k and column rank l, or if T strongly preserves column rank k, then T is a (P, Q)-block-transformation.*

The proof of this theorem relies upon eight lemmas which now follow.

Lemma 3.7. *Let $2 \leq k \leq m \leq n$. Let $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ be a linear transformation that preserves column rank k. If T does not preserve column rank 1, then there is some column rank 1 matrix whose image has column rank at least 2.*

Proof. Suppose that T does not preserve column rank 1 and $c(T(A)) \leq 1$ for all A with $c(A) = 1$. Then, there is some cell $E_{i,j}$ such that $T(E_{i,j}) = O$. Without loss of generality, assume that $T(E_{1,1}) = O$. Since

$$c(E_{1,1} + E_{2,2} + \cdots + E_{k,k}) = k$$

and T preserves column rank k, we have

$$\begin{aligned} & c(T(E_{2,2} + E_{3,3} + \cdots + E_{k,k})) \\ = & \; c(T(E_{1,1} + E_{2,2} + \cdots + E_{k,k})) \\ = & \; k. \end{aligned}$$

Let $X = T(E_{2,2} + \cdots + E_{k,k})$ then we can choose a set of cells $Y = \{F_1, F_2, \cdots, F_k\}$ such that $X \sqsupseteq F_1 + F_2 + \cdots + F_k$, with $c(F_1 + F_2 + \cdots + F_k) = k$. Since $T(E_{2,2} + \cdots + E_{k,k}) = X$, there is some cell in $\{E_{2,2}, \cdots, E_{k,k}\}$ whose image under T dominates two cells in Y, a contradiction. This contradiction establishes the lemma. ■

Recall that the matrix J is the matrix whose entries are all ones.

Lemma 3.8. *Let $2 \leq k \leq m \leq n$ and $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ be a linear transformation that preserves column rank k. If T does not preserve column rank 1, then $c(T(I)) \leq k$.*

Proof. By Lemma 3.7, if T does not preserve column rank 1, then there is some column rank 1 matrix whose image has column rank 2 or more. Without loss of generality, we may assume that $T(E_{1,1}) \sqsupseteq E_{1,1} + E_{2,2}$. Suppose that $c(T(I)) \geq k + 1$. Then,

$$c(T(I)[3, \cdots, p|3, \cdots, q]) \geq k - 1.$$

Without loss of generality, we may assume that $T(I)[3, \cdots, p|3, \cdots, q] \sqsupseteq E_{3,3} + E_{4,4} + \cdots + E_{k+1,k+1}$. Thus, there are $k-1$ cells, $F_3, F_4, \cdots, F_{k+1}$ such that $T(F_3 + F_4 + \cdots + F_{k+1}) \sqsupseteq E_{3,3} + E_{4,4} + \cdots + E_{k+1,k+1}$. Then,

$$T(E_{1,1} + F_3 + F_4 + \cdots + F_{k+1}) \sqsupseteq I_{k+1}.$$

But, $c(E_{1,1} + F_3 + F_4 + \cdots + F_{k+1}) = k$ while $c(T(E_{1,1} + F_3 + F_4 + \cdots + F_{k+1})) \geq k+1$, a contradiction. Thus, $c(T(I)) \leq k$. ■

Lemma 3.9. *Let $1 \leq k < l \leq m \leq n$. Let $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{p,q}(\mathbb{B})$ be a linear transformation that preserves column rank k, column rank l, then T preserves column rank 1.*

Proof. Suppose that T does not preserve column rank 1. By Lemma 3.7, there is some column rank 1 matrix whose image has column rank at least 2. Let A be such a column rank 1 matrix. Without loss of generality, we may assume that $T(E_{1,1}) \sqsupseteq E_{1,1} + E_{2,2}$. Now, by Lemma 3.8, if $B = T(I)$ is in the image of T, $c(B) \leq k < l$. But if we take $B = T(I_l)$, then $T(I_l)$ must have column rank l, a contradiction.

That is, $0 < c(T(A)) \leq 1$. Since A was an arbitrary column rank 1 matrix, T preserves column rank 1. ■

Lemma 3.10. *Let $1 \leq k \leq m \leq n$. If $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation that preserves column rank k, column rank $k+2$ and T maps cell into cell, then T strongly preserves column rank $k+1$.*

Proof. Let $A \in \mathbb{M}_{m,n}(\mathbb{B})$.

Case 1. Suppose that $c(A) = k+1$ and $c(T(A)) \geq k+2$. Let $A_1, A_2, \cdots, A_{k+1}$ be matrices of column rank 1 such that $A = A_1 + A_2 + \cdots + A_{k+1}$. Without loss of generality we may assume that

$$T(A) \sqsupseteq E_{1,1} + E_{2,2} + \cdots + E_{k+2,k+2}$$

and, since the image of some A_i must have column rank at least 2, we may assume that $c(T(A_1 + A_2 + \cdots + A_i)) \geq i + 1$, for every $i = 1, 2, \cdots k+1$. But then

$$c(A_1 + A_2 + \cdots + A_k) = k$$

while

$$c(T(A_1 + A_2 + \cdots + A_k)) \geq k + 1,$$

a contradiction, Thus if

$$c(A) = k + 1,$$

$$c(T(A)) \leq k + 1.$$

Case 2. Suppose that $c(A) = k + 1$ and $c(T(A)) = s \leq k$. Without loss of generality, we may assume that $A = E_{1,1} + E_{2,2} + \cdots + E_{k+1,k+1}$ and $T(A) \sqsupseteq E_{1,1} +$

$E_{2,2}+\cdots+E_{s,s}$. Then there are s members of $\{T(E_{1,1}), T(E_{2,2}), \cdots, T(E_{k+1,k+1})\}$ whose sum dominates $E_{1,1}+E_{2,2}+\cdots+E_{s,s}$. Say, without loss of generality, that

$$T(E_{1,1}+E_{2,2}+\cdots+E_{s,s}) \sqsupseteq E_{1,1}+E_{2,2}+\cdots+E_{s,s}.$$

Now, $c(A+E_{k+2,k+2}) = k+2$ so that $c(T(A+E_{k+2,k+2})) = k+2$. But since

$$\begin{aligned} & c(T(A+E_{k+2,k+2})) \\ &= c((T(A)+T(E_{k+2,k+2})) \\ &\leq c(T(A))+c(T(E_{k+2,k+2})), \end{aligned}$$

it follows that $c(T(E_{k+2,k+2})) \geq k+2-s$ and there are s members of $\{T(E_{1,1}), T(E_{2,2}), \cdots, T(E_{k+1,k+1})\}$ whose sum together with $T(E_{k+2,k+2})$ has column rank $k+2$, say

$$c(T(E_{1,1}+E_{2,2}+\cdots+E_{s,s}+E_{k+2,k+2})) = k+2.$$

Since $s \leq k$, $c(E_{1,1}+E_{2,2}+\cdots+E_{s,s}+E_{k+2,k+2}) \leq k+1$ and $c(T(E_{1,1}+E_{2,2}+\cdots+E_{s,s}+E_{k+2,k+2})) = k+2$. By Case 1, we again arrive at a contradiction.

Therefore T strongly preserves column rank $k+1$. ■

Lemma 3.11. *Let $1 \leq k < r, s$. If $c(E_{1,1}+\cdots+E_{k,k}+A) \geq k+1$ and $A[k+1,\cdots,r|k+1,\cdots,s] = O$, then there is some $i, 1 \leq i \leq k$, such that $c(E_{1,1}+\cdots+E_{i-1,i-1}+E_{i+1,i+1}+\cdots+E_{k,k}+A) \geq k+1$.*

Proof. Suppose that $B = E_{1,1}+\cdots+E_{k,k}+A$ and $c(B) \geq k+1$. Then there are $k+1$ cells $F_1, F_2, \cdots, F_{k+1}$ such that

$$B \sqsupseteq F_1+F_2+\cdots+F_{k+1}$$

and

$$c(F_1+F_2+\cdots+F_{k+1}) = k+1.$$

If $F_1+F_2+\cdots+F_{k+1} \sqsupseteq I_k \oplus O$ then one cell F_j must be a cell $E_{a,b}$ where $a, b \geq k+1$, which contradicts the assumption

$$A[k+1,\cdots,r|k+1,\cdots,s] = O.$$

Thus $F_1 + F_2 + \cdots + F_{k+1}$ does not dominate $I_k \oplus O$. That is, there is some $i, 1 \leq i \leq k$, such that

$$c(E_{1,1} + \cdots + E_{i-1,i-1} + E_{i+1,i+1} + \cdots + E_{k,k} + A) \geq k + 1.$$

■

Lemma 3.12. *Let $2 \leq k+1 < m \leq n$. If $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation that preserves column rank k, column rank $k+1$ and T maps cell into cell, then T preserves column rank 1.*

Proof. If $k = 1$, the lemma vacuously holds.
Suppose that $k \geq 2$.

Suppose that T does not preserve column rank 1. Then there is some matrix of column rank 1 whose image has column rank at least 2. Without loss of generality, we may assume that

$$T(E_{1,1} + E_{2,1}) \sqsupseteq E_{1,1} + E_{2,2}.$$

By Lemma 3.8 we have that $c(T(J)) \leq k + 2$. Since T preserves column rank $k + 1$,

$$c(T(J)) \geq k + 1.$$

Thus, $c(T(J)) = k + i$ for either $i = 1$ or $i = 2$. Now, we may assume that for some r, s with $r + s = k + i$,

$$T(J)[r+1, \cdots, p|s+1, \cdots, q] = O.$$

Further, we may assume, without loss of generality, that there are $k + i$ cells $F_1, F_2, \cdots, F_{k+i}$ such that $T(F_l) \sqsupseteq E_{l,k+i-l+1}$ for $l = 1, \cdots, k+i$. Suppose the image of one of the cells in $F_1, F_2, \cdots, F_{k+i}$ dominates more than one cell in $\{E_{1,k+i}, E_{2,k+i-1}, \cdots, E_{k+1,i}\}$. Say, without loss of generality, that

$$T(F_1) \sqsupseteq E_{1,k+i} + E_{2,k+i-1},$$

then,

$$T(F_1 + F_2 + F_3 + \cdots + F_{k+1}) \sqsupseteq E_{1,k+i} + E_{2,k+i-1} + \cdots + E_{k+1,i},$$

a contradiction,
since

$$c(F_1 + F_2 + F_3 + \cdots + F_{k+1}) \leq k,$$

and hence

$$c(T(F_1 + F_2 + F_3 + \cdots + F_{k+1})) \leq k,$$

and

$$c(E_{1,k+i} + E_{2,k+i-1} + \cdots + E_{k+1,i}) = k + 1.$$

It follows that for each $j = 1, \cdots, k+1$, if $T(F_l) \sqsupseteq E_{j,k+i-j+1}$ then $l = j$ since $T(F_j) \sqsupseteq E_{j,k+i-j+1}$ is unique. Further, by permuting we may assume that

$$F_1 + F_2 + \cdots + F_k \sqsubseteq \begin{bmatrix} J_k & O_{k,n-k} \\ O_{m-k,k} & O_{m-k,n-k} \end{bmatrix}.$$

Now, let $O \neq A \in \mathbb{M}_{m,n}(\mathbb{B})$ have column rank 1, and suppose that $A[1, 2, \cdots, k|1, 2, \cdots, n] = O$ and $A[1, \cdots m|1, \cdots, k] = O$. So that

$$A = \begin{bmatrix} O_k & O_{k,n-k} \\ O_{m-k,k} & A_1 \end{bmatrix}.$$

If $T(A)[k+1, \cdots, p|1, i] = O$, then, since

$$c(F_1 + \cdots + F_k + A) = k + 1,$$

$$c(T(F_1 + \cdots + F_k + A)) = k + 1.$$

Applying Lemma 3.11, we have that there is some j such that

$$c(T(F_1 + \cdots + F_{j-1} + F_{j+1} + \cdots + F_k + A)) = k + 1.$$

But

$$c(F_1 + \cdots + F_{j-1} + F_{j+1} + \cdots + F_k + A) = k$$

while

$$c(T(F_1 + \cdots + F_{j-1} + F_{j+1} + \cdots + F_k + A)) = k + 1,$$

a contradiction. So we must have that

$$T(E_{k+1,1})[k+1, \cdots, p|1, i] \neq O.$$

If $T(E_{k+1,1})[k+1,\cdots,p|1,i]\neq O$ then

$$c(T(F_1+\cdots+F_k+E_{k+1,1}))=k+1,$$

a contradiction since $c(F_1+\cdots+F_k+E_{k+1,1})=k$. Suppose that the $(k,i+1)$ entry of $T(E_{k,k+1})$ is nonzero, then,

$$c(T(F_1+\cdots+F_{k-1}+E_{k,k+1}+E_{k+1,k+1}))=k+1,$$

a contradiction, since

$$c(F_1+\cdots+F_{k-1}+E_{k,k+1}+E_{k+1,k+1})=k.$$

Consider $T(F_1+\cdots+F_{k-1}+E_{k,k+1}+E_{k+1,k+2})$. This must have column rank $k+1$ and dominates $E_{1,k+i}+E_{2,k+i-1}+\cdots+E_{k-1,i+2}+E_{k+1,j}$ for some $j\in\{1,i\}$. Thus, by Lemma 3.11, there is some cell in $\{F_1,\cdots,F_{k-1}\}$, say F_j such that

$$c(T(F_1+\cdots+F_{j-1}+F_{j+1}+\cdots+F_{k-1}+E_{k,k+1}+E_{k+1,k+2}))=k+1.$$

But

$$c(F_1+\cdots+F_{j-1}+F_{j+1}+\cdots+F_{k-1}+E_{k,k+1}+E_{k+1,k+2})=k,$$

a contradiction.

It follows that T must preserve column rank 1. ■

Lemma 3.13. *Let $2\leq k\leq m\leq n$. If $T:\mathbb{M}_{m,n}(\mathbb{B})\rightarrow\mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation that strongly preserves column rank k, then T preserves column rank $k-1$.*

Proof. If $k=2$, the lemma holds. Suppose that $k\geq 3$.

Let $A\in\mathbb{M}_{m,n}(\mathbb{B})$ and $c(A)=k-1$, and suppose that $c(T(A))=s<k-1$. Without loss of generality, we may assume that

$$c(T(E_{1,1}+\cdots+E_{k-1,k-1}))=s<k-1.$$

Since $c(T(E_{1,1}+\cdots+E_{k,k}))=k$, we have that $c(T(E_{k,k}))\geq k-s$. Without loss of generality we may assume that

$$T(E_{1,1}+\cdots+E_{k,k})\sqsupseteq E_{1,1}+\cdots+E_{k,k}$$

and that

$$T(E_{k,k}) \sqsupseteq E_{t+1,t+1} + \cdots + E_{k,k}$$

for some $t \leq s$. Then, there are t cells $\{E_{i_1,i_1}, \cdots, E_{i_t,i_t}\}$ in $\{E_{1,1}, \cdots, E_{k,k}\}$ such that

$$T(E_{i_1,i_1} + \cdots + E_{i_t,i_t}) \sqsupseteq E_{1,1} + \cdots + E_{t,t}.$$

Then

$$T(E_{i_1,i_1} + \cdots + E_{i_t,i_t} + E_{k,k}) \sqsupseteq E_{1,1} + \cdots + E_{k,k}.$$

Thus

$$c(T(E_{i_1,i_1} + \cdots + E_{i_t,i_t} + E_{k,k})) = k.$$

But

$$c(E_{1,1} + \cdots + E_{t,t} + E_{k,k}) = t + 1 \leq s + 1 < (k-1) + 1 = k,$$

which contradicts the assumption of T. Hence $c(T(A)) \geq k - 1$. Further, $c(T(A)) \leq k - 1$, since T strongly preserves column rank k. Thus, T preserves column rank $k - 1$. ■

Lemma 3.14. *Let $2 \leq k < m \leq n$. If $T : \mathbb{M}_{m,n}(\mathbb{B}) \to \mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation that strongly preserves column rank k, then T preserves column rank 1.*

Proof. By Lemma 3.13, T preserves column rank $k - 1$. By Lemma 3.9, T preserves column rank 1. ■

We are now ready to prove the main theorem of this section.

Proof of Theorem 3.6. Case 1. Suppose T preserve column rank k and l. Then by Lemma 3.9 T preserves column rank 1. By Lemma 3.4, T strongly preserves column rank 1. By Theorem 3.3, T is a (P, Q)-block-transformation. Case 2. Suppose T strongly column rank k. Then by Lemma 3.14, T preserve column rank 1. By lemma 3.4 T strongly preserves column rank 1. By Theorem 3.3, T is a (P, Q)-block-transformation. ■

As a concluding remark, we have the characterization the linear transformations that preserve column rank between $m \times n$ Boolean matrices and $p \times q$ Boolean matrices.

References

[1] L. B. Beasley, A. E. Guterman, Y. B. Jun and S. Z. Song, *Linear preservers of extremes of rank inequalities over semirings: Row and column ranks*, Linear Algebra and Its Applications, **413**, (2006), 495-509.

[2] L. B. Beasley and N. J. Pullman, *Boolean-rank-preserving operators and Boolean rank-1-spaces*, Linear Algebra and Appl., **59**(1984), 55-77.

[3] L. B. Beasley and N. J. Pullman, *Semiring rank versus column rank*, Linear Algebra Appl., **101**(1988), 33-48.

[4] L. B. Beasley, S. Z. Song, K. T. Kang and B. K. Sarma, *Column ranks and their preservers over nonnegative real matrices*, Linear Algebra Appl., **399**(2005), 3-16.

[5] R. Brualdi and H. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, New York, 1991.

[6] S. G. Hwang and S. Z. Song, *Spanning column ranks and their preservers of nonnegative matrices*, Linear Algebra Appl., **254**(1997), 485-495.

[7] K. H. Kim, *Boolean Matrix Theory and Applications*, Pure and Applied Mathematics, Vol. 70, Marcel Dekker, New York, (1982).

[8] C. K. Li and S. Pierce, *Linear preserver problems*, Amer. Math. Monthly, **108**(2001), 591-605.

[9] S. Pierce and others, *A Survey of Linear Preserver Problems*, Linear and Multilinear Algebra, **33**(1992), 1-119.

[10] P. S. S. N. V. P. Rao and K. P. S. B. Rao, *On generalized inverses of Boolean matrices*, Linear Algebra and Its Applications, **11**(1975), 135-153.

[11] S. Z. Song, *Linear operators that preserve column rank of Boolean matrices*, Proc. American Math. Soc. **119** (1993), 1085-1088.

[12] S. Z. Song and L. B. Beasley, *Linear transformations that preserve term rank between different matrix spaces*, J. Koean Math. Soc. **50**(2013), 127-136.

<國文抄錄>

서로 다른 이항 부울 행렬 공간 사이의 열 계수 보존자 연구

반환상의 행렬이론은 1981년 이후 비어슬리와 풀만 교수를 비롯하여 많은 선형 대수학자들이 연구해왔다. 그래서 다양한 대수적 구조상에서 행렬함수들을 보존하는 선형연산자에 대한 많은 논문들이 발표되었다. 그렇지만 서로 다른 행렬 공간 사이에서 행렬함수를 보존하는 선형변한에 대한 연구는 없었다.

우선 $\mathbb{F}$를 체 라하고 $\mathbb{M}_{m,n}(\mathbb{F})$를 체 $\mathbb{F}$상에서 원소를 갖는 모든 $m\times n$행렬들의 벡터 공간이라고 하자. 지난 100여 년 동안에 다음의 문제를 연구하는데 많은 노력이 있었다.: 선형연산자 $T:\mathbb{M}_{m,n}(\mathbb{F})\rightarrow\mathbb{M}_{m,n}(\mathbb{F})$가 어떤 함수나 어떤 집합을 불변하게 남기는 경우에 T의 형태를 규명하라.

이러한 문제 연구를 선형보존자 문제라고 부른다.

이 선형연산자의 연구는 1897년에 시작되었는데, 프로베니우스가 복소수 행렬들과 실 대칭 행렬들 위에서 행렬식의 값은 보존하는 선형연산자를 규명한 것이 이 연구의 시작이다.

본 논문에서는 $m\times n$ 부울 행렬들에서 $p\times q$ 부울 계수들로 보내지면서, 행렬의 열 계수를 보존하는 선형변환들을 생각한다. 이렇게 서로 다른 부울 행렬 공간들에서 열 계수를 보존하는 선형연산자의 형태를 규명하였다. 이 결과는 동일한 행렬공간 상에서 열 계수를 보존하는 선형연산자에 관한 결과들을 확장시키게 된다. 주요 정리는 다음과 같다.:

정리. $1<k<l\leq m\leq n$이고 $k+1<m$이라 하자. $T:\mathbb{M}_{m,n}(\mathbb{B})\rightarrow\mathbb{M}_{p,q}(\mathbb{B})$가 열 계수 k와 l을 보존하거나, 열 계수 k를 강하게 보존하는 선형 변환이라고 가정하면, T는 적당한 순환 행렬들 P와 Q를 사용하여 $T(A)=P(A\oplus O)Q$ 형태로 나타남을 밝혔다. 또, 그 역도 성립함을 알 수 있다.

감사의 글

 2012년 많은 추억이 깃든 대학을 졸업하면서 저는 대학원생활을 시작하게 되었습니다. 그리고 2년여의 시간이 흐른 지금 감사의 글을 이렇게 써 내려가고 있습니다. 감사함으로 가득 찬 대학원 생활을 보낸 것 같아 감사의 글을 끝으로 마지막이라 생각하니 아쉬움 또한 크게 남는 것 같습니다.

 무엇하나 만만한 것이 없던 지난 2년여의 시간은 여러 가지로 힘들었습니다. 그러나 주변을 둘러보아도 대학원생활을 하는 데에 있어서 저에게는 도움을 주는 언니, 오빠들로 가득 차 있었음에 남들보다 행복한 생활이 아니었나 하고 다시 생각하게 됩니다. 학문적 배움은 물론 사회생활을 못해본 저에게 인생을 살아가는 중요한 가르침 또한 얻을 수 있게끔 도와주었습니다. 또 석사생활을 마치고 앞으로의 제 삶에서 많은 사람들에게 도움을 줄 수 있는 사람이 되자고 다짐도 하게 되었습니다.

 먼저 송석준 교수님께 마음 깊이 감사의 말씀을 드립니다. 저의 지지부진한 수업준비에도 불구하고 항시 따뜻한 마음과 애정이 담긴 격려로서 화답해주셔서 정말 감사합니다. 바쁘신 와중에도 제 논문이 좀 더 발전 할 수 있도록 함께 고민해주시고 지도해주셔서 두려움 없이 교수님 방에 찾아 뵐 수 있었습니다. 그렇게 적극적은 논문 쓰는 자세 또한 배워나가게 된 것 같습니다. 그리고 교수님께서 보여주신 온화한 마음과 학생을 존중하는 모습은 더욱 더 많은 것을 느끼게 해주셨습니다.

 이곳에서 저는 많은 교수님의 은혜를 입었습니다. 특히 수학이라는 넓고 깊은 학문에 대해 다양한 관찰력을 접하게 해주신 양영오 · 방은숙 · 정승달 · 윤용식 · 유상욱 · 진현성 교수님, 이지순 · 강경태 · 정민주 선생님께도 감사의 말씀을 드립니다. 대학원 입학 전부터 많은 걱정이던 제게 큰 위로와 힘이 되 준 희란언니, 함께 입학하면서 동생처럼 아껴주고 도와준 승표오빠, 방학동안 부족한 공부를 옆에서 끝까지 도와준 수산언니, 재우오빠, 모든 일에 열심인 모습을 보여준 현아언니 그리고 우철오빠, 여림언니, 명효오빠에게도 모두 감사의 마음을 전합니다. 덕분에 외롭지 않고 이쁨 받는 대학원 생활로 기억에 남을 것 같습니다.

 언제나 저를 이해해주고 용기를 주며 사랑해주는 든든한 친구들 수지, 영은이에게도 고마움을 전합니다. 자주보진 못해도 서로가 힘이 되길 바라는 세화, 슬비, 유정, 소영이 그리고 대학교 생활도 잊을 수 없게 해준 수학과 동기들과 자연과학대학 단위 08학번 친구들에게도 고마움을 전합니다. 그 밖에도 이름을 일일이 거론하지 못한 많은 분들께도 이글을 빌어 마음을 전합니다. 마지막으로 제 평생의 후원자이자 언제나 갚지 못할 사랑으로 보살펴주고 믿어주는 우리 엄마, 우리 아빠와 자랑스럽고 대견한 사랑하는 동생들에게도 감사의 마음과 진심어린 사랑의 마음을 전합니다.

2013년 12월