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TERM RANK PRESERVERS BETWEEN DIFFERENT FUZZY MATRIX SPACES

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CONTENTS

<**Abstract**>

TERM RANK PRESERVERS BETWEEN DIFFERENT FUZZY MATRIX SPACES

In this paper we consider linear transformations from $m \times n$ fuzzy matrices into $p \times q$ fuzzy matrices that preserve term rank. We study linear transformation that preserve term rank between different fuzzy matrix spaces. This results extend the results on the linear transformation from $m \times n$ binary Boolean matrices into $p \times q$ binary Boolean matrices that preserve term rank.

The term rank of a matrix A is the minimal number k such that all the nonzero entries of A are contained in h rows and $k-h$ columns. The term rank of a matrix A is denoted by $\tau(A)$.

Let $\mathbb{R}$ be the field of reals, let $\mathcal{F} = \{\alpha \in \mathbb{R} | 0 \leq \alpha \leq 1\}$ denote a subset of reals. Define $a + b = max\{a, b\}$ and $a \cdot b = min\{a, b\}$ for all $a, b \in \mathcal{F}$. Thus $(\mathcal{F}, +, \cdot)$ is a commutative antinegative semiring. Then $(\mathcal{F}, +, \cdot)$ is called a fuzzy semiring.

For a linear transformation $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{p,q}(\mathcal{F})$, first, we say that T preserves term rank k if $\tau(T(X)) = k$ whenever $\tau(X) = k$ for all $X \in \mathbb{M}_{m,n}(\mathcal{F})$. Second, T strongly preserves term rank k provided that $\tau(T(X)) = k$ if and only if $\tau(X) = k$ for all $X \in \mathbb{M}_{m,n}(\mathcal{F})$. Finally, we say that T preserves term rank if it preserves term rank k for every $k(\leq m)$.

We characterize the linear transformation that preserve term rank of fuzzy matrices. The following is the main theorem:

Theorem. Let $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{p,q}(\mathcal{F})$ be a linear transformation. Then the following are equivalent:

1. T preserves term rank;
2. T preserves term rank k and term rank h, with $1 \leq k < h \leq m \leq n$ and $k + 1 \leq m$;
3. T strongly preserves term rank g, with $1 \leq g \leq m \leq n$;
4. T is of the form : $T(X) = P[(X \circ B) \oplus O]Q$ or $P[(X \circ B)^t \oplus O]Q$ for some permutation matrices P and Q.

1 Introduction

There are many papers on linear operators on a matrix space that preserve matrix functions over various algebraic structures. But there are few papers of linear transformations from one matrix space into another matrix space that preserve matrix functions over an algebraic structure. In this paper we consider linear transformations from $m \times n$ fuzzy matrices into $p \times q$ fuzzy matrices that preserve term rank.

A *semiring* [2] is a set $\mathbb{S}$ equipped with two binary operations $+$ and $\cdot$ such that $(\mathbb{S}, +)$ is a commutative monoid with identity element 0 and $(\mathbb{S}, \cdot)$ is a monoid with identity element 1. In addition, the operations $+$ and $\cdot$ are connected by distributivity of $\cdot$ over $+$, and 0 annihilates $\mathbb{S}$.

Hereafter, $\mathbb{S}$ will be denote an arbitrary commutative and antinegative semiring. For all x, y $\in \mathbb{S}$, we supress the dot of x $\cdot$ y, and simply write xy. Let $\mathbb{M}_{m,n}(\mathbb{S})$ and $\mathbb{M}_{p,q}(\mathbb{S})$ be the set of all $m \times n$ and $p \times q$ matrices respectively with entries in a $\mathbb{S}$. Algebraic operations on $\mathbb{M}_{m,n}(\mathbb{S})$ and $\mathbb{M}_{p,q}(\mathbb{S})$ are defined as if the underlying scalars were in a field.

The *term rank*, $\tau(A)$, of a matrix A is the minimal number k such that all the nonzero entries of A are contained in h rows and $k - h$ columns. Term rank plays a central role in combinatorial matrix theory and has many applications in network and graph theory (see [4]). And the *line* means rows or columns.

From now on we will assume that $2 \leq m \leq n$. It follows that $1 \leq \tau(A) \leq m$ for all nonzero $A \in \mathbb{M}_{m,n}(\mathbb{S})$.

Let $\mathbb{N}_k^{(r,s)}$ denote the set of all matrices in $\mathbb{M}_{r,s}(\mathbb{S})$ whose term rank is k.

Let $T : \mathbb{M}_{m,n}(\mathbb{S}) \to \mathbb{M}_{p,q}(\mathbb{S})$ be a linear transformation. If f is a function defined on $\mathbb{M}_{m,n}(\mathbb{S})$ and on $\mathbb{M}_{p,q}(\mathbb{S})$, then T *preserves* the function f if $f(T(A)) = f(A)$ for all $A \in \mathbb{M}_{m,n}(\mathbb{S})$. If $\mathbb{X}$ is a subset of $\mathbb{M}_{m,n}(\mathbb{S})$ and $\mathbb{Y}$ is a subset of $\mathbb{M}_{p,q}(\mathbb{S})$, then T *preserves* the pair $(\mathbb{X}, \mathbb{Y})$ if $A \in \mathbb{X}$ implies $T(A) \in \mathbb{Y}$. Further, T *strongly preserves* the pair $(\mathbb{X}, \mathbb{Y})$ if $A \in \mathbb{X}$ if and only if $T(A) \in \mathbb{Y}$. Further, we say that T *(strongly) preserves term rank* k if T (strongly) preserves the pair $(\mathbb{N}_k^{(r,s)},\mathbb{N}_k^{(p,q)})$.

Beasley and Pullman ([2]) have characterized linear operators on the m×n Boolean matrices that preserve term rank, and the following are main results of their work: for a linear operator on the m×n Boolean matrices,

$$T \text{ preserves term rank if and only if } T \text{ preserves term ranks 1 and 2;} \tag{1.1}$$

$$T \text{ preserves term rank if and only if } T \text{ strongly preserves term rank 1 or } m. \tag{1.2}$$

Kang, Song and Beasley ([5]) also have characterized linear operators on the m×n matrices over commutative antinegative semiring that preserve term rank, and the following are main results of their work: for a linear operator on the m×n commutative antinegative semiring matrices,

$$T \text{ preserves term rank if and only if } T \text{ preserves term ranks 1 and } k. \tag{1.3}$$

Song and Beasley ([7]) have obtained the characterizations of the linear transformation from the $m \times n$ Boolean matrices into $p \times q$ Boolean matrices.

Note that if $1 \leq k \leq m \leq n$ and $T : \mathbb{M}_{m,n}(\mathbb{S}) \to \mathbb{M}_{p,q}(\mathbb{S})$ preserves term rank k then necessarily $k \leq min(p,q)$.

In this paper, we extend the results of Song and Beasley ([7]) to the fuzzy matrices. A sectional summary is as follows: Some definitions and preliminaries are presented in Section 2. Section 3 generalizes the result in ([7]) by showing that linear transformation T from $m \times n$ fuzzy matrices into $p \times q$ fuzzy matrices preserves term rank if and only if T preserves term ranks k and h, where $1 \leq k < h \leq m \leq n$. And we have other characterizations.

2 Preliminaries

In this section, we give some definitions and basic results for our main results.

Definition 2.1. *A* **semiring** $\mathbb{S}$ *consist of a set* $\mathbb{S}$ *and two binary operations, addition and multiplication, such that ;*

- $\mathbb{S}$ *is an Abelian monoid under addition (identity denoted by 0) ;*
- $\mathbb{S}$ *is a semigroup under multiplication (identity, if any, denoted by 1);*
- *multiplication is distributive over addition on both sides ;*
- $s0 = 0s = 0$ *for all* $s \in \mathbb{S}$

In particular, a semiring $\mathbb{S}$ is called **antinegative** if 0 is the only element to have an additive inverse.

The following are some examples of antinegative semirings which occur in combinatorics. Let $\mathbb{B} = \{0, 1\}$. Then $(\mathbb{B}, +, \cdot)$ is an antinegative semiring (the *binary Boolean semiring*) if arithmetic in $\mathbb{B}$ follows the usual rules except that $1 + 1 = 1$. And $\mathbb{Z}^+$, the nonnegative integers, is an antinegative semiring too.

Definition 2.2. *Let* $\mathbb{R}$ *be the field of reals, let* $\mathcal{F} = \{\alpha \in \mathbb{R} | 0 \leq \alpha \leq 1\}$ *denote a subset of reals. Define*

$a + b = max\{a, b\}$ *and* $a \cdot b = min\{a, b\}$

for all $a, b \in \mathcal{F}$. *Thus* $(\mathcal{F}, +, \cdot)$ *is a commutative antinegative semiring. Then* $(\mathcal{F}, +, \cdot)$ *is called a* **fuzzy semiring**.

Let $\mathbb{M}_{m,n}(\mathcal{F})$ *denote the set of all* $m \times n$ *matrices with entries in a fuzzy semiring* $\mathcal{F}$. *We call a matrix in* $\mathbb{M}_{m,n}(\mathcal{F})$ *as a* **fuzzy matrix**.

Definition 2.3. *The matrix* $A^{(m,n)}$ *denotes a matrix in* $\mathbb{M}_{m,n}(\mathcal{F})$, $O^{(m,n)}$ *is the* $m \times n$ *zero matrix,* I_n *is the* $n \times n$ *identity matrix,* $I_k^{(m,n)} = I_k \oplus O_{m-k,n-k}$, *and* $J^{(m,n)}$ *is the* $m \times n$ *matrix all of whose entries are* 1. *Let* $E_{i,j}^{(m,n)}$ *be the* $m \times n$ *matrix whose* (i,j)*th entry is* 1 *and whose other entries are all* 0, *and we call* $E_{i,j}^{(m,n)}$ *a* **cell**. *An* $m \times n$ *matrix* $L^{(m,n)}$ *is called a* **full line matrix** *if*

$$L^{(m,n)} = \sum_{l=1}^{n} E_{i,l}^{(m,n)} \quad or \quad L^{(m,n)} = \sum_{k=1}^{m} E_{k,j}^{(m,n)}$$

for some $i \in \{1, \ldots, m\}$ *or for some* $j \in \{1, \ldots, n\}$; $R_i^{(m,n)} = \sum_{l=1}^{n} E_{i,l}^{(m,n)}$ *is the* i*th full row matrix and* $C_j^{(m,n)} = \sum_{k=1}^{m} E_{k,j}^{(m,n)}$ *is the* j*th full column matrix. We will suppress the subscripts or superscripts on these matrices when the orders are evident from the context and we write* A, O, I, I_k, J, $E_{i,j}$, L, R_i *and* C_j *respectively.*

Definition 2.4. *A* **line** *of matrix* $A \in \mathbb{M}_{m,n}(\mathcal{F})$ *is a row or a column of the matrix* A.

Definition 2.5. *A matrix* $A \in \mathbb{M}_{m,n}(\mathcal{F})$ *has* **term rank** k ($\tau(A) = k$) *if the least number of lines needed to include all nonzero elements of* A *is equal to* k.

Lemma 2.6. *For matrices* A *and* B *in* $\mathbb{M}_{m,n}(\mathcal{F})$, *we have* $\tau(A+B) \leq \tau(A) + \tau(B)$.

Proof. If $\tau(A) = r$, $\tau(B) = s$, then there exist r lines for A and s lines for B which covers all nonzero entries of A and B respectively. If these lines are all different, then $\tau(A+B) = r+s$. But if there were the same lines for the covering *of the nonzero entries for A and B, then $\tau(A+B) < r+s = \tau(A)+\tau(B)$. Thus $\tau(A+B) \leq \tau(A)+\tau(B)$.* ∎

Lemma 2.7. *For matrices A and B in $\mathbb{M}_{m,n}(\mathcal{F})$, we have $\tau(A) \leq \tau(A+B)$.*

Proof. If $\tau(A) = r$, then there exist r lines that cover all nonzero entries of A. If these lines cover all nonzero entries of B, then $\tau(A) = \tau(A+B)$. But if not, $\tau(A) < \tau(A+B)$. Thus, $\tau(A) \leq \tau(A+B)$ ∎

Definition 2.8. *If A and B are matrices in $\mathbb{M}_{m,n}(\mathcal{F})$, we say that B* **dominates** *A (written $A \sqsubseteq B$ or $B \sqsupseteq A$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i and j. This provides a reflexive and transitive relation on $\mathbb{M}_{m,n}(\mathcal{F})$.*

Lemma 2.9. *For matrices A and B in $\mathbb{M}_{m,n}(\mathcal{F})$, $A \sqsubseteq B$ implies that $\tau(A) \leq \tau(B)$.*

Proof. If $\tau(B) = r$, then there exist r lines that cover all nonzero entries of B. Since $A \sqsubseteq B$, these lines cover all nonzero entries of A. Thus $\tau(A) \leq r = \tau(B)$.

∎

Definition 2.10. *For any matrix A and lists L_1 and L_2 of row and column indices respectively, $A(L_1 \mid L_2)$ denotes the submatrix formed by omitting the rows L_1 and columns L_2 from A and $A[L_1 \mid L_2]$ denotes the submatrix formed by choosing the rows L_1 and columns L_2 from A.*

Definition 2.11. *For matrices A and B in $\mathbb{M}_{m,n}(\mathcal{F})$, the matrix $A \circ B$ denotes the* **Hadamard** *or* **Schur product**. *That is, the $(i,j)^{th}$ entry of $A \circ B$ is $a_{i,j}b_{i,j}$.*

Definition 2.12. *If $\mathcal{F}$ is a fuzzy semiring, $1 \le m,n$ and $1 \le p,q$, and $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{p,q}(\mathcal{F})$, then T is a (P,Q,B)***-block-transformation** *if there are permutation matrices $P \in \mathbb{M}_p(\mathcal{F})$ and $Q \in \mathbb{M}_q(\mathcal{F})$, and $B \in \mathbb{M}_{m,n}(\mathcal{F})$ with $b_{i,j}$ are nonzero, such that*

- *$m \le p$ and $n \le q$, and $T(A) = P[(A \circ B) \oplus O]Q$ for all $A \in \mathbb{M}_{m,n}(\mathcal{F})$ or*
- *$m \le q$ and $n \le p$, and $T(A) = P[(A \circ B)^t \oplus O]Q$ for all $A \in \mathbb{M}_{m,n}(\mathcal{F})$.*

Definition 2.13. *If $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{p,q}(\mathcal{F})$ is a (P,Q,B)-block-transformation and $B = J$, then T is a (P,Q)***-block-transformation**.

3 Characterizations of term rank preservers of fuzzy matrices.

In this section, we give the lemmas and theorems for the linear transformation that preserve term rank of fuzzy matrices. We also give suitable example. As their results, we have characterization of term rank preservers of fuzzy matrices between different fuzzy matrix spaces, which are contained in Theorem 3.18. These results extend those results Boolean matrix in [7].

Definition 3.1. *For a linear transformation $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{p,q}(\mathcal{F})$, we say that T*

(1) **preserves term rank** k *if* $\tau(T(X)) = k$ *whenever* $\tau(X) = k$ *for all* $X \in \mathbb{M}_{m,n}(\mathcal{F})$*, or equivalently if* T *preserves the pair* $(\mathbb{N}_k^{(r,s)},\mathbb{N}_k^{(p,q)})$*;*

(2) **strongly preserves term rank** k *if* $\tau(T(X)) = k$ *if and only if* $\tau(X) = k$ *for all* $X \in \mathbb{M}_{m,n}(\mathcal{F})$*, or equivalently if* T *strongly preserves the pair* $(\mathbb{N}_k^{(r,s)},\mathbb{N}_k^{(p,q)})$*;*

(3) **preserves term rank** *if it preserves term rank* k *for every* $k(\leq m)$*.*

Example 3.2. *Let* $T : \mathbb{M}_{2,3}(\mathcal{F}) \to \mathbb{M}_{3,4}(\mathcal{F})$ *is a* (P, Q, B)*-block-transformation, and*

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}.$$

Then for $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & \frac{1}{5} & \frac{1}{3} \end{pmatrix} \in \mathbb{M}_{2,3}(\mathcal{F})$, *we have* $A \circ B = \begin{pmatrix} \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{5} & \frac{1}{6} \end{pmatrix}$

and $T(A) = P[(A \circ B) \oplus O]Q = P[\begin{pmatrix} \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{5} & \frac{1}{6} \end{pmatrix} \oplus [O]_{1\times 1}]Q$

$$=P\begin{pmatrix} \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 0 \\ 0 & \frac{1}{5} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{3} & 0 & \frac{1}{2} \\ 0 & \frac{1}{5} & 0 & \frac{1}{6} \end{pmatrix}.$$

Thus $\tau(A) = 2$, *and* $\tau(T(A)) = 2$.

Lemma 3.3. *Let* $1 \leq m, n$ *and* $1 \leq p, q$ *and* $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{m,n}(\mathcal{F})$. *If* T *is a* (P, Q, B)*-block-transformation, then* T *strongly preserves term rank k.*

Proof. Assume that T is a (P, Q, B)-block-transformation, and A$\in \mathbb{M}_{m,n}(\mathcal{F})$ with $\tau(A) = k$ with $1 \leq k \leq m$. Then $T(A) = P[(A \circ B) \oplus O]Q$ or $T(A) = P[(A \circ B)^t \oplus O]Q$.

Case 1. Let $T(A) = P[(A\circ B)\oplus O]Q$. Since B has no zeros, $\tau(A\circ B) = \tau(A)$. And $\tau((A\circ B)\oplus O) = \tau(A\circ B)$. Moreover the permuting rows and columns does not change the term rank, $\tau(T(A)) = \tau(P[(A\circ B)\oplus O]Q) = \tau((A\circ B)\oplus O)$. Thus T preserves term rank k. If $\tau(T(A)) = k$, then $\tau(T(A)) = \tau(P[(A \circ B) \oplus O]Q) = \tau((A \circ B) \oplus O) = \tau(A \circ B) = \tau(A)$.Thus $\tau(A) = k$. That is, T strongly preserves term rank k.

Case 2. Let $T(A) = P[(A \circ B)^t \oplus O]Q$. As in Case 1, a parallel argument shows the same results. That is, T strongly preserves term rank k. ■

Theorem 3.4. *Let* $1 \leq m, n$ *and* $1 \leq p, q$ *and* $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{m,n}(\mathcal{F})$. *Then* T *strongly preserves term rank 1 if and only if* T *is a* (P, Q, B)*-block-transformation. (Necessarily, either* $m \leq p$ *and* $n \leq q$, *or* $m \leq q$ *and* $n \leq p$.*)*

Proof. If T is a (P, Q, B)-block-transformation, then T strongly preserves term rank 1 by Lemma 3.3.

Assume that T strongly preserves term rank 1. Then, the image of each line in $\mathbb{M}_{m,n}(\mathcal{F})$ is a line in $\mathbb{M}_{m,n}(\mathcal{F})$. We may assume that either $T(R_1^{(m,n)}) \sqsubseteq R_1^{(p,q)}$ or $T(R_1^{(m,n)}) \sqsubseteq C_1^{(p,q)}$.

Case 1. $T(R_1^{(m,n)}) \sqsubseteq R_1^{(p,q)}$. Suppose that $T(C_j^{(m,n)}) \sqsubseteq R_i^{(p,q)}$. Then, since $E_{1,j}^{(m,n)}$ is in both $R_1^{(m,n)}$ and $C_j^{(m,n)}$ and since $T(E_{1,j}^{(m,n)}) \neq O$, we must have $i = 1$. But then, for $j \neq k$ $T(E_{2,j}^{(m,n)} + E_{1,k}^{(m,n)}) \sqsubseteq R_1^{(m,n)}$ and hence, has term rank 1. But $\tau(E_{2,j}^{(m,n)} + E_{1,k}^{(m,n)}) = 2$, a contradiction. Thus the image of any column is dominated by a column. Similarly, the image of any row is dominated by a row. Further, since the sum of two rows (columns) has term rank 2, the image of distinct rows (columns) must be dominated by distinct columns. Let $\phi : \{1, \cdots m\} \to \{1, \cdots, p\}$ be a mapping defined by $\phi(i) = j$ if $T(R_i^{(m,n)}) \sqsubseteq R_j^{(p,q)}$ and define $\theta : \{1, \cdots n\} \to \{1, \cdots, p\}$ by $\theta(i) = j$ if $T(C_i^{(m,n)}) \sqsubseteq C_j^{(p,q)}$. Then, it is easily seen that ϕ and θ are one-to-one mappings, and hence, $m \leq p$ and $n \leq q$. Let $\phi' : \{1, \cdots, p\} \to \{1, \cdots, p\}$ and $\theta' : \{1, \cdots, q\} \to \{1, \cdots, q\}$ be one-to-one mappings such that $\phi' \mid_{\{1, \cdots m\}} = \phi$ and $\theta' \mid_{\{1, \cdots n\}} = \theta$. Let $P_{\phi'}$ and $Q_{\theta'}$ denote the permutation matrices corresponding to the permutations ϕ' and θ'.

In this case we have that $m \leq p$ and $n \leq q$, there is some nonzero $b_{i,j} \in \mathcal{F}$ such that $B = [b_{i,j}]$, $T(E_{i,j}) = b_{i,j}(P_{\phi'}[E_{r,s} \oplus O]Q_{\theta'})$ for every cell $E_{i,j}$. Thus,

$$\begin{aligned} T(A) &= T\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}E_{i,j}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}T(E_{i,j}) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}b_{i,j}(P[E_{i,j} \oplus O]Q) = P[(A \circ B) \oplus O]Q \end{aligned}$$

for every $A \in \mathbb{M}_{m,n}(\mathcal{F})$. That is, T is a (P, Q, B)-block-transformation.

Case 2. $T(R_1^{(m,n)}) \sqsubseteq C_1^{(p,q)}$. As in case 1, a parallel argument shows that

$m \leq q$ and $n \leq p$. Then we have $T(E_{i,j}) = b_{i,j}(P_{\phi'}[E_{i,j} \oplus O]^t Q_{\theta'})$ for all $E_{i,j}$. Thus

$$\begin{aligned} T(A) &= T\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}E_{i,j}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}T(E_{i,j}) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}b_{i,j}(P[E_{i,j} \oplus O]^t Q) = P[(A \circ B)^t \oplus O]Q \end{aligned}$$

for every $A \in \mathbb{M}_{m,n}(\mathcal{F})$, and consequently that T is a (P, Q, B)-block-transformation.

■

Lemma 3.5. *Let $2 \leq k \leq m \leq n$. If $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{m,n}(\mathcal{F})$ is a linear transformation that preserves term rank k and term rank 1, then T strongly preserves term rank 1.*

Proof. Case 1. Assume that k=2. For $A \in \mathbb{M}_{m,n}(\mathcal{F})$ with $\tau(A) = 1$, $\tau(T(A)) = 1$.

For $B \in \mathbb{M}_{m,n}(\mathcal{F})$ with $\tau(T(B)) = 1$, assume $\tau(B) \neq 1$. Then $\tau(B) \geq 2$. But $\tau(B) \neq 2$ since $\tau(B) = 2$ implies $\tau(T(B)) = 2$, a contradiction. Thus $\tau(B) \geq 3$. Let $B_1 \sqsubseteq B$ such that $\tau(B_1) = 2$ and $B = B_1 + B_2$ with $\tau(B_2) \geq 1$. Then $T(B_1) \sqsubseteq T(B_1) + T(B_2) = T(B_1 + B_2) = T(B)$. Thus $2 = \tau(T(B_1)) \leq \tau(T(B)) = 1$ by Lemma 2.9. It leads a contradiction. That is, T strongly preserves term rank 1.

Case 2. Assume that $k \geq 3$. Suppose a term rank 2 matrix is mapped to a term rank 1 matrix. Without loss of generality, $\tau(T(E_{1,1} + E_{2,2})) = 1$. But then, since T preserves term rank 1, $\tau(T(E_{1,1} + E_{2,2} + E_{3,3} + \cdots + E_{k,k})) = \tau(T(E_{1,1} + E_{2,2}) + T(E_{3,3}) + \cdots + T(E_{k,k})) \leq \tau(T(E_{1,1} + E_{2,2})) + \tau(T(E_{3,3})) + \cdots + \tau(T(E_{k,k}))) = 1 + (k-2) < k$, a contradiction. Thus, T strongly preserves term rank 1. ■

Corollary 3.6. *Let $1 < k \le m, n$ and $1 \le p, q$ and $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{m,n}(\mathcal{F})$ be a linear transformation. Then T preserves term rank 1 and term rank k if and only if T is a (P, Q, B)-block-transformation.*

Proof. By Lemma 3.5, T strongly preserves term rank 1. By Theorem 3.4, the corollary follows. ∎

Lemma 3.7. *Let $2 \le k \le m \le n$. Let $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{m,n}(\mathcal{F})$ be a linear transformation that preserves term rank k. If T does not preserve term tank 1, then there is some term rank 1 matrix whose image has term rank at least 2.*

Proof. Suppose that T does not preserve term rank 1 and $\tau(T(A)) \le 1$ for all A with $\tau(A) = 1$. Then, there is some cell $E_{i,j}$ such that $T(E_{i,j}) = O$. Without loss of generality, assume that $T(E_{1,1}) = O$. Since $\tau(E_{1,1} + E_{2,2} + \cdots + E_{k,k}) = k$ and T preserves term rank k, we have $\tau(T(E_{2,2} + E_{3,3} + \cdots + E_{k,k})) = \tau(T(E_{1,1} + E_{2,2} + \cdots + E_{k,k})) = k$. Let $X = T(E_{2,2} + \cdots + E_{k,k})$ then we can choose a set of cells $C = \{F_1, F_2, \cdots, F_k\}$ such that $X \sqsupseteq F_i$ for all $i = 1, \cdots, k$, and $\tau(F_1 + F_2 + \cdots + F_k) = k$. Since $T(E_{2,2} + \cdots + E_{k,k}) = X$, there is some cell in $\{E_{2,2}, \cdots, E_{k,k}\}$ whose image under T dominates two cells in C, a contradiction. This contradiction establishes the lemma. ∎

Lemma 3.8. *Let $1 \le k \le m \le n$. Let $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{m,n}(\mathcal{F})$ be a linear transformation that preserves term rank k. If $A \in \mathbb{M}_{m,n}(\mathcal{F})$ and $\tau(A) \le k$ then $\tau(T(A)) \le k$.*

Proof. If $\tau(A) = k$, then $\tau(T(A)) = k$ since T preserves term rank k. Suppose that $\tau(A) = h < k$, and $\tau(T(A)) > k$. Then there exist a matrix B such that

$\tau(A+B) = k$ and hence $\tau(T(A+B)) = k$, but by Lemma 2.7, $k = \tau(T(A+B)) = \tau(T(A) + T(B)) \geq \tau(T(A)) > k$, a contradiction. Thus $\tau(T(A)) \leq k$. ■

Recall that the matrix J is the matrix whose entries are all ones.

Lemma 3.9. *Let $2 \leq k \leq m \leq n$ and $T : \mathbb{M}_{m,n}(\mathcal{F}) \rightarrow \mathbb{M}_{m,n}(\mathcal{F})$ be a linear transformation that preserves term rank k. If T does not preserve term rank 1, then $\tau(T(J)) \leq k+2$.*

Proof. By Lemma 3.7, if T does not preserve term rank 1, then there is some term rank 1 matrix whose image has term rank 2 or more. Without loss of generality, we may assume that $T(E_{1,1} + E_{1,2}) \sqsupseteq b_{1,1}E_{1,1} + b_{2,2}E_{2,2}$ with all $b_{i,j}$ is nonzero.

Suppose that $\tau(T(J)) \geq k+3$. Then, $\tau(T(J)[3, \cdots, p|3, \cdots, q]) \geq k-1$. Without loss of generality, we may assume that $T(J)[3, \cdots, p|3, \cdots, q] \sqsupseteq b_{3,3}E_{3,3} + b_{4,4}E_{4,4} + \cdots + b_{k+1,k+1}E_{k+1,k+1}$, all $b_{i,j}$ are nonzero. Thus, there are $k-1$ cells, $F_3, F_4, \cdots, F_{k+1}$ such that $T(F_3+F_4+\cdots+F_{k+1}) \sqsupseteq b_{3,3}E_{3,3}+b_{4,4}E_{4,4}+\cdots+b_{k+1,k+1}E_{k+1,k+1}$. Then, $T(E_{1,1}+E_{1,2}+F_3+F_4+\cdots+F_{k+1}) \sqsupseteq D \circ I_{k+1}$ with $d_{i,i}$ are entries of D and nonzero. But, $\tau(E_{1,1} + E_{1,2} + F_3 + F_4 + \cdots + F_{k+1}) \leq k$ while $\tau(T(E_{1,1} + E_{1,2} + F_3 + F_4 + \cdots + F_{k+1})) \geq k+1$, a contradiction. Thus, $\tau(T(J)) \leq k+2$. ■

Lemma 3.10. *Let $1 \leq k$, $k+3 \leq h \leq m \leq n$. Let $T : \mathbb{M}_{m,n}(\mathcal{F}) \rightarrow \mathbb{M}_{m,n}(\mathcal{F})$ be a linear transformation that preserves term rank k and term rank h, then T preserves term rank 1.*

Proof. Suppose that T does not preserve term rank 1. By Lemma 3.7, there is some term rank 1 matrix whose image has term rank at least 2. Let A be such a term rank 1 matrix. Then, A is dominated by a row or column and the image of the sum of two cells in that line has term rank at least two. Without loss of generality, we may assume that $T(E_{1,1}+E_{1,2}) \sqsupseteq b_{1,1}E_{1,1}+b_{2,2}E_{2,2}$, $b_{i,j}$ are nonzero. Now, by Lemma 3.9, if $B = T(C)$ is in the image of T, $\tau(B) \leq k+2 < h$. But if we take $B = T(I_h)$, then $T(I_h)$ must have term rank h, a contradiction.

That is, $\tau(T(A)) \leq 1$. Since A was an arbitrary term rank 1 matrix, T preserves term rank 1. ∎

Lemma 3.11. *Let $1 \leq k \leq m \leq n$. If $T : \mathbb{M}_{m,n}(\mathcal{F}) \rightarrow \mathbb{M}_{m,n}(\mathcal{F})$ is a linear transformation that preserves term rank k and term rank $k+2$, then T strongly preserves term rank $k+1$.*

Proof. Let $A \in \mathbb{M}_{m,n}(\mathcal{F})$.

Case 1. Suppose that $\tau(A) = k+1$ and $\tau(T(A)) \geq k+2$. Let $A_1, A_2, \cdots, A_{k+1}$ be matrices of term rank 1 such that $A = A_1 + A_2 + \cdots + A_{k+1}$. Without loss of generality we may assume that $T(A) \sqsupseteq b_{1,1}E_{1,1} + b_{2,2}E_{2,2} + \cdots + b_{k+2,k+2}E_{k+2,k+2}$ with all $b_{i,j}$ are nonzero, and since the image of some A_i must have term rank at least 2, we may assume that $\tau(T(A_1 + A_2 + \cdots + A_i)) \geq i+1$, for every $i = 1, 2, \cdots k+1$. But then $\tau(A_1 + A_2 + \cdots + A_k) = k$ while $\tau(T(A_1 + A_2 + \cdots + A_k)) \geq k+1$, a contradiction, Thus if $\tau(A) = k+1$, $\tau(T(A)) \leq k+1$.

Case 2. Suppose that $\tau(A) = k+1$ and $\tau(T(A)) = s \leq k$. Without loss of generality, we may assume that $A = b_{1,1}E_{1,1} + b_{2,2}E_{2,2} + \cdots + b_{k+1,k+1}E_{k+1,k+1}$ and $T(A) \sqsupseteq b_{1,1}E_{1,1} + b_{2,2}E_{2,2} + \cdots + b_{s,s}E_{s,s}$. Then there are s members of $\{T(b_{1,1}E_{1,1}), b_{2,2}T(E_{2,2}), \cdots, T(b_{k+1,k+1}E_{k+1,k+1})\}$ whose sum dominates $b_{1,1}E_{1,1} + b_{2,2}E_{2,2} + \cdots + b_{s,s}E_{s,s}$. Say, without loss of generality, that $T(b_{1,1}E_{1,1} + b_{2,2}E_{2,2} + \cdots + b_{s,s}E_{s,s}) \sqsupseteq b_{1,1}E_{1,1} + b_{2,2}E_{2,2} + \cdots + b_{s,s}E_{s,s}$. Now, $\tau(A + b_{k+2,k+2}E_{k+2,k+2}) = k+$

2 so that $\tau(T(A+b_{k+2,k+2}E_{k+2,k+2}))=k+2$. But since $\tau(T(A+b_{k+2,k+2}E_{k+2,k+2}))=\tau((T(A)+T(b_{k+2,k+2}E_{k+2,k+2}))\le\tau(T(A))+\tau(T(b_{k+2,k+2}E_{k+2,k+2}))$, it follows that $\tau(T(b_{k+2,k+2}E_{k+2,k+2}))\ge k+2-s$ and there are s members of $\{T(b_{1,1}E_{1,1}),T(b_{2,2}E_{2,2}),\cdots,T(b_{k+1,k+1}E_{k+1,k+1})\}$ whose sum together with $T(b_{k+2,k+2}E_{k+2,k+2})$ has term rank $k+2$, say $\tau(T(b_{1,1}E_{1,1}+b_{1,1}E_{2,2}+\cdots+b_{s,s}E_{s,s}+b_{k+2,k+2}E_{k+2,k+2}))=k+2$. Since $s\le k$, $\tau(b_{1,1}E_{1,1}+b_{2,2}E_{2,2}+\cdots+b_{s,s}E_{s,s}+b_{k+2,k+2}E_{k+2,k+2})\le k+1$ and $\tau(T(b_{1,1}E_{1,1}+b_{2,2}E_{2,2}+\cdots+b_{s,s}E_{s,s}+b_{k+2,k+2}E_{k+2,k+2}))=k+2$. By Case 1, we again arrive at a contradiction.

Therefore T strongly preserves term rank $k+1$. ∎

Lemma 3.12. *Let $1\le k\le r,s$. If $\tau(b_{1,1}E_{1,1}+\cdots+b_{k,k}E_{k,k}+A)\ge k+1$ with all $b_{p,q}$ are nonzero and $A[k+1,\cdots,r|k+1,\cdots,s]=O$, then there is some $i,1\le i\le k$, such that $\tau(b_{1,1}E_{1,1}+\cdots+b_{i-1,i-1}E_{i-1,i-1}+b_{i+1,i+1}E_{i+1,i+1}+\cdots+b_{k,k}E_{k,k}+A)\ge k+1$.*

Proof. Suppose that $B=b_{1,1}E_{1,1}+\cdots+b_{k,k}E_{k,k}+A$ with all $b_{p,q}$ are nonzero and $\tau(B)\ge k+1$. Then there are $k+1$ cells $F_1,F_2,\cdots,F_{k+1}$ such that $B\sqsupseteq F_1+F_2+\cdots+F_{k+1}$ and $\tau(F_1+F_2+\cdots+F_{k+1})=k+1$. If $F_1+F_2+\cdots+F_{k+1}\sqsupseteq I_k\oplus O$ then one cell F_j must be a cell $E_{a,b}$ where $a,b\ge k+1$, which contradicts the assumption $A[k+1,\cdots,r|k+1,\cdots,s]=O$. Thus $F_1+F_2+\cdots+F_{k+1}$ does not dominate $I_k\oplus O$. That is, there is some $i,1\le i\le k$, such that $\tau(b_{1,1}E_{1,1}+\cdots+b_{i-1,i-1}E_{i-1,i-1}+b_{i+1,i+1}E_{i+1,i+1}+\cdots+b_{k,k}E_{k,k}+A)\ge k+1$. ∎

Lemma 3.13. *Let $2\le k+1\le m\le n$. If $T:\mathbb{M}_{m,n}(\mathcal{F})\to\mathbb{M}_{m,n}(\mathcal{F})$ is a linear transformation that preserves term rank k and term rank $k+1$, then T preserves term rank 1.*

Proof. If $k = 1$, the lemma holds. Suppose that $k \geq 2$.

Suppose that T does not preserve term rank 1. Then there is some matrix of term rank 1 whose image has term rank at least 2. Without loss of generality, we may assume that $T(E_{1,1} + E_{1,2}) \sqsupseteq b_{1,1}E_{1,1} + b_{2,2}E_{2,2}$ with all $b_{i,j}$ are nonzero. By Lemma 3.9 we have that $\tau(T(J)) \leq k + 2$. Since T preserves term rank $k + 1$, $\tau(T(J)) \geq k + 1$.

Thus, $\tau(T(J)) = k + i$ for either $i = 1$ or $i = 2$. Now, we may assume that for some r, s with $r + s = k + i$, $T(J)[r + 1, \cdots, p|s + 1, \cdots, q] = O$. Further, we may assume, without loss of generality, that there are $k + i$ term rank 1 matrices $c_1F_1, c_2F_2, \cdots, c_{k+i}F_{k+i}$ with all c_i are nonzero such that $T(c_lF_l) \sqsupseteq b_{1,k+i-l+1}E_{l,k+i-l+1}$ for $l = 1, \cdots, k+i$. Suppose the image of one of the term rank 1 matrices in $c_1F_1, c_2F_2, \cdots, c_{k+i}F_{k+i}$ dominates more than one cell in $\{b_{1,k+i}E_{1,k+i}$, $b_{2,k+i-1}E_{2,k+i-1}, \cdots, b_{k+1,i}E_{k+1,i}\}$. Say, without loss of generality, that $T(c_1F_1) \sqsupseteq b_{1,k+i}E_{1,k+i} + b_{2,k+i-1}E_{2,k+i-1}$, then, $T(c_1F_1 + c_3F_3 + \cdots + c_{k+1}F_{k+1}) \sqsupseteq b_{1,k+i}E_{1,k+i} + b_{2,k+i-1}E_{2,k+i-1} + \cdots + b_{k+1,i}E_{k+1,i}$, a contradiction since $\tau(c_1F_1 + c_3F_3 + \cdots + c_{k+1}F_{k+1}) \leq k$, and hence $\tau(T(c_1F_1 + c_3F_3 + \cdots + c_{k+1}F_{k+1})) \leq k$, and $\tau(b_{1,k+i}E_{1,k+i} + b_{2,k+i-1}E_{2,k+i-1} + \cdots + b_{k+1,i}E_{k+1,i}) = k+1$. It follows that for each $j = 1, \cdots, k+1$, if $T(c_lF_l) \sqsupseteq b_{j,k+i-j+1}E_{j,k+i-j+1}$ then $l = j$ since $T(c_jF_j) \sqsupseteq b_{j,k+i-j+1}E_{j,k+i-j+1}$ is unique. Further, by permuting we may assume that

$$c_1F_1 + c_2F_2 + \cdots + c_kF_k \sqsubseteq \begin{bmatrix} J_k & O_{k,n-k} \\ O_{m-k,k} & O_{m-k,n-k} \end{bmatrix}.$$

Now, let $O \neq A \in \mathbb{M}_{m,n}(\mathcal{F})$ have term rank 1, and suppose that

$A[1, 2, \cdots, k|1, 2, \cdots, n] = O$ and $A[1, \cdots m|1, \cdots, k] = O$.

So that $A = \begin{bmatrix} O_k & O_{k,n-k} \\ O_{m-k,k} & A_1 \end{bmatrix}$.

If $T(A)[k + 1, \cdots, p|k + 1, \cdots, q] = O$, then, since $\tau(c_1F_1 + \cdots + c_kF_k + A) =$

$k+1$, $\tau(T(c_1F_1+\cdots+c_kF_k+A)) = k+1$. Applying Lemma 3.12, we have that there is some j such that $\tau(T(c_1F_1 + \cdots + c_{j-1}F_{j-1} + c_{j+1}F_{j+1} + \cdots + c_kF_k + A)) = k + 1$. But $\tau(c_1F_1 + \cdots + c_{j-1}F_{j-1} + c_{j+1}F_{j+1} + \cdots + c_kF_k + A) = k$ while $\tau(T(c_1F_1+\cdots+c_{j-1}F_{j-1}+c_{j+1}F_{j+1}+\cdots+c_kF_k+A)) = k+1$, a contradiction. So we can say that $T(b_{k+1,k+11}E_{k+1,k+1})[k+1|k+1] \neq O$ and $T(b_{k+1,k+2}E_{k+1,k+2})[k+1|k+2] \neq O$. If $T(b_{k,k+1}E_{k,k+1})[k+1,\cdots,p|k+1,\cdots,q] \neq O$ then $\tau(T(c_1F_1 + \cdots + c_kF_k + b_{k,k+1}E_{k,k+1})) \geq \tau(b_{1,k+1}E_{1,k+1} + b_{2,k}E_{2,k} + b_{3,k-1}E_{3,k-1} + \cdots + b_{k-1,3}E_{k-1,3} + b_{k,2}E_{k,2} + T(E_{k,k+1})) \geq k+1$, a contradiction since $\tau(c_1F_1 + \cdots + c_kF_k + b_{k,k+1}E_{k,k+1}) = k$. Thus, $T(b_{k,k+1}E_{k,k+1})[k+1,\cdots,p|k+1,\cdots,q] = O$. Suppose that $(k,1),(k,2)and(k,k+2),\cdots,(k,q)$ entries of $T(b_{k,k+1}E_{k,k+1})$ is nonzero, then, $\tau(T(c_1F_1+\cdots+c_{k-1}F_{k-1}+b_{k,k+1}E_{k,k+1}+b_{k+1,k+1}E_{k+1,k+1})) \geq k+1$, a contradiction, since $\tau(c_1F_1+\cdots+c_{k-1}F_{k-1}+b_{k,k+1}E_{k,k+1}+b_{k+1,k+1}E_{k+1,k+1}) = k$.

Consider $T(c_1F_1+\cdots+c_{k-1}F_{k-1}+b_{k,k+1}E_{k,k+1}+b_{k+1,k+2}E_{k+1,k+2})$. This must have term rank $k+1$ and dominates $b_{1,k+1}E_{1,k+1} + b_{2,k}E_{2,k} + \cdots + b_{k-1,3}E_{k-1,3} + b_{k+1,k+2}E_{k+1,k+2}$. Thus, by Lemma 3.12, there is some term rank 1 matrix in $\{c_1F_1,\cdots,c_{k-1}F_{k-1}\}$, say c_jF_j such that $\tau(T(c_1F_1 + \cdots + c_{j-1}F_{j-1} + c_{j+1}F_{j+1} + \cdots + c_{k-1}F_{k-1} + b_{k,k+1}E_{k,k+1} + b_{k+1,k+2}E_{k+1,k+2})) = k+1$. But $\tau(c_1F_1 + \cdots + c_{j-1}F_{j-1} + c_{j+1}F_{j+1} + \cdots + c_{k-1}F_{k-1} + b_{k,k+1}E_{k,k+1} + b_{k+1,k+2}E_{k+1,k+2}) = k$, a contradiction.

It follows that T must preserve term rank 1. ∎

Lemma 3.14. *Let $2 \leq k \leq m \leq n$. If $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{m,n}(\mathcal{F})$ is a linear transformation that strongly preserves term rank k, Then T preserves term rank $k-1$.*

Proof. If $k = 2$, the lemma holds. Suppose that $k \geq 3$.

Let $A \in \mathbb{M}_{m,n}(\mathcal{F})$ and $\tau(A) = k-1$, and suppose that $\tau(T(A)) = s < k-1$. Without loss of generality, we may assume that $\tau(T(E_{1,1}+\cdots+E_{k-1,k-1})) = s < k-1$. Since $\tau(T(E_{1,1}+\cdots+E_{k,k})) = k$, we have that $\tau(T(E_{k,k})) \geq k-s$. Without loss of generality we may assume that $T(E_{1,1}+\cdots+E_{k,k}) \sqsupseteq b_{1,1}E_{1,1}+\cdots+b_{k,k}E_{k,k}$ with all $b_{i,j}$ are nonzero and that $T(E_{k,k}) \sqsupseteq b_{t+1,t+1}E_{t+1,t+1} + \cdots + b_{k,k}E_{k,k}$ for some $t \leq s$. Then, there are t cells $\{E_{i_1,i_1}, \cdots, E_{i_t,i_t}\}$ in $\{E_{1,1}, \cdots, E_{k,k}\}$ such that $T(E_{i_1,i_1} + \cdots + E_{i_t,i_t}) \sqsupseteq b_{1,1}E_{1,1} + \cdots + b_{t,t}E_{t,t}$. Then $T(E_{i_1,i_1} + \cdots + E_{i_t,i_t} + E_{k,k}) \sqsupseteq b_{1,1}E_{1,1} + \cdots + b_{k,k}E_{k,k}$. Thus $\tau(T(E_{i_1,i_1} + \cdots + E_{i_t,i_t} + E_{k,k})) = k$. But $\tau(E_{1,1} + \cdots + E_{t,t} + E_{k,k}) = t+1 \leq s+1 < (k-1)+1 = k$, which contradicts the assumption of T. Hence $\tau(T(A)) \geq k-1$. Further, $\tau(T(A)) \leq k-1$, since T strongly preserves term rank k. Thus, T preserves term rank $k-1$. ■

Lemma 3.15. *Let $2 \leq k \leq m \leq n$. If $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{m,n}(\mathcal{F})$ is a linear transformation that strongly preserves term rank k, then T preserves term rank 1.*

Proof. By Lemma 3.14, T preserves term rank $k-1$. By Lemma 3.13, T preserves term rank 1. ■

Lemma 3.16. *Let $1 \leq k < h \leq m \leq n$ and $k+1 \leq m$. If $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{m,n}(\mathcal{F})$ is a linear transformation that preserves term rank k and term rank h, then T is a (P, Q, B)-block-transformation.*

Proof. Case 1. If $h = k+1$, T preserves term rank 1 by Lemma 3.13.

Case 2. Assume $h = k+2$. By Lemma 3.11, T preserves term rank $k+1$. Thus, T preserves term rank 1 by Lemma 3.13.

Case 3. If $h \geq k+3$, T preserves term rank 1 by Lemma 3.10.

Then T preserves term rank 1 by Cases 1,2 and 3. Thus, by Lemma 3.5, T strongly preserves term rank 1. By Theorem 3.4, the lemma follows. ∎

Lemma 3.17. *Let $1 \leq k \leq m \leq n$. If $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{m,n}(\mathcal{F})$ is a linear transformation that strongly preserves term rank k, then T is a (P,Q,B)-block-transformation.*

Proof. By Lemma 3.15, T preserves term rank 1. By Lemma 3.5, T strongly preserves term rank 1. By Theorem 3.4, the lemma follows. ∎

This is the main theorem :

Theorem 3.18. *Let $T : \mathbb{M}_{m,n}(\mathcal{F}) \to \mathbb{M}_{p,q}(\mathcal{F})$ be a linear transformation. Then the following are equivalent*:

1. *T preserves term rank*;
2. *T preserves term rank k and term rank h, with $1 \leq k \leq h \leq m \leq n$ and $k+1 < m$*;
3. *T strongly preserves term rank g, with $1 \leq g \leq m \leq n$*;
4. *T is a (P,Q,B)-block transformation.*

Proof. It is obvious that 1 implies 2 and 3. And 4 implies 1, 2 and 3 by Lemma 3.3. In order to show that 2 implies 4, assume that T preserves term rank k and term rank h, with $1 \leq k < h \leq m \leq n$. Thus, by Lemma 3.16, T is a (P,Q,B)-block transformation. In order to show that 3 implies 4, if we apply Lemma 3.17, T is a (P,Q,B)-block transformation.

∎

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2012년 8월, 대학원 입학 후 벌써 2년이란 시간이 흘렀습니다. 막막해 보였던 석사 과정이 끝을 보이고 논문도 마무리하게 되었습니다. 지난 시간을 돌이켜 보면 많은 것들이 떠오르지만, 역시 가장 생각나는 것은 제가 도움 받아온 감사한 분들의 얼굴인 것 같습니다.

우선 부족한 저에게 큰 가르침을 주식 항상 격려해주신 저의 지도교수님, 송석준 교수님께 감사드립니다. 교수님 덕분에 공부뿐만 아니라 삶에 대한 태도나 습관 등 많은 것을 배웠습니다. 교수님의 제자인 것이 자랑스럽습니다.

2년의 석사 과정을 보람 있게 보낼 수 있도록 지도해주신 방은숙 교수님, 양영오 교수님, 정승달 교수님, 윤용식 교수님, 유상욱 교수님, 진현성 교수님, 강경태 선생님께도 감사 인사를 드리고 싶습니다. 옳은 방향으로 앞으로 나갈 수 있도록 피와 살이 되는 조언도 해 주시고, 때로는 인생선배로서 친근하게 살아가는 이야기도 들려주시는 교수님들. 교수님들이 아니셨으면 제 석사과정이 이토록 좋은 기억으로 남을 수 있었을까요?

공부하다 불현듯 불안해지고 답답해질 때마다 진심어린 토닥거림으로 저를 위로해주는 친구, 선배님, 후배들 감사합니다. 수다 떨면서 맛있는 음식을 먹으며 당신들과 함께 하는 시간은 과거에도, 지금도, 앞으로도 제게 가장 소중한 시간 중 하나일 거에요.

대학원 와서 새로 생긴 인연들에게도 진심으로 감사합니다. 함께 공부하는 사이라 서로의 힘든 점도 너무 잘 알고 있어서 존재만으로도 힘이 될뿐더러, 어려울 때는 서로 기대고 도움도 받으면서 2년을 무사히 보낼 수 있었던 것 같습니다. 대학원 온 덕분에 정말 좋은 사람들을 많이 알게 되었네요. 여러분에게 저 또한 오래 함께 하고 싶은 좋은 인연이었으면 좋겠습니다.

마지막으로, 항상 저를 믿고 응원해주시는 부모님, 할머니 할아버지, 외할머니 외할아버지, 동생, 그리고 광주, 경기도, 사천, 뉴질랜드에 계시는 저의 가족들. 늦은 나이에 공부하는 제게 힘이 되어주셔서 감사합니다. 저는 행복한 사람이에요. 사랑합니다.