Generalized Obata theorem on a foliated Riemannian manifold

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Contents

⟨**Abstract**⟩

Generalized Obata theorem

on a foliated Riemannian manifold

Let $(M, g_M, \mathcal{F})$ be a complete, connected Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M. Then $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$, where $S^q(1/c)$ is the q-sphere of radius $1/c$ in $(q+1)$-dimensional Euclidean space and G is a discrete subgroup of the orthogonal group $O(q)$, if and only if there exists a non-constant basic function f such that $\nabla_X df = -c^2 f X^b$ for all normal vector fields X, where c is a positive constant. Moreover, when M admits a transversal conformal field $\bar{Y}$, i.e., $\theta(Y)g_Q = 2f_Y g_Q, (f_Y \neq 0)$, we study several applications of the generalized Obata theorem.

1 Introduction

Let (M, g_M) be a compact Einstein manifold of dimesion $n \geq 2$ with constant sectional curvature c^2. Then M. Obata ([13]) proved that the following conditions $(C_1) \sim (C_4)$ are equivalent to each other:

(C_1) M is isometric to a sphere $S^n(1/c)$ with radius $1/c$ in the (n+1)-dimensional Euclidean space.

(C_2) M admits an infinitesimal non-isometric conformal transformation.

(C_3) M admits a non-constant function f satisfying

$$\nabla^2 f = -c^2 f g_M.$$

(C_4) M admits a non-constant function f satisfying

$$\Delta f = nc^2 f.$$

In 2002, J. M. Lee and K. Richadson ([8]) proved that the equivalence between the above conditions (C_1) and (C_4) for Riemannian foliations. That is,

Theorem 1.1 ([8]) *Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. Suppose that there exists a positive constant c such that the transversal Ricci operator ρ^∇ satisfies $\rho^\nabla(X) \geq c^2(q-1)X$ for every normal vector field X. Then the smallest nonzero eigenvalue λ_B of the basic Laplacian satisfies*

$$\lambda_B \geq c^2 q.$$

The equality holds if and only if: $(M, \mathcal{F})$ *is transversally isometric to* $(S^q(1/c), G)$*, where* G *is the discrete subgroup of the orthogonal group* $O(q)$ *acting on the q-sphere* $S^q(1/c)$ *with radius* $1/c$.

In 2008, S. D. Jung and M. J. Jung ([5]) proved the equivalence between (C_1) and (C_2) for Riemannian foliations. That is,

Theorem 1.2 ([5]) *Let* $(M, g_M, \mathcal{F})$ *be as in Theorem 1.1 and* $\rho^\nabla(X) \geq \frac{\sigma^\nabla}{q} X (\sigma^\nabla \neq 0)$ *for any normal vector field* X*, where* σ^∇ *is the transversal scalar curvature. If* M *admits a transversal non-isometric conformal field, then* $(M, \mathcal{F})$ *is transversally isometric to* $(S^q(1/c), G)$*, where* G *is the discrete subgroup of the orthogonal group* $O(q)$ *acting on the q-sphere* $S^q(1/c)$ *with radius* $1/c$*, where* $c^2 = \frac{\sigma^\nabla}{q(q-1)}$.

In this thesis, we discuss the relationship between (C_1) and (C_3) for Riemannian foliations, so called a generalized Obata theorem. Moreover, we study several applications related to the generalized Obata theorem.

The thesis is organized as following: In Section 2, we review definitions and properties of a Riemannian foliation. In Section 3, we define the tensors E^∇ and Z^∇ on the normal bundle Q as follows: $E^\nabla(X) = \rho^\nabla(X) - \frac{\sigma^\nabla}{q} X$ and $Z^\nabla(X,Y)Z = R^\nabla(X,Y)Z - \frac{\sigma^\nabla}{q(q-1)}(g_Q(Y,Z)X - g_Q(X,Z)Y)$ for any normal vector fields X, Y, Z. When M admits a transversal conformal field, we prove the integral formulas about E^∇ and Z^∇, respectively. In Section 4, we prove the equivalence between (C_1) and (C_3) for Riemannian foliations. That is,

Theorem 1.3 *Let $(M, g_M, \mathcal{F})$ be a complete, connected Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M, and let c be a positive real number. Then the following are equivalent:*

(1) There exists a non-constant basic function f such that $\nabla_X df = -c^2 f X^b$ for all normal vectors X, where X^b is the g_M-dual form of X.

(2) $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$, where G is the discrete subgroup of the orthogonal group $O(q)$ acting on the q-sphere $S^q(1/c)$ with radius $1/c$ in Euclidean space $\mathbb{R}^{q+1}$.

Consequently, we have the following theorem.

Theorem 1.4 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transversally Einstein foliation of codimesion $q \geq 2$ and a bundle-like g_M. Then following conditions $(F_1) \sim (F_4)$ are equivalent to each other:*

(F_1) $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$, where G is the discrete subgroup of the orthogonal group $O(q)$ acting on the q-sphere $S^q(1/c)$ with radius $1/c$ in Euclidean space $\mathbb{R}^{q+1}$.

(F_2) M admits a transversal non-isometric conformal field.

(F_3) M admits a non-constant basic function f satisfying

$$\nabla_X df = -c^2 f X^b$$

for all normal vectors X, where X^b is the g_M-dual form of X.

(F_4) *M admits a non-constant basic function f satisfying*

$$\Delta_B f = qc^2 f.$$

In the last Section, we study several applications of the generalized Obata theorem.

2 Riemannian foliation

In this section, we review definitions and properties of Riemannian foliation. Let M^{n+q} be a smooth manifold of dimension $n+q$. For the readers who study the foliated manifolds, we give the proofs of theorems which are already known.

Definition 2.1 A family $\mathcal{F} \equiv \{l_\alpha\}_{\alpha\in A}$ of connected subsets of a manifold M^{n+q} is called a n-dimensional (or codimension q) *foliation* if

(1) $M = \cup_\alpha l_\alpha$,

(2) $l_\alpha \cap l_\beta = \varnothing$, for any $\alpha \neq \beta$,

(3) for any point $p \in M$, there exist a C^r–chart (φ_i, U_i) such that if $U_i \cap l_\alpha \neq \varnothing$, then the connected component of $U_i \cap l_\alpha$ is homeomorphic to A_c, where

$$A_c = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^q | y = \text{constant}\}.$$

Here (φ_i, U_i) is called a *distinguished* (or *foliated*) chart.

Remark. From (3) in Definition 2.1, we know that on $U_i \cap U_j \neq \varnothing$, the coordinate change $\varphi_j^{-1} \circ \varphi_i : \varphi_i^{-1}(U_i \cap U_j) \to \varphi_j^{-1}(U_i \cap U_j)$ has the form

$$\varphi_j^{-1} \circ \varphi_i(x,y) = (\varphi_{ij}(x,y), \gamma_{ij}(y)), \tag{2.1}$$

where $\varphi_{ij} : \mathbb{R}^{n+q} \to \mathbb{R}^p$ is a differential map and $\gamma_{ij} : \mathbb{R}^q \to \mathbb{R}^q$ is a diffeomorphism.

Let $(M, g_M, \mathcal{F})$ be a $(n+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a Riemannian metric g_M. Let TM be the tangent bunlde of M, L the tangent bundle of $\mathcal{F}$ and then L is the integrable subbundle of TM, i.e.,

$$X, Y \in \Gamma L \Longrightarrow [X,Y] \in \Gamma L.$$

Let $Q = TM/L$ be the corresponding normal bundle of $\mathcal{F}$. Then the metric g_M defines a splitting σ in the exact sequence of vector bundles

$$0 \longrightarrow L \longrightarrow TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \longrightarrow 0\ , \tag{2.2}$$

where $\pi : TM \to Q$ is a projection and $\sigma : Q \to L^{\perp}$ is a bundle map satisfying $\pi \circ \sigma = id$. Thus $g_M = g_L \oplus g_{L^{\perp}}$ induces a metric g_Q on Q , that is,

$$g_Q(s,t) = g_M(\sigma(s), \sigma(t)) \tag{2.3}$$

for any $s, t \in \Gamma Q$. So we have an identification $L^{\perp}$ with Q via an isometric splitting $(Q, g_Q) \cong (L^{\perp}, g_{L^{\perp}})$.

Definition 2.2 A Riemannian metric g_Q on Q of a foliation $\mathcal{F}$ is *holonomy invariant* if

$$\theta(X) g_Q = 0 \tag{2.4}$$

for any $X \in \Gamma L$. Here $\theta(X)$ is the transverse Lie derivative, which is defined by $\theta(X)s = \pi[X, Y_s]$, where $Y_s = \sigma(s)$.

Definition 2.3 A foliation $\mathcal{F}$ is *Riemannian* if there exists a holonomy invariant metric g_Q on Q. A metric g_M is a *bundle-like* metric with respect to $\mathcal{F}$ if the induced metric g_Q is holonomy invariant.

Theorem 2.4 ([21]) *Let $\mathcal{F}$ be a foliation on (M, g_M). Then the following conditions are equivalent:*

(1) *$\mathcal{F}$ is Riemannian and g_M is a bundle-like metric.*

(2) *There exists an orthonomal adapted frame $\{E_i, E_a\}$ such that*

$$g_M(\nabla^M_{E_a} E_i, E_b) + g_M(\nabla^M_{E_b} E_i, E_a) = 0,$$

where ∇^M be the Levi-Civita connection on M.

(3) *All geodesics orthogonal to a leaf at one point are orthogonal to each leaf at every point.*

Definition 2.5 The *transverse Levi-Civita connection* ∇^Q on the normal bundle Q is defined by

$$\nabla^Q_X s = \begin{cases} \pi([X, Y_s]) & \forall X \in \Gamma L, \\ \pi(\nabla^M_X Y_s) & \forall X \in \Gamma L^\perp, \end{cases} \tag{2.5}$$

where $Y_s = \sigma(s)$.

Theorem 2.6 ([20]) *The transverse Levi-Civita connection $\nabla^Q \equiv \nabla$ is metrical and torsion-free with respect to ∇. That is, $\nabla_X g_Q = 0$ for all $X \in \Gamma TM$ and $T^\nabla = 0$, where for any $Y, Z \in \Gamma TM$,*

$$T^\nabla(Y, Z) = \nabla_Y \pi(Z) - \nabla_Z \pi(Y) - \pi[Y, Z] = 0.$$

Proof. For all $X \in \Gamma TM$ and $s, t \in \Gamma Q$,

$$\begin{aligned} 2g_Q(\nabla_X s, t) &= X g_Q(s,t) + Y_s g_Q(t, \pi(X)) - Y_t g_Q(\pi(X), s) \\ &+ g_Q(\pi([X, Y_s]), t) - g_Q(\pi([Y_s, Y_t]), X) + g_Q(\pi([Y_t, X]), s), \end{aligned}$$

where $Y_s = \sigma(s)$ and $Y_t = \sigma(t)$.

Then by a direct calculation, we have

$$(\nabla_X g_Q)(s,t) = X g_Q(s,t) - g_Q(\nabla_X s, t) - g_Q(s, \nabla_X t) = 0.$$

Now, we prove the torsion-freeness. For $X \in \Gamma L$, $Y \in \Gamma TM$ we have $\pi(X) = 0$ and

$$T^\nabla(X,Y) = \nabla_X \pi(Y) - \pi[X,Y] = 0.$$

For $Y, Z \in \Gamma Q$, we have

$$T^\nabla(Y,Z) = \pi(\nabla^M_Y Z) - \pi(\nabla^M_Z Y) - \pi[Y,Z] = \pi(T(Y,Z)) = 0,$$

where T is the (vanishing) torsion of ∇^M. Finally the bilinearity and skew symmetry of T^∇ imply the desired result. □

Let the transversal curvature tensor R^∇ of ∇ is defined by

$$R^\nabla(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \tag{2.6}$$

for any $X, Y \in \Gamma TM$.

Proposition 2.7 ([21]) *Let $(M, g_M, \mathcal{F})$ be a complete, connected Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M.*

$$(1)\ i(X)R^\nabla = 0, \qquad (2)\ \theta(X)R^\nabla = 0$$

for any $X \in \Gamma L$, where $i(X)$ is the interior product.

Proof. (1) Let $Y \in \Gamma TM$ and $s \in \Gamma Q$. Then

$$\begin{aligned} R^{\nabla}(X,Y)s &= \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \\ &= \theta(X)\nabla_Y s - \nabla_Y \theta(X)s - \nabla_{\theta(X)Y} s \\ &= (\theta(X)\nabla)_Y s = 0. \end{aligned}$$

(2) Let $Y, Z \in \Gamma TM$ and $s \in \Gamma Q$. Then

$$\begin{aligned} &(\theta(X)R^{\nabla})(Y,Z)s \\ =\ & \theta(X)R^{\nabla}(Y,Z)s - R^{\nabla}(\theta(X)Y,Z)s - R^{\nabla}(Y,\theta(X)Z)s - R^{\nabla}(Y,Z)\theta(s) \\ =\ & \theta(X)\{\nabla_Y \nabla_X s - \nabla_Z \nabla_Y s - \nabla_{[Y,Z]} s\} - \{\nabla_{\theta(X)Y} \nabla_Z s - \nabla_Z \nabla_{\theta(X)Y} s - \nabla_{[\theta(X)Y,Z]} s\} \\ -\ & \{\nabla_Y \nabla_{\theta(X)Z} s - \nabla_{\theta(X)Z} \nabla_Y s - \nabla_{[Y,\theta(X)Z]} s\} - \{\nabla_Y \nabla_Z \theta(X)s - \nabla_Z \nabla_Y \theta(X)s - \nabla_{[Y,Z]} \theta(X)s\} \\ =\ & -\nabla_{\theta(X)[Y,Z]} s + \nabla_{[\theta(X)Y,Z]} s + \nabla_{[Y,\theta(X)Z]} s = (-\nabla_{[X,[Y,Z]]}) + \nabla_{[[X,Y],Z]} + \nabla_{[Y,[X,Z]]})s = 0. \end{aligned}$$

□

Definition 2.8 The *transversal Ricci operator* ρ^{∇} and the *transversal scalar curvature* σ^{∇} with respect to ∇ are defined by

$$\rho^{\nabla}(s) = \sum_a R^{\nabla}(s, E_a)E_a, \quad \sigma^{\nabla} = g_Q(\rho^{\nabla}(E_a), E_a),$$

where $\{E_a\}$ is a local orthonomal basic frame of Q.

Definition 2.9 The foliation $\mathcal{F}$ is said to be (transversally) *Einsteinian* if

$$\rho^{\nabla} = \frac{1}{q}\sigma^{\nabla} \cdot id \tag{2.7}$$

with constant transversal scalar curvature σ^{∇}.

Definition 2.10 The *mean curvature form* κ of $\mathcal{F}$ is given by

$$\kappa(X) = g_Q(\sum_{i=1}^{n} \pi(\nabla^M_{E_i} E_i), X) \tag{2.8}$$

for any $X \in \Gamma Q$, where $\{E_i\}_{i=1,\cdots,n}$ is a local orthonormal basis of L. The foliation $\mathcal{F}$ is said to be *minimal* (or *harmonic*) if $\kappa = 0$.

Definition 2.11 Let $\mathcal{F}$ be an arbitrary foliation on a manifold M. A differential form ω is *basic* if for any $X \in \Gamma L$,

$$i(X)\omega = 0, \ \theta(X)\omega = 0. \tag{2.9}$$

Locally, the basic $r-$form ω is expressed by

$$\omega = \sum_{a_1 < \cdots < a_r} \omega_{a_1 \cdots a_r} dy^{a_1} \wedge \cdots \wedge dy^{a_r}, \tag{2.10}$$

where $\frac{\partial \omega_{a_1 \cdots a_r}}{\partial x^j} = 0$ for all $j = 1, \cdots, n$. Let $\Omega^r_B(\mathcal{F})$ be the space of all basic r–forms. Then ([1])

$$\Omega^*(M) = \Omega^*_B(\mathcal{F}) \oplus \Omega^*_B(\mathcal{F})^\perp.$$

Let ω_B be the basic part of the form ω. From now on, κ_B is the basic part of the mean curvature form κ.

Theorem 2.12 ([1]) *For a Riemannian foliation $\mathcal{F}$ on a compact manifold, κ_B is closed, i.e., $d\kappa_B = 0$.*

Definition 2.13 The *basic Laplacian* Δ_B acting on $\Omega^*_B(\mathcal{F})$ by

$$\Delta_B = d_B \delta_B + \delta_B d_B, \tag{2.11}$$

where δ_B is the formal adjoint operator of $d_B = d|_{\Omega_B^*(\mathcal{F})}$, which are locally given by

$$d_B = \sum_a \theta^a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_a i(E_a)\nabla_{E_a} + i(\kappa_B^\sharp), \tag{2.12}$$

where $\kappa_B^\sharp$ is the g_Q–dual vector of κ_B, $\{E_a\}$ is a local orthonormal basic frame of Q and θ^a is a g_Q–dual 1–form to E_a.

Definition 2.14 A vector field $Y \in M$ is an *infinitesimal automorphism* of $\mathcal{F}$ if

$$[Y, Z] \in \Gamma L \quad \forall Z \in \Gamma L.$$

Let $V(\mathcal{F})$ be the space of all infinitesimal automorphism, i.e.,

$$V(\mathcal{F}) = \{Y \in TM | [Y, Z] \in \Gamma L, \quad \forall Z \in \Gamma L\}.$$

Now we put

$$\bar{V}(\mathcal{F}) = \{\bar{Y} = \pi(Y) | Y \in V(\mathcal{F})\}.$$

It is trivial that an elements s of $\bar{V}(\mathcal{F})$ satisfies $\nabla_X s = 0$ for all $X \in \Gamma L$.

Theorem 2.15 ([22]) **(Transversal divergence theorem)** *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a transversally oriented foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then*

$$\int_M \operatorname{div}_\nabla \bar{X} = \int_M g_Q(\bar{X}, \kappa_B^\sharp) \tag{2.13}$$

for all $X \in V(\mathcal{F})$, where $\operatorname{div}_\nabla X$ denotes the transversal divergence of X with respect to the connection ∇.

Proof. Let $\{E_i\}$ and $\{E_a\}$ be orthonormal basis of L and Q, respectively. Then for any $X \in V(\mathcal{F})$,

$$\begin{aligned}
\operatorname{div} X &= \sum_i g_M(\nabla^M_{E_i} X, E_i) + \sum_a g_M(\nabla^M_{E_a} X, E_a) \\
&= -\sum_i g_Q(\bar{X}, \pi(\nabla^M_{E_i} E_i)) + \sum_a g_Q(\pi(\nabla^M_{E_a} X), E_a) \\
&= -g_Q(\bar{X}, \kappa_B^\sharp) + g_Q(\nabla_{E_a} \bar{X}, E_a) \\
&= -g_Q(\bar{X}, \kappa_B^\sharp) + \operatorname{div}_\nabla \bar{X},
\end{aligned}$$

where $\bar{X} = \pi(X)$. By the divergence theorem, we have

$$0 = \int_M \operatorname{div} X = \int_M \operatorname{div}_\nabla \bar{X} - \int_M g_Q(\bar{X}, \kappa_B^\sharp).$$

This completes the proof of this Theorem. □

Now we define an operator $A_Y : \Gamma Q \to \Gamma Q$ for any $Y \in V(\mathcal{F})$ by

$$A_Y s = \theta(Y)s - \nabla_Y s. \tag{2.14}$$

Then it is proved ([7]) that, for any vector field $Y \in V(\mathcal{F})$,

$$A_Y s = -\nabla_{Y_s} \bar{Y}, \tag{2.15}$$

where Y_s is the vector field such that $\pi(Y_s) = s$. So A_Y depends only on $\bar{Y} = \pi(Y)$ and is a linear operator. Moreover, A_Y extends in an obvious way to tensors of any type on Q ([19]). In particular, for any basic 1-form $\phi \in \Omega^1_B(\mathcal{F})$, the operator A_Y is given by

$$(A_Y \phi)(s) = -\phi(A_Y s) \tag{2.16}$$

for any $s \in \Gamma Q$. We define $\nabla^*_{tr} \nabla_{tr} : \Omega^r_B(\mathcal{F}) \to \Omega^r_B(\mathcal{F})$ by

$$\nabla^*_{tr} \nabla_{tr} \phi = -\sum_a \nabla^2_{E_a, E_a} \phi + \nabla_{\kappa_B^\sharp} \phi, \tag{2.17}$$

where $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla^M_X Y}$ for any $X, Y \in \Gamma TM$.

Proposition 2.16 ([4]) *The operator $\nabla^*_{tr}\nabla_{tr}$ is positive definite and formally self adjoint on the space of basic forms, i.e.,*

$$\int < \nabla^*_{tr}\nabla_{tr}\varphi, \psi > = \int < \nabla_{tr}\varphi, \nabla_{tr}\psi >,$$

where $< \nabla_{tr}\varphi, \nabla_{tr}\psi > = \sum_a < \nabla_{E_a}\varphi, \nabla_{E_a}\psi >$.

Proof. Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ with the property that $(\nabla E_a)_x = 0$ for all a. Then we have at the point that for any φ and ψ,

$$\begin{aligned} < \nabla^*_{tr}\nabla_{tr}\varphi, \psi > &= -\sum_a < \nabla_{E_a}\nabla_{E_a}\varphi, \psi > + < \nabla_{\kappa_B^\sharp}\varphi, \psi > \\ &= -\sum_a E_a < \nabla_{E_a}\varphi, \psi > + \sum_a < \nabla_{E_a}\varphi, \nabla_{E_a}\psi > + < \nabla_{\kappa_B^\sharp}\varphi, \psi > . \end{aligned}$$

Now, we define $v \in \bar{V}(\mathcal{F})$ by $g_Q(v, w) = < \nabla_w\varphi, \psi >$ for all $w \in \Gamma Q$. Then

$$\mathrm{div}_\nabla(v) = \sum_a g_Q(\nabla_{E_a}v, E_a) = \sum_a E_a g_Q(v, E_a) = \sum_a E_a < \nabla_{E_a}\varphi, \psi > .$$

By the transversal divergence theorem on the foliated Riemannian manifold, we have

$$\int \mathrm{div}_\nabla(v) = \int < v, \kappa_B^\sharp > = \int < \nabla_{\kappa_B^\sharp}\varphi, \psi > .$$

Hence the proof follows. □

Theorem 2.17 ([4]) *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. Then for any basic form $\phi \in \Omega^r_B(\mathcal{F})$,*

$$\Delta_B\phi = \nabla^*_{tr}\nabla_{tr}\phi + F(\phi) + A_{\kappa_B^\sharp}\phi, \quad \phi \in \Omega^r_B(\mathcal{F}), \tag{2.18}$$

where $F(\phi) = \sum_{a,b} \theta^a \wedge i(E_b)R^\nabla(E_b, E_a)\phi$. If ϕ is a basic 1–form, then $F(\phi)^\sharp = \rho^\nabla(\phi^\sharp)$.

For any vector field $X \in V(\mathcal{F})$, if we put $\Delta_B \bar{X} = (\Delta_B \phi)^\sharp$, where $\phi^\sharp = \bar{X}$, then we have the following corollary.

Corollary 2.18 ([5]) *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. Then for any vector field $X \in V(\mathcal{F})$,*

$$\Delta_B \bar{X} = \nabla_{tr}^* \nabla_{tr} \bar{X} + \rho^\nabla(\bar{X}) - A^t_{\kappa_B^\sharp} \bar{X}, \tag{2.19}$$

where A^t is an adjoint operator of A.

Proof. Let $\phi^\sharp = \bar{X}$. From (2.16), we have

$$(A_{\kappa_B^\sharp} \phi)^\sharp = -A^t_{\kappa_B^\sharp} \phi^\sharp = -A^t_{\kappa_B^\sharp} \bar{X}.$$

From Theorem 2.17, the proof follows. □

3 Integral formulas

In this section, we define the tensors E^{∇} and Z^{∇} on the normal bundle Q. Also, we have prove the integral formulas for E^{∇} and Z^{∇}. Let $(M, g_M, \mathcal{F})$ be a $(n+q)$-dimensional closed, oriented Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M.

Lemma 3.1 ([5]) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M such that $\delta_B \kappa_B = 0$. Then for any basic function f, we have*

$$\int_M f^r \kappa_B^{\sharp}(f) = 0 \tag{3.1}$$

for any integer $r \neq -1$. For $r = -1$, it holds only if $f > 0$ or $f < 0$.

Proof. In case of $r \neq -1$, we have

$$\begin{aligned}
\int_M f^r \kappa_B^{\sharp}(f) &= \int_M f^r g_Q(\kappa_B, d_B f) = \frac{1}{r+1}\int_M g_Q(\kappa_B, d_B f^{r+1}) \\
&= \frac{1}{r+1}\int_M g_Q(\delta_B \kappa_B, f^{r+1}) = 0.
\end{aligned}$$

In case of $r = -1$, we have for any basic function $f > 0$

$$\int_M \frac{1}{f}\kappa_B^{\sharp}(f) = \int_M g_Q(\kappa_B^{\sharp}, d_B \ln f) = 0,$$

which completes the proof. □

Proposition 3.2 ([5]) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. Then for any vector field*

$X \in V(\mathcal{F})$,

$$g_Q(\Delta_B\bar{X},\bar{X}) - 2\mathrm{Ric}^\nabla(\bar{X},\bar{X}) - \frac{1}{2}|\theta(X)g_Q - \frac{2}{q}\mathrm{div}_\nabla\bar{X}|^2 + \frac{q-2}{q}(\mathrm{div}_\nabla\bar{X})^2 + g_Q(A_{\kappa_B^\sharp}\bar{X},\bar{X}) - \mathrm{div}_\nabla(A_X\bar{X}) - \mathrm{div}_\nabla((\mathrm{div}_\nabla\bar{X})\bar{X}) = 0,$$

where $\mathrm{Ric}^\nabla(X,Y) = g_Q(\rho^\nabla(X),Y)$ *for any vector fields* $X,Y \in \Gamma Q$.

Lemma 3.3 ([5]) *Let* $(M,g_M,\mathcal{F})$ *be a closed, oriented Riemannian manifold with a foliation* $\mathcal{F}$ *of codimension* q *and a bundle-like metric* g_M. *Then for any vector field* $X \in V(\mathcal{F})$,

$$\int_M \{g_Q(A_{\kappa_B^\sharp}\bar{X},\bar{X}) + \mathrm{div}_\nabla(A_X\bar{X})\} = -\int_M Xg_Q(\kappa_B^\sharp,\bar{X}), \qquad (3.2)$$

$$\int_M \mathrm{divv}_\nabla((\mathrm{div}_\nabla\bar{X})\bar{X}) = \int_M (\mathrm{div}_\nabla\bar{X})g_Q(\bar{X},\kappa_B^\sharp). \qquad (3.3)$$

Proof. From (2.13) and (2.15), equation (3.2) is proved. Equation (3.3) follows from the transversal divergence theorem (2.13). □

Proposition 3.4 ([3]) *Let* $(M,g_M,\mathcal{F})$ *be a closed, oriented Riemannian manifold with a foliation* $\mathcal{F}$ *of codimension* q *and a bundle-like metric* g_M. *Then for any basic function* f, *we have*

$$\int_M [g_Q(\Delta_B\nabla f,\nabla f) - 2\mathrm{Ric}^\nabla(\nabla f,\nabla f) - 2|\nabla\nabla f + \frac{1}{q}\{\Delta_B f - \kappa_B^\sharp(f)\}g_Q|^2 + \frac{q-2}{q}\{\Delta_B f - \kappa_B^\sharp(f)\}^2 + 2g_Q(A_{\kappa_B^\sharp}\nabla f,\nabla f) + 2\kappa_B^\sharp(f)\Delta_B f - \kappa_B^\sharp(f)^2] = 0,$$

where ∇f *is the transversal gradient of* f.

Proof. We first compute $\theta(\nabla f)g_Q = 2\nabla\nabla f$. Let $\{E_a\}$ be a local orthonormal basic frame of Q. Then

$$\begin{aligned}(\theta(\nabla f)g_Q)(E_a,E_b) &= g_Q(\nabla_a\nabla f,E_b)+g_Q(\nabla_b\nabla f,E_a)\\ &= \sum_c\{g_Q(\nabla_a(\nabla_c f)E_c,E_b)+g_Q(\nabla_b(\nabla_c f)E_c,E_a)\}\\ &= \sum_c\{(\nabla_a\nabla_c f)g_Q(E_c,E_b)+(\nabla_b\nabla_c f)g_Q(E_c,E_a)\}\\ &= 2\nabla_a\nabla_b f,\end{aligned}$$

where $\nabla_a = \nabla_{E_a}$. Since $\int_M Y(f) = \int_M f(\delta_B\phi)$ for any $Y \in V(\mathcal{F})$ and $\phi^\sharp = Y$, we have

$$\int_M (\nabla f)g_Q(\kappa_B^\sharp,\nabla f) = \int_M \kappa_B^\sharp(f)\Delta_B f.$$

Note that $\mathrm{div}_\nabla \nabla f = -\delta_T d_B f = -\Delta_B f + \kappa_B^\sharp(f)$, where $\delta_T\phi = -\sum_a i(E_a)\nabla_{E_a}\phi$. So if we put $\bar{X} = \nabla f$ in (3.3), then

$$\begin{aligned}\int_M \mathrm{div}_\nabla((\mathrm{div}_\nabla\nabla f)\nabla f) &= \int_M(\mathrm{div}_\nabla\nabla f)g_Q(\nabla f,\kappa_B^\sharp)\\ &= -\int_M\{\Delta_B f-\kappa_B^\sharp(f)\}\kappa_B^\sharp(f).\end{aligned}$$

If we put $\bar{X} = \nabla f$ in Proposition 3.2, then the proof follows. □

Lemma 3.5 ([3]) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. Then for any basic function f, we have*

$$\int_M g_Q(A_{\kappa_B^\sharp}\nabla f,\nabla f) = -\int_M \kappa_B^\sharp(f)\Delta_B f + \frac{1}{2}\int_M \kappa_B^\sharp(|d_B f|^2).$$

Proof. Note that for any basic 1-form ϕ,

$$(A_Y\phi)^\sharp = -A_Y^t\phi^\sharp$$

for any vector field $Y \in V(\mathcal{F})$.

From (2.15), we have

$$\begin{aligned}\int_M g_Q(A_{\kappa_B^\sharp}\nabla f, \nabla f) &= -\int_M g_Q(\nabla f, (A_{\kappa_B^\sharp}d_B f)^\sharp) = -\int_M g_Q(d_B f, A_{\kappa_B^\sharp}d_B f)\\ &= -\int_M g_Q(\theta(\kappa_B^\sharp)d_B f, d_B f) + \int_M g_Q(\nabla_{\kappa_B^\sharp}d_B f, d_B f).\end{aligned}$$

Since $\theta(\kappa_B^\sharp)d_B f = d_B i(\kappa_B^\sharp)d_B f$, we have

$$\begin{aligned}\int_M g_Q(\theta(\kappa_B^\sharp)d_B f, d_B f) &= \int_M g_Q(d_B i(\kappa_B^\sharp)d_B f, d_B f)\\ &= \int_M g_Q(i(\kappa_B^\sharp)d_B f, \Delta_B f)\\ &= \int_M \kappa_B^\sharp(f)\Delta_B f,\end{aligned}$$

which completes the proof. □

Theorem 3.6 ([3]) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle - like metric g_M such that $\delta_B\kappa_B = 0$. If a basic function f satisfies $(\Delta_B - \kappa_B^\sharp)f = \lambda f$, then*

$$\frac{q-1}{q}\lambda\int_M |\nabla f|^2 - \int_M \mathrm{Ric}^\nabla(\nabla f, \nabla f) - \int_M |\nabla\nabla f + \frac{\lambda}{q} f g_Q|^2 = 0. \tag{3.4}$$

Proof. Since $\Delta_B d_B f = d_B \Delta_B f$, we have

$$\begin{aligned}\int_M g_Q(\Delta_B \nabla f, \nabla f) &= \int_M g_Q(\Delta_B d_B f, d_B f) \\ &= \int_M g_Q(d_B \Delta_B f, d_B f) \\ &= \int_M g_Q(d_B(\lambda f + \kappa_B^\sharp(f)), d_B f) \\ &= \lambda \int_M |d_B f|^2 + \int_M \kappa_B^\sharp(f) \Delta_B f.\end{aligned} \tag{3.5}$$

From Lemma 3.5 and (3.5), we have

$$\begin{aligned}&\int_M \{g_Q(\Delta_B \nabla f, \nabla f) + 2g_Q(A_{\kappa_B^\sharp} \nabla f, \nabla f) + 2\kappa_B^\sharp(f)\Delta_B f - \kappa_B^\sharp(f)^2\} \\ &= \lambda \int_M |d_B f|^2 + \lambda \int_M f\kappa_B^\sharp(f) + \int_M \kappa_B^\sharp(|d_B f|^2).\end{aligned}$$

Since $\Delta_B f - \kappa_B^\sharp(f) = \delta_T d_B f$, we have

$$\int_M \{\Delta_B f - \kappa_B^\sharp(f)\}^2 = \int_M g_Q(\delta_T d_B f, \lambda f) = \int_M \lambda |d_B f|^2.$$

From Proposition 3.4, we have

$$\begin{aligned}&\frac{2(q-1)}{q}\lambda \int_M |\nabla f|^2 + \lambda \int_M f\kappa_B^\sharp(f) + \int_M \kappa_B^\sharp(|d_B f|^2) - 2\int_M \mathrm{Ric}^\nabla(\nabla f, \nabla f) \\ &- 2\int_M |\nabla\nabla f + \frac{\lambda}{q} f g_Q|^2 = 0.\end{aligned} \tag{3.6}$$

Since $\delta_B \kappa_B = 0$, from Lemma 3.1, we have

$$\int_M f\kappa_B^\sharp(f) = 0 = \int_M \kappa_B^\sharp(|d_B f|^2).$$

Hence the proof follows from (3.6). □

Definition 3.7 If a vector field $Y \in V(\mathcal{F})$ satisfies $\theta(Y)g_Q = 2f_Y g_Q$, for a basic scale function f_Y depending on Y, then $\bar{Y}$ is called a *transversal conformal field* of $\mathcal{F}$ with a scale function f_Y. In particular, if $f_Y = 0$, then $\bar{Y}$ is called a *transversal killing filed* of $\mathcal{F}$.

Remark. 1. If $\bar{Y}$ is a transversal conformal field of $\mathcal{F}$ with a scale function f_Y, then

$$f_Y = \frac{1}{q}\mathrm{div}_\nabla \bar{Y}. \tag{3.7}$$

2. Note that $\bar{Y}$ is a transversal conformal field with a scale function f_Y if and only if

$$g_Q(\nabla_X \bar{Y}, Z) + g_Q(\nabla_Z \bar{Y}, X) = 2 f_Y g_Q(X, Z) \tag{3.8}$$

for any $X, Z \in Q$.

Lemma 3.8 ([5]) *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. If $\bar{Y} \in \bar{V}(\mathcal{F})$ is a transversal conformal field with a scale function f_Y, then*

$$g_Q((\theta(Y)R^\nabla)(E_a, E_b)E_c, E_d) = \delta_b^d \nabla_a f_c - \delta_b^c \nabla_a f_d - \delta_a^d \nabla_b f_c + \delta_a^c \nabla_b f_d, \tag{3.9}$$

$$(\theta(Y)\mathrm{Ric}^\nabla)(E_a, E_b) = -(q-2)\nabla_a f_b + (\Delta_B f_Y - \kappa_B^\sharp(f_Y))\delta_a^b, \tag{3.10}$$

$$\theta(Y)\sigma^\nabla = 2(q-1)(\Delta_B f_Y - \kappa_B^\sharp(f_Y)) - 2 f_Y \sigma^\nabla, \tag{3.11}$$

where $\nabla_a = \nabla_{E_a}, f_a = \nabla_a f_Y$ and $\mathrm{Ric}^\nabla(X, Y) = g_Q(\rho^\nabla(X), Y)$ for any $X, Y \in Q$.

Now we define the tensors E^∇ and Z^∇ respectively by

$$E^\nabla(X) = \rho^\nabla(X) - \tfrac{\sigma^\nabla}{q} X, \tag{3.12}$$

$$Z^\nabla(X, Y)Z = R^\nabla(X, Y)Z - \tfrac{\sigma^\nabla}{q(q-1)}(g_Q(Y, Z)X - g_Q(X, Z)Y) \tag{3.13}$$

for any fields $X, Y, Z \in \Gamma Q$. Then we have the following lemma (cf. [3]).

Lemma 3.9 *Let* $(M, g_M, \mathcal{F})$ *be a Riemannian manifold with a foliation* $\mathcal{F}$ *of codimension* q *and a bundle-like metric* g_M*. Then*

$$\mathrm{tr}E^\nabla = 0, \tag{3.14}$$

$$\textstyle\sum_a Z^\nabla(X, E_a)E_a = E^\nabla(X) \quad \forall X \in \Gamma Q, \tag{3.15}$$

$$|E^\nabla|^2 = |\rho^\nabla|^2 - \tfrac{(\sigma^\nabla)^2}{q}, \tag{3.16}$$

$$|Z^\nabla|^2 = |R^\nabla|^2 - \tfrac{2(\sigma^\nabla)^2}{q(q-1)}, \tag{3.17}$$

$$\mathrm{div}_\nabla E^\nabla = \tfrac{q-2}{2q}\nabla\sigma^\nabla, \tag{3.18}$$

where $\mathrm{tr}E^\nabla = \sum_a g_Q(E^\nabla(E_a), E_a)$.

Proof. From (3.12), we have

$$\begin{aligned} |E^\nabla|^2 &= \sum_a g_Q(E^\nabla(E_a), E^\nabla(E_a)) \\ &= \sum_a g_Q(\rho^\nabla(E_a) - \frac{\sigma^\nabla}{q}E_a, \rho^\nabla(E_a) - \frac{\sigma^\nabla}{q}E_a) \\ &= |\rho^\nabla|^2 - \frac{(\sigma^\nabla)^2}{q}. \end{aligned}$$

From (3.13), we have

$$\begin{aligned} |Z^\nabla|^2 &= \sum_{a,b,c} g_Q(Z^\nabla(E_a, E_b)E_c, Z^\nabla(E_a, E_b)E_c) \\ &= |R^\nabla|^2 - \frac{2\sigma^\nabla}{q(q-1)}\sum_{a,b,c}\{(g_Q(R^\nabla(E_a, E_c)E_c, E_a) - g_Q(R^\nabla(E_c, E_b)E_c, E_b))\} \\ &\quad + \frac{2(\sigma^\nabla)^2}{q^2(q-1)^2}\sum_{a,b}(\delta_a^a\delta_b^b - \delta_a^b\delta_a^b) \\ &= |R^\nabla|^2 - \frac{2(\sigma^\nabla)^2}{q(q-1)}. \end{aligned}$$

Since $Y(\sigma^\nabla) = 2\sum_a g_Q((\nabla_{E_a}\rho^\nabla)(Y), E_a)$ for any $Y \in \Gamma Q$, we have

$$\begin{aligned} \mathrm{div}_\nabla E^\nabla &= \sum_a (\nabla_{E_a} E^\nabla)(E_a) = \sum_a \nabla_{E_a} E^\nabla(E_a) \\ &= \sum_a \nabla_{E_a}\rho^\nabla(E_a) - \frac{1}{q}\sum_a (\nabla_{E_a}\sigma^\nabla)E_a \\ &= \frac{1}{2}\nabla\sigma^\nabla - \frac{1}{q}\nabla\sigma^\nabla = \frac{q-2}{2q}\nabla\sigma^\nabla. \end{aligned}$$

From (3.12) and (3.13), others follows. □

Lemma 3.10 *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. If $\bar{Y} \in \bar{V}(\mathcal{F})$ is a transversal conformal field with a scale function f_Y, then*

$$(\theta(Y)E^\nabla)(E_a, E_b) = -(q-2)[\nabla_a f_b + \frac{1}{q}\{\Delta_B f_Y - \kappa_B^\sharp(f_Y)\}\delta_a^b], \tag{3.19}$$

$$\begin{aligned} g_Q((\theta(Y)Z^\nabla)(E_a, E_b)E_c, E_d) &= \delta_b^d\nabla_a f_c - \delta_b^c\nabla_a f_d - \delta_a^d\nabla_b f_c + \delta_a^c\nabla_b f_d \\ &\quad - \frac{2}{q}(\Delta_B f_Y - \kappa_B^\sharp(f_Y))(\delta_a^d\delta_b^c - \delta_b^d\delta_a^c), \end{aligned} \tag{3.20}$$

$$\sum_a g_Q(E^\nabla(\theta(Y)E_a), E^\nabla(E_a)) = -f_Y|E^\nabla|^2, \tag{3.21}$$

$$\theta(Y)|E^\nabla|^2 = -2(q-2)g_Q(\nabla\nabla f_Y, E^\nabla) - 4f_Y|E^\nabla|^2, \tag{3.22}$$

$$\sum_{a,b,c} g_Q(Z^\nabla(\theta(Y)E_a, E_b)E_c, Z^\nabla(E_a, E_b)E_c) = -f_Y|Z^\nabla|^2, \tag{3.23}$$

$$\theta(Y)|Z^\nabla|^2 = -8g_Q(\nabla\nabla f_Y, E^\nabla) - 4f_Y|Z^\nabla|^2. \tag{3.24}$$

Proof. From (3.10), (3.11) and (3.12), we have

$$\begin{aligned}
(\theta(Y)E^{\nabla})(E_a,E_b) &= \theta(Y)E^{\nabla}(E_a,E_b) - E^{\nabla}(\theta(Y)E_a,E_b) - E^{\nabla}(E_a,\theta(Y)E_a) \\
&= \theta(Y)\{\mathrm{Ric}^{\nabla}(E_a,E_b) - \frac{\sigma^{\nabla}}{q}g_Q(E_a,E_b)\} \\
&\quad -\mathrm{Ric}^{\nabla}(\theta(Y)E_a,E_b) + \frac{\sigma^{\nabla}}{q}g_Q(\theta(Y)E_a,E_b) \\
&\quad -\mathrm{Ric}^{\nabla}(E_a,\theta(Y)E_b) + \frac{\sigma^{\nabla}}{q}g_Q(E_a,\theta(Y)E_b) \\
&= (\theta(Y)\mathrm{Ric}^{\nabla})(E_a,E_b) - \frac{1}{q}(\theta(Y)\sigma^{\nabla})\delta_a^b - \frac{2}{q}f_Y\sigma^{\nabla}\delta_a^b \\
&= -(q-2)[\nabla_a f_b + \frac{1}{q}\{\Delta_B f_Y - \kappa_B^{\sharp}(f_Y)\}\delta_a^b].
\end{aligned}$$

From (3.13), we have

$$\begin{aligned}
(\theta(Y)Z^{\nabla})(E_a,E_b)E_c &= \theta(Y)Z^{\nabla}(E_a,E_b)E_c - Z^{\nabla}(\theta(Y)E_a,E_b)E_c \\
&\quad -Z^{\nabla}(E_a,\theta(Y)E_b)E_c - Z^{\nabla}(E_a,E_b)E_c \\
&= (\theta(Y)R^{\nabla})(E_a,E_b)E_c - \frac{1}{q(q-1)}(\theta(Y)\sigma^{\nabla})(\delta_b^c E_a - \delta_a^c E_b) \\
&\quad -\frac{2f_Y\sigma^{\nabla}}{q(q-1)}(\delta_b^c E_a - \delta_a^c E_b).
\end{aligned}$$

Then (3.20) follows from (3.9) and (3.11).

By a direct calculation, we have

$$\begin{aligned}
\sum_a g_Q(E^{\nabla}(\theta(Y)E_a),E^{\nabla}(E_a)) &= \sum_{a,b} g_Q(\theta(Y)E_a,E_b)g_Q(E^{\nabla}(E_a),E^{\nabla}(E_b)) \\
&= \sum_{a,b}\{-2f_Y\delta_a^b - g_Q(E_a,\theta(Y)E_b)\}g_Q(E^{\nabla}(E_a),E^{\nabla}(E_b)) \\
&= -2f_Y|E^{\nabla}|^2 - \sum_a g_Q(E^{\nabla}(\theta(Y)E_a),E^{\nabla}(E_a)),
\end{aligned}$$

which proves (3.21).

From (3.19), (3.21) and tr$E^\nabla = 0$, we have

$$\begin{aligned}
\theta(Y)|E^\nabla|^2 &= \sum_a \theta(Y) g_Q(E^\nabla(E_a), E^\nabla(E_a)) \\
&= 2\sum_a g_Q((\theta(Y)E^\nabla)E_a, E^\nabla(E_a)) \\
&= 2\sum_{a,b} g_Q((\theta(Y)E^\nabla)E_a, E_b) g_Q(E^\nabla(E_a), E_b) \\
&= 2\sum_{a,b}\{(\theta(Y)E^\nabla)(E_a, E_b) - (\theta(Y)g_Q)(E^\nabla(E_a), E_b)\} g_Q(E^\nabla(E_a), E_b) \\
&= -2(q-2)\sum_{a,b}[(\nabla_a f_b) + \frac{1}{q}\{\nabla_B f_Y - \kappa^\sharp_B(f_Y)\}\delta^b_a] g_Q(E^\nabla(E_a), E_b) - 4 f_Y |E^\nabla|^2 \\
&= -2(q-2) g_Q(\nabla\nabla f_Y, E^\nabla) - 4 f_Y |E^\nabla|^2,
\end{aligned}$$

which proves (3.22).

By a direct calculation, we have

$$\begin{aligned}
&\sum_{a,b,c} g_Q(Z^\nabla(\theta(Y)E_a, E_b)E_c, Z^\nabla(E_a, E_b)E_c) \\
= &\sum_{a,b,c,d} g_Q(\theta(Y)E_a, E_d) g_Q(Z^\nabla(E_d, E_b)E_c, Z^\nabla(E_a, E_b)E_c) \\
= &\sum_{a,b,c,d}\{-(\theta(Y)g_Q)(E_a, E_d) - g_Q(E_a, \theta(Y)E_d)\} g_Q(Z^\nabla(E_d, E_b)E_c, Z^\nabla(E_a, E_b)E_c) \\
= &-2 f_Y |Z^\nabla|^2 - \sum_{a,b,c} g_Q(Z^\nabla(\theta(Y)E_a, E_b)E_c, Z^\nabla(E_a, E_b)E_c),
\end{aligned}$$

which proves (3.23).

From (3.20) and tr$E^\nabla = 0$, we have

$$\begin{aligned}
&\sum_{a,b,c} g_Q((\theta(Y)Z^\nabla)(E_a, E_b)E_c, Z^\nabla(E_a, E_b)E_c) \\
= &\sum_{a,b,c,d} g_Q((\theta(Y)Z^\nabla)(E_a, E_b)E_c, E_d) g_Q(Z^\nabla(E_a, E_b)E_c, E_d)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{a,b,c,d}[\delta_b^d\nabla_a f_c - \delta_b^c\nabla_a f_d - \delta_a^d\nabla_b f_c + \delta_a^c\nabla_b f_d - \frac{2}{q}\{\Delta_B f_Y - \kappa_B^\sharp(f_Y)\}(\delta_b^c\delta_a^d - \delta_a^c\delta_b^d)] \\
&\quad g_Q(Z^\nabla(E_a,E_b)E_c,E_d) \\
&= -4\sum_{a,c}\nabla_a f_c g_Q(Z^\nabla(E_a,E_b)E_b,E_c) - \frac{4}{q}\{\Delta_B f_Y - \kappa_B^\sharp(f_Y)\}\sum_a g_Q(E^\nabla(E_a),E_a) \\
&= -4g_Q(\nabla\nabla f_Y, E^\nabla).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\theta(Y)|Z^\nabla|^2 &= \sum_{a,b,c}\theta(Y)g_Q(Z^\nabla(E_a,E_b)E_c, Z^\nabla(E_a,E_b)E_c) \\
&= \sum_{a,b,c}(\theta(Y)g_Q)(Z^\nabla(E_a,E_b)E_c, Z^\nabla(E_a,E_b)E_c) \\
&\quad +2\sum_{a,b,c}g_Q((\theta(Y)Z^\nabla)(E_a,E_b)E_c, Z^\nabla(E_a,E_b)E_c) \\
&\quad +2\sum_{a,b,c}g_Q(Z^\nabla(\theta(Y)E_a,E_b)E_c, Z^\nabla(E_a,E_b)E_c) \\
&\quad +2\sum_{a,b,c}g_Q(Z^\nabla(E_a,\theta(Y)E_b)E_c, Z^\nabla(E_a,E_b)E_c) \\
&\quad +2\sum_{a,b,c}g_Q(Z^\nabla(E_a,E_b)\theta(Y)E_c, Z^\nabla(E_a,E_b)E_c) \\
&= -8g_Q(\nabla\nabla f_Y, E^\nabla) - 4f_Y|Z^\nabla|^2,
\end{aligned}
$$

which proves (3.24). □

Proposition 3.11 ([3]) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M such that $\delta_B\kappa_B = 0$. Assume that the transversal scalar curvature σ^∇ is constant. If M admits a transversal conformal field $\bar{Y}$ with a non-zero scale function f_Y, then*

$$
\int_M\{g_Q(E^\nabla(\nabla f_Y),\nabla f_Y) + |\nabla\nabla f_Y + \frac{\sigma^\nabla}{q(q-1)}f_Y g_Q|^2\} = 0. \tag{3.25}
$$

Proof. Since σ^∇ is constant, from (3.11), $(\Delta_B - \kappa_B^\sharp) f_Y = \frac{\sigma^\nabla}{q-1} f_Y$. If we let $\lambda = \frac{\sigma^\nabla}{q-1}$, then (3.25) follows from Theorem 3.6. □

Lemma 3.12 ([3]) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. Assume that the transversal scalar curvature σ^∇ is constant. Then for any function f,*

$$\mathrm{div}_\nabla (E^\nabla (f \nabla f)) = g_Q(E^\nabla(\nabla f), \nabla f) + f g_Q(E^\nabla, \nabla\nabla f), \tag{3.26}$$

where $E^\nabla(X, Y) = g_Q(E^\nabla(X), Y)$ for all $X, Y \in \Gamma Q$.

Proof. Since σ^∇ is constant, $\mathrm{div}_\nabla E^\nabla = 0$. Hence

$$\begin{aligned}
\mathrm{div}_\nabla (E^\nabla (f\nabla f)) &= \sum_a g_Q(\nabla_{E_a}(E^\nabla(f\nabla f)), E_a) \\
&= \sum_a g_Q((\nabla_{E_a} E^\nabla)(f\nabla f), E_a) + \sum_a g_Q(E^\nabla(\nabla_{E_a}(f\nabla f)), E_a) \\
&= \sum_a g_Q((\nabla_{E_a} E^\nabla)(E_a), f\nabla f) + \sum_a g_Q(E^\nabla(E_a), \nabla_{E_a}(f\nabla f)) \\
&= \sum_a g_Q(E^\nabla(E_a), \nabla_{E_a} f \nabla f) + f \sum_a g_Q(E^\nabla(E_a), \nabla_{E_a}\nabla f) \\
&= g_Q(E^\nabla(\nabla f), \nabla f) + f g_Q(E^\nabla, \nabla\nabla f).
\end{aligned}$$

□

Proposition 3.13 ([3]) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M. Assume that the transversal scalar curvature σ^∇ is constant. If M admits a transversal conformal field*

$\bar{Y}$ *with a non-zero scale function* f_Y, *then*

$$(q-2)\int_M g_Q(E^\nabla(\nabla f_Y),\nabla f_Y) = \int_M \{2{f_Y}^2|E^\nabla|^2 + \frac{1}{2}f_Y\theta(Y)|E^\nabla|^2\} + (q-2)\int_M g_Q(E^\nabla(f_Y\kappa_B^\sharp),\nabla f_Y). \quad (3.27)$$

Proof. From (3.26) and the transversal divergence theorem, we have

$$\int_M g_Q(E^\nabla(\nabla f_Y),\nabla f_Y) = \int_M g_Q(E^\nabla(f_Y\kappa_B^\sharp),\nabla f_Y) - \int_M f_Y g_Q(E^\nabla,\nabla\nabla f_Y). \quad (3.28)$$

From (3.22), we get

$$(q-2)\int_M f_Y g_Q(\nabla\nabla f_Y, E^\nabla) = -2\int_M {f_Y}^2|E^\nabla|^2 - \frac{1}{2}\int_M f_Y\theta(Y)|E^\nabla|^2.$$

Hence the proof follows from (3.28). □

Proposition 3.14 *Let* $(M, g_M, \mathcal{F})$ *be a closed, oriented Riemannian manifold with a foliation* $\mathcal{F}$ *of codimension* q *and a it bundle-like metric* g_M. *Assume that the transversal scalar curvature* σ^∇ *is constant. If* M *admits a transversal conformal field* $\bar{Y}$ *with a non-zero scale function* f_Y, *then*

$$\int_M g_Q(E^\nabla(\nabla f_Y),\nabla f_Y) = \int_M \{\frac{1}{2}{f_Y}^2|Z^\nabla|^2 + \frac{1}{8}f_Y\theta(Y)|Z^\nabla|^2\} + \int_M g_Q(E^\nabla(f_Y\kappa_B^\sharp),\nabla f_Y). \quad (3.29)$$

Proof. From (3.24), we have

$$\int_M g_Q(\nabla\nabla f_Y, E^\nabla) = -\frac{1}{2}\int_M f_Y|Z^\nabla|^2 - \frac{1}{8}\int_M \theta(Y)|Z^\nabla|^2.$$

Hence the proof follows from (3.28). □

4 The generalized Obata theorem

Definition 4.1 Let G be a discrete group. A Riemannian foliation $(M, \mathcal{F})$ is *transversally isometric* to (W, G), where G acts by isometries on a Riemannaian manifold (W, g_W), if there exists a homeomorphism $\eta : W/G \to M/\mathcal{F}$ that is *locally covered by isometries*. That is, given any $x \in M$, there exists a local smooth transversal V containing x and a neighborhood U in W and an isometry $\phi : U \to V$ such that the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{\phi} & V \\ {\scriptstyle P\circ i}\downarrow & & \downarrow{\scriptstyle \tilde{P}\circ j} \\ W/G & \xrightarrow{\eta} & M/\mathcal{F} \end{array}$$

where $i : U \to W$ and $j : V \to M$ are inclusions and $P : W \to W/G$ and $\tilde{P} : M \to M/\mathcal{F}$ are the projections.

Now, we prove the generalized Obata theorem.

Theorem 4.2 *Let $(M, g_M, \mathcal{F})$ be a complete, connected Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M, and let c be a positive real number. Then the following are equivalent:*

(1) *There exists a non-constant basic function f such that $\nabla_X df = -c^2 f X^b$ for all vectors $X \in L^\perp$, where X^b is the g_M-dual form of X.*

(2) *$(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$, where G is the discrete subgroup of the orthogonal group $O(q)$ acts by isometries on the last q coordinates of the q-sphere $S^q(1/c)$ of radius $1/c$ in Euclidean space $\mathbb{R}^{q+1}$.*

Proof. It is clear that the second condition implies the first, because if f is the first coordinate function in $\mathbb{R}^{q+1}$ considered as a function on the sphere $S^q(1/c)$, it satisfies the first condition.

Conversely, assume that the first condition is satisfied for the basic funtion f. This implies that for each $x \in M$,

$$-c^2 f(x) g_{L_x^\perp} = \nabla^2 f|_{L_x^\perp}, \tag{4.1}$$

where $L_x^\perp$ is the normal space to the leaf through $x \in M$ and $g_{L_x^\perp} = g_{L^\perp}|_{L_x^\perp}$ is the metric restricted to $L_x^\perp$. For any unit speed geodesic $\gamma : [0, \beta) \to M$ that is normal to the leaves of the foliation,

$$\begin{aligned} -c^2(f \circ \gamma) &= -c^2(f \circ \gamma) g_M(\gamma', \gamma') \\ &= g_M((\nabla^2 f)(\gamma'), \gamma') = g_M(\nabla_{\gamma'} \nabla f, \gamma') \\ &= g_M(\nabla f, \gamma')' - g_M(\nabla f, \nabla_{\gamma'} \gamma') \\ &= (f \circ \gamma)'', \end{aligned}$$

where ∇f is the transversal gradient of f. Thus

$$(f \circ \gamma)(t) = A \cos(ct) + B \sin(ct)$$

for some constants A and B.

Let $\gamma(0) = x_0 \in M$ be either a global maximum or global minimum of f on M. Then $A = f(x_0)$ and $B = 0$. Thus

$$f(\gamma(t)) = f(x_0) \cos(ct) \tag{4.2}$$

for any unit speed geodesic γ orthogonal to the leaf l_{x_0} through x_0, and the maximum and minimum values along γ must have opposite signs. Suppose that we choose the geodesic so that it connects an absolute maximum x_0 with an absolute minimum x_1; such a normal geodesic can always be found (see [9]). Since the metric is bundle-like, every geodesic with initial velocity in $L^{\perp}$ is guaranteed to be orthogonal to $L^{\perp}$ at all poinis ([18]).

We prove the theorem by four Steps. Let $M_s = \{l_y | \mathrm{dist}(l_{x_0}, l_y) = s\}$ for any non-negative real number s.

Step 1. $M_0 = \{l_{x_0}\}$ *and* $M_{\frac{\pi}{c}} = \{l_{x_1}\}$.

Since the nondegeneracy of the normal Hessian implies that each maximum and minimum of $f \circ \gamma$ occurs at an isolated closed leaf of $(M, \mathcal{F})$, the set $f^{-1}(-f(x_0))$ must be a discrete union of closed leaves, and $l_{x_1} \subset f^{-1}(-f(x_0))$. Note that $f^{-1}([f(x_0), -f(x_0)]) = M$ and the normal exponential map is surjective ([9]). Hence $f^{-1}(-f(x_0))$ is a single closed leaf, say l_{x_1}, so that all normal geodesics through x_0 meet l_{x_1} at the exact distance $\frac{\pi}{c}$. Similarly, $f^{-1}(f(x_0)) = l_{x_0}$.

Step 2. M_s $(0 < s < \pi/c)$ *is diffeomorphic to the unit normal sphere bundle of* $l_{x_0} \subset M$.

Given any leaf l of M that is neither l_{x_0} nor l_{x_1}, there exists a minimal normal geodesic connecting it to l_{x_0} by completeness. In fact, there exists such a minimal normal geodesic through x_0, and its initial velocity lies in $L^{\perp}_{x_0}$. By equation (4.2), the gradient of f is nonzero at each $\gamma(t)$ for $0 < t < \pi/c$ and is parallel to $\gamma'(t)$. Since geodesics are determined by velocity at a single point, it is impossible that two geodesics with initial velocities through x_0 meet at the same point unless that point has distance at least π/c from x_0. Thus, the normal exponential map $\exp^{\perp}_{x_0} : L^{\perp}_{x_0} \to M$ is injective on the ball $B_{\pi/c} := B_{\pi/c}(x_0) \subset L^{\perp}_{x_0}$. This discussion is independent of the initial point of l_{x_0} chosen, because for a bundle-like metric the distance from a point x_0 on one leaf closure to another is independent of the choice $x_0 \in l_{x_0}$ (see [9]). We have $\bigcup_{x \in l_{x_0}} \exp^{\perp}_x \overline{(B_{\pi/c}(x))} = M$. By the preceding discussion, $\bigcup_{x \in l_{x_0}} \exp^{\perp}_x \partial\overline{(B_s(x))} = M_s$. Since $\partial\overline{B_s(x)}$ is the unit sphere on q-dimensional Euclidean space, M_s $(0 < s < \pi/c)$ is diffeomorphic to the unit normal sphere bundle of $l_{x_0} \subset M$.

Let $B^{+}_{\pi/c}$ denote the one-point compactification of $B_{\pi/c}$ and G be an orthogonal transformaitons at x_0 on $L^{\perp}_{x_0}$ ([12]).

Step 3. *$M/\mathcal{F}$ is homeomorphic to S/G, where $S = B^{+}_{\pi/c}$ is a sphere.*

Since G at x_0 acts by orthogonal transformations on $L^{\perp}_{x_0}$, $M_s \bigcap \exp^{\perp}_{x_0}(L^{\perp}_{x_0})$ is isometric to $\partial\overline{B_s(x_0)}/G$ by the indeced metric $g_{L^{\perp}}$ on $L^{\perp}_{x_0}$, and so leaf space $M_s/\mathcal{F}$ is diffeomorphic to $\partial\overline{B_s(x_0)}/G$, for $0 \le s < \pi/c$. Then $(M \setminus l_{x_1})/\mathcal{F}$ is diffeomorphic to $B_{\pi/c}/G$ by the map

$$\eta : B_{\pi/c}/G \to (M \setminus l_{x_1})/\mathcal{F}$$

defined by $\eta(O_\xi) = l_{\exp^{\perp}_{x_0}(\xi)}$, where ξ in $B_{\pi/c} \subset L^{\perp}_{x_0}$, O_ξ is the G-orbit of ξ in $B_{\pi/c}$ and $l_{\exp^{\perp}_{x_0}(\xi)}$ is the leaf containing $\exp^{\perp}_{x_0}(\xi)$. Now, we define

$$\bar{\eta} : B^{+}_{\pi/c}/G \to M/\mathcal{F}$$

by $\bar{\eta}\mid_{B_{\frac{\pi}{c}}/G} = \eta$ and $\bar{\eta}(\infty) = l_{x_1}$. Then $M/\mathcal{F}$ is homeomorphic to S/G.

Step 4. *$M/\mathcal{F}$ is transversally isometric to $S^q(1/c)/G$.*

Let v and w be any two nonzero orthonormal vectors in $L^\perp_{x_0}$, and let W_s denote the $L^\perp$-parallel translate of $w = W_0$ along the geodesic $\gamma(s)$ with initial velocity v; thus $W_s \in L^\perp_{\gamma(s)}$ is a well-defined vector at each $\gamma(s)$ for $0 \le s < \pi/c$. We see that W_s is tangent to M_s for $s \in (0, \pi/c)$.

First, we prove the following.

(i) $csW_0(y_j) = \sin(cs)W_s(y_j)$, for $0 < s < \pi/c$.

Let (y_j) be geodesic normal coordinates for the normal ball $\exp^\perp_{x_0}(B_{\pi/c}(x_0))$. Suppose that these coordinates are chosen at x_0 such that $y_1(\gamma(s)) = s$ and each of $\frac{\partial}{\partial y_j}$ for $j > 1$ is orthogonal to $v = \gamma'(0)$ at $x_0 = 0$. We extend s to be the function $s(y) = \sqrt{\sum y_j^2}$ and write $y_j = s\theta_j$, so that each θ_j is independent of s. Thus, $\gamma'(s)(\theta_j) = 0$ and $W_s(s) = 0$. Further, we let $\frac{\partial}{\partial s}$ denote the radial vector field, which agrees with $\gamma'(s)$ along γ. In the calculations that follow, we extend $y_j, \theta_j, \frac{\partial}{\partial s}$ to be well-defined and basic in a small neighborhood of the transversal $\exp^\perp_{x_0}(B_{\pi/c})$. From the calculation of f above, we see that $\nabla f = -c\sin(cs)f(x_0)\frac{\partial}{\partial s}$.

Since ∇ is torsion-free and $\nabla_{\gamma'(s)}W_s = 0$ by construction,

$$\begin{aligned}
\pi\left[\frac{\partial}{\partial s}, W_s\right] &= -\nabla_{W_s}\frac{\partial}{\partial s} = \frac{1}{c\sin(cs)f(x_0)}\nabla_{W_s}\nabla f \\
&= -\frac{c^2}{c\sin(cs)f(x_0)}f(\gamma(s))W_s \\
&= -\frac{c\cos(cs)}{\sin(cs)}W_s.
\end{aligned}$$

On the other hand, since θ_j is locally defined basic function, for $0 < s < \pi/c$,

$$\frac{d}{ds}W_s(\theta_j) = \frac{\partial}{\partial s}W_s(\theta_j) = \left[\frac{\partial}{\partial s}, W_s\right](\theta_j) = \pi\left[\frac{\partial}{\partial s}, W_s\right](\theta_j) = -\frac{c\cos(cs)}{\sin(cs)}W_s(\theta_j).$$

Solving the differential equation above, we have

$$W_s(\theta_j) = \frac{1}{\sin(cs)}W_{\pi/2c}(\theta_j),\ 0 < s < \pi/c. \tag{4.3}$$

Since $W_s(s) = 0$ and $y_j = s\theta_j$, we have

$$W_s(y_j) = sW_s(\theta_j)$$

for $0 < s < \pi/c$. Then, for all j,

$$\begin{aligned} W_0(y_j) &= \lim_{s->0} W_s(y_j) = \frac{1}{c}W_{\frac{\pi}{2c}}(\theta_j) \\ &= \frac{\sin(cs)}{c}W_s(\theta_j) = \frac{\sin(cs)}{cs}W_s(y_j). \end{aligned}$$

Next, we prove the isometry property.

(ii) $\eta^* g_{L^\perp} = g_s$, where g_s is the standard metric metric of $S^q(1/c)$. Note that since the vectors $\frac{\partial}{\partial\theta_j}$ for $j > 1$ form a basis of the tangent space for $M_s \cap \exp^\perp_{x_0}(B_{\pi/c})$ at $\gamma(s)$ with $s > 0$, the equation above uniquely defines the vector W_s in terms of W_0. Since the metric of the sphere $S^q(1/c)$ satisfies the same hypothesis, a corresponding fact is true for geodesic normal coordinates on $S^q(1/c)$.

We now show that the equation above implies that the pullback of the metric $g_{L^\perp}$ to $B_{\pi/c}$ is the same as the standard metric g_S corresponding to geodesic normal coordinates on $S^q(1/c)$. As above, let W_s denote the parallel displacement of W_0 along $\gamma(s)$, and let $\overline{W_s}$ denote the parallel displacement of W_0 along the geodesic in $(B_{\pi/c}, g_S)$ with unit tangent vector v. Then

$$d\eta(\overline{W_s})(\theta_j) = \overline{W_s}(\theta_j \circ \eta) = \frac{c}{\sin(cs)}\overline{W_0}(s\theta_j \circ \eta) = \frac{c}{\sin(cs)}W_0(s\theta_j) = W_s(\theta_j).$$

Namely we have $d\eta(\overline{W_s}) = W_s$. Thus we have

$$|d\eta(\overline{W_s})| = |W_s| = |W_0| = |\overline{W_s}|.$$

We may reverse the roles of x_0 and x_1 and obtain a similar result.

Now, given any point $x \in M$, there is a minimal geodesic connecting this point to a point x_0' on the leaf containing x_0. If $x \notin l_{x_1}$, the above analysis shows that the map $\exp^{\perp}_{x_0'}$ restricted to $(B_{\pi/c}(x_0'), g_S)$ is an isometry onto its image, and that image contains x. Further, the map $\exp^{\perp}_{x_0'}$ locally covers the map $\overline{\eta} : (B^{+}_{\pi/c}/G, g_S) \to (M/\mathcal{F}, g_{L^{\perp}})$. If $x \notin l_{x_0}$, a similar fact is true for $\exp^{\perp}_{x_1'}$. Thus the map $\overline{\eta}$ is locally covered by isometries, and we conclude that $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$. □

5 Applications of the generalized Obata theorem

In this section, we give several applications of the generalized Obata theorem.

Theorem 5.1 ([3]) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M such that $\delta_B \kappa_B = 0$. Assume that the transversal scalar curvature $\sigma^\nabla (\neq 0)$ is constant. If M admits a transversal conformal field $\bar{Y}$ with a non-zero scale function f_Y such that*

$$\int_M g_Q(E^\nabla(\nabla f_Y), \nabla f_Y) \geq 0, \tag{5.1}$$

then $(M, \mathcal{F})$ is transversally isometric to a sphere $(S^q(1/c), G)$, where $c^2 = \frac{\sigma^\nabla}{q(q-1)}$ and G is a discrete subgroup of $O(q)$.

Proof. From Proposition 3.11, we have

$$\nabla\nabla f_Y = -\frac{\sigma^\nabla}{q(q-1)} f_Y g_Q.$$

Since σ^∇ is constant, the transversal scalar curvature σ^∇ is non - negative ([5]). Therefore, $\frac{\sigma^\nabla}{q(q-1)}$ is positive. By the generalized Obata theorem, the proof is completed. □

Theorem 5.2 ([3]) *Let $(M, g_M, \mathcal{F})$ as in Theorem 5.1, except that $\mathcal{F}$ is minimal. If M admits a transversal conformal field $\bar{Y}$ with a non-zero scale function f_Y such that*

$$\theta(Y)|E^\nabla|^2 = 0, \tag{5.2}$$

then $(M, \mathcal{F})$ is transversally isometric to a sphere $(S^q(1/c), G)$, where $c^2 = \frac{\sigma^\nabla}{q(q-1)}$ and G is a discrete subgroup of $O(q)$.

Proof. From Proposition 3.13, the minimality of $\mathcal{F}$ and $\theta(Y)|E^\nabla|^2 = 0$ imply that

$$(q-2)\int_M g_Q(E^\nabla(\nabla f_Y), \nabla f_Y) = 2\int_M {f_Y}^2 |E^\nabla|^2.$$

For $q \geq 3$, the proof follows from Theorem 5.1. For $q = 2$, $|E^\nabla|^2 = 0$. So $\mathcal{F}$ is transversally Einstein. Hence the proof follows from Theorem 1.4. □

Corollary 5.3 ([3]) *Let $(M, g_M, \mathcal{F})$ as in Theorem 5.1, except that $\mathcal{F}$ is minimal. If M admits a transversal conformal field $\bar{Y}$ with non-zero scale function f_Y such that*

$$\theta(Y)|\rho^\nabla|^2 = 0, \tag{5.3}$$

then $(M, \mathcal{F})$ is transversally isometric to a sphere $(S^q(1/c), G)$, where $c^2 = \frac{\sigma^\nabla}{q(q-1)}$ and G is a discrete subgroup of $O(q)$.

Proof. Since σ^∇ is constant, $\theta(Y)\sigma^\nabla = 0$. From (3.12), we have

$$\theta(Y)|E^\nabla|^2 = \theta(Y)|\rho^\nabla|^2 = 0.$$

Hence the proof follows from Theorem 5.2. □

If $|\rho^\nabla|^2$ is constant, then $\theta(Y)|\rho^\nabla|^2 = 0$. Hence we have the following corollary from Corollary 5.3.

Corollary 5.4 *Let $(M, g_M, \mathcal{F})$ as in Theorem 5.1, except that $\mathcal{F}$ is minimal. If M admits a transversal conformal field $\bar{Y}$ with a non-zero scale function f_Y such that $|\rho^\nabla|^2$ is constant, then $(M, \mathcal{F})$ is transversally isometric to a sphere $(S^q(1/c), G)$, where $c^2 = \frac{\sigma^\nabla}{q(q-1)}$ and G is a discrete subgroup of $O(q)$.*

Theorem 5.5 *Let* $(M, g_M, \mathcal{F})$ *as in Theorem 5.1, except that* $\mathcal{F}$ *is minimal. If* M *admits a transversal conformal field* $\bar{Y}$ *with a non-zero scale function* f_Y *such that*

$$\theta(Y)|E^\nabla|^2 = t f_Y |E^\nabla|^2 \qquad (t > -4), \tag{5.4}$$

then $(M, \mathcal{F})$ *is transversally isometric to a sphere* $(S^q(1/c), G)$*, where* $c^2 = \frac{\sigma^\nabla}{q(q-1)}$ *and* G *is a discrete subgroup of* $O(q)$.

Proof. From Proposition 3.13, we have

$$2(q-2)\int_M g_Q(E^\nabla(\nabla f_Y), \nabla f_Y) = (4+t)\int_M {f_Y}^2 |E^\nabla|^2.$$

Since $t > -4$, $2(q-2)\int_M g_Q(E^\nabla(\nabla f_Y), \nabla f_Y) \geq 0$. For $q \geq 3$, the proof follows from Theorem 5.1. For $q = 2$, $|E^\nabla|^2 = 0$. So $\mathcal{F}$ is transversally Einstein. Hence the proof follows from Theorem 1.4. □

Theorem 5.6 *Let* $(M, g_M, \mathcal{F})$ *as in Theorem 5.1, except that* $\mathcal{F}$ *is minimal. If* M *admits a transversal conformal field* $\bar{Y}$ *with a non-zero scale function* f_Y *such that*

$$\theta(Y)|R^\nabla|^2 = 0, \tag{5.5}$$

then $(M, \mathcal{F})$ *is transversally isometric to a sphere* $(S^q(1/c), G)$*, where* $c^2 = \frac{\sigma^\nabla}{q(q-1)}$ *and* G *is a discrete subgroup of* $O(q)$.

Proof. Since $\theta(Y)|Z^\nabla|^2 = \theta(Y)|R^\nabla|^2 = 0$. From Proposition 3.14, we have

$$\int_M g_Q(E^\nabla(\nabla f_Y), \nabla f_Y) = \frac{1}{2}\int_M {f_Y}^2 |Z^\nabla|^2 \geq 0.$$

Hence the proof follows from Theorem 5.1. □

If $|R^\nabla|^2$ is constant, then $\theta(Y)|R^\nabla|^2 = 0$. Hence we have the following corollary from Theorem 5.6.

Corollary 5.7 *Let* $(M, g_M, \mathcal{F})$ *as in Theorem 5.1, except that* $\mathcal{F}$ *is minimal. If* M *admits a transversal conformal field* $\bar{Y}$ *with a non-zero scale function* f_Y *such that* $|R^\nabla|^2$ *is constant, then* $(M, \mathcal{F})$ *is transversally isometric to a sphere* $(S^q(1/c), G)$*, where* $c^2 = \frac{\sigma^\nabla}{q(q-1)}$ *and* G *is a discrete subgroup of* $O(q)$.

Theorem 5.8 *Let* $(M, g_M, \mathcal{F})$ *as in Theorem 5.1, except that* $\mathcal{F}$ *is minimal. If* M *admits a transversal conformal field* $\bar{Y}$ *with a non-zero scale function* f_Y *such that*

$$\theta(Y)|Z^\nabla|^2 = t f_Y |Z^\nabla|^2 \qquad (t \geq -4), \tag{5.6}$$

then $(M, \mathcal{F})$ *is transversally isometric to a sphere* $(S^q(1/c), G)$*, where* $c^2 = \frac{\sigma^\nabla}{q(q-1)}$ *and* G *is a discrete subgroup of* $O(q)$.

Proof. From Proposition 3.14, we have

$$\int_M g_Q(E^\nabla(\nabla f_Y), \nabla f_Y) = \frac{4+t}{8} \int_M {f_Y}^2 |Z^\nabla|^2.$$

Since $t \geq -4$, $\int_M g_Q(E^\nabla(\nabla f_Y), \nabla f_Y) \geq 0$. Hence the proof follows from Theorem 5.1. □

Theorem 5.9 *Let* $(M, g_M, \mathcal{F})$ *be a closed, oriented Riemannian manifold with foliation* $\mathcal{F}$ *of codimension* q *and a bundle-like metric* g_M*. Assume that the transversal scalar curvature* $\sigma^\nabla (\neq 0)$ *is constant. If* M *admits a transversal conformal field* $\bar{Y}$ *with* *a non-zero scale function f_Y, then*

$$f_Y{}^2(\sigma^\nabla)^2 \le q(q-1)^2|\nabla\nabla f_Y|^2. \tag{5.7}$$

Equality holds if and only if $(M,\mathcal{F})$ is transversally isometric to a sphere $(S^q(1/c),G)$, where $c^2 = \frac{\sigma^\nabla}{q(q-1)}$ and G is a discrete subgroup of $O(q)$.

Proof. Since σ^∇ is constant, $(\Delta_B - \kappa_B^\sharp)f_Y = \frac{\sigma^\nabla}{q-1}f_Y$ from (3.11). Hence, we have

$$\begin{aligned} 0 &\le |\nabla\nabla f_Y + \frac{\sigma^\nabla}{q(q-1)}f_Y g_Q|^2 \\ &= |\nabla\nabla f_Y|^2 + \frac{2\sigma^\nabla}{q(q-1)}f_Y\sum_{a,b}\nabla_a\nabla_b f_Y\delta_a^b + \frac{(\sigma^\nabla)^2}{q(q-1)^2}f_Y{}^2 \\ &= |\nabla\nabla f_Y|^2 - \frac{(\sigma^\nabla)^2}{q(q-1)^2}f_Y{}^2, \end{aligned}$$

which proves (5.7). Equality holds if and only if

$$\nabla\nabla f_Y = -\frac{\sigma^\nabla}{q(q-1)}f_Y g_Q.$$

By the generalized Obata theorem, the proof is completed. □

Proposition 5.10 *Let $(M,g_M,\mathcal{F})$ be a closed, oriented Riemannian manifold with foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. Assume that the transversal scalar curvature $\sigma^\nabla(\ne 0)$ is constant. If M admits a transversal conformal field $\bar{Y}$ with a non-zero scale function f_Y, then*

$$\int_M|\nabla f_Y|^2 = \frac{\sigma^\nabla}{q-1}\int_M f_Y{}^2 + \int_M \kappa_B^\sharp(f_Y)f_Y. \tag{5.8}$$

Proof. By a direct calculation, we have

$$\frac{1}{2}\Delta_B f_Y{}^2 = (\Delta_B f_Y)f_Y - |\nabla f_Y|^2 = \frac{\sigma^\nabla}{q-1}f_Y{}^2 + \kappa_B^\sharp(f_Y)f_Y - |\nabla f_Y|^2. \tag{5.9}$$

Since for any basic function f,

$$\int_M \Delta_B f = 0,$$

by integrality (5.9), the proof follows. □

Theorem 5.11 *Let* $(M, g_M, \mathcal{F})$ *as in Theorem 5.1. If* M *admits a transversal conformal field* $\bar{Y}$ *with a non-zero scale function* f_Y*, then*

$$\int_M \mathrm{Ric}^\nabla(\nabla f_Y, \nabla f_Y) \le \frac{(\sigma^\nabla)^2}{q(q-1)} \int_M {f_Y}^2 + \frac{\sigma^\nabla}{q} \int_M \kappa_B^\sharp(f) f. \tag{5.10}$$

Equality holds if and only if $(M, \mathcal{F})$ *is transversally isometric to a sphere* $(S^q(1/c), G)$*, where* $c^2 = \frac{\sigma^\nabla}{q(q-1)}$ *and* G *is a discrete subgroup of* $O(q)$*.*

Proof. From Proposition 3.11, we have

$$\int_M g_Q(E^\nabla(\nabla f_Y), \nabla f_Y) \le 0. \tag{5.11}$$

By definition of E^∇, (5.11) can be rewritten as

$$\int_M \mathrm{Ric}^\nabla(\nabla f_Y, \nabla f_Y) - \frac{\sigma^\nabla}{q} \int_M |\nabla f_Y|^2 \le 0.$$

From Proposition 5.10, (5.10) is proved. Equality holds if and only if

$$\nabla\nabla f_Y = -\frac{\sigma^\nabla}{q(q-1)} f_Y g_Q.$$

By the generalized Obata theorem, the proof is completed. □

Remark. The existence of the bundle-like metric g_M for $(M, \mathcal{F})$ such that κ is basic, i.e., $\kappa \in \Omega_B^1(\mathcal{F})$, is proved in ([2]). In ([10,11]), for any bundle-like metric g_M with

$\kappa \in \Omega^1_B(\mathcal{F})$, it is proved that there exists another bundle-like metric $\tilde{g}_M$ for which the mean curvature form $\tilde{\kappa}$ is basic-harmonic. Hence all theorems in section 5 hold without the condition $\delta_B \kappa_B = 0$.

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감사의 글

제 논문의 마지막 장을 쓰면서 드.디.어. 라는 말과 함께 입가에 미소가 지워지지 않습니다. 힘들어서 포기 할까 라는 생각을 했던 제가 우습기만 합니다. 언젠가는 이렇게 좋은 일이 다가오는데 말이죠.

오늘이 있기까지 천학 비재한 저를 연구에 매진할 수 있도록 아낌없는 격려와 지도를 해주신 정승달 교수님께 진심으로 감사드립니다. 교수님의 제자로써 부족함이 없는 연구자가 될 수 있도록 노력하겠습니다. 그리고 대학 입학부터 대학원 졸업까지 열정적인 강의와 많은 지식함량에 도움을 주신 제주대 수학과 양영오 교수님, 송석준 교수님, 방은숙 교수님, 윤용식 교수님, 유상욱 교수님, 진현성 교수님께 깊은 감사를 드립니다. 또 제 논문을 심사 해주시면서 소중한 조언을 해주셨던 경희대 김병학 교수님께도 감사의 인사를 전합니다.

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2014년 6월 마지막날

<국문초록>

엽층 리만 다양체에서의 일반화된 Obata 정리

$(M, g_M, \mathcal{F})$는 엽층 $\mathcal{F}$의 여차원이 $q \geqq 2$ 이고 bundle-like 계량 g_M을 가지는 완비 연결 리만 다양체라고 하자. $(M, \mathcal{F})$이 $(q+1)$ 차원인 유클리드 공간에서 직교그룹 $O(q)$의 이산 부분그룹 G에 대하여 반지름 $1/c$인 q-구면 $(S^q(1/c), G)$와 횡단적 등장사상일 필요충분조건은 임의의 법벡터장 X와 양의 상수 c에 대하여 $\nabla_X df = -c^2 f X^b$를 만족하는 상수가 아닌 기본 함수 f가 존재할 때이다. 더욱이 M이 횡단적 등각장 $\overline{Y}$를 허용할 때, 즉, $\theta(Y) g_Q = 2 f_Y g_Q$, $(f_Y \neq 0)$인 경우에 대하여 일반화된 Obata 정리의 응용을 연구하였다.

博士學位論文

Generalized Obata theorem on a foliated Riemannian manifold

濟州大學校 大學院

數 學 科

李　今　蘭

2014年　8月

Generalized Obata theorem on a foliated Riemannian manifold

指導敎授 鄭 承 達

李 今 蘭

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李今蘭의 理學 博士學位 論文을 認准함

審査委員長 ______________________

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濟州大學校 大學院

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