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SPECTRAL CRITERION FOR MEMBERSHIP IN $\mathbb{A}^{2}_{1,1}(r)$

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1. INTRODUCTION

Let $\mathcal{H}$ denote a separable, infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $1_{\mathcal{H}}$ and is closed in the weak* operator topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains T and $1_{\mathcal{H}}$ and is closed in the weak* operator topology. Moreover, let $Q_{\mathcal{A}_T}$ denote the quotient space $\mathcal{C}_1(\mathcal{H})/\perp_{\mathcal{A}_T}$, where $\mathcal{C}_1(\mathcal{H})$ is the trace class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and $\perp_{\mathcal{A}_T}$ denotes the preannihilator of $\mathcal{A}_T$ in $\mathcal{C}_1(\mathcal{H})$. For a brief notation, we shall denote $Q_{\mathcal{A}_T}$ by Q_T. One knows that $\mathcal{A}_T$ is the dual space of Q_T and that the duality is given by

$$\langle A, [L] \rangle = \operatorname{tr}(AL), \quad A \in \mathcal{A}_T, [L] \in Q_T.$$

The Banach space Q_T is called a predual of $\mathcal{A}_T$. For x and y in $\mathcal{H}$, we can write $x \otimes y$ for the rank one operator in $\mathcal{C}_1(\mathcal{H})$ defined by

$$(x \otimes y)(u) = (u, y)x, \qquad \forall u \in \mathcal{H}.$$

Before giving the main result, we recall some ideas introduced in [1], [2], [3], [4], [5] and [7]. Recall that any contraction T can be written as a direct sum $T = T_1 \oplus T_2$, where T_1 is a completely nonunitary contraction and T_2 is a unitary operator. If T_2 is absolutely continuous or acts on the space (0), T will be called an *absolutely continuous contraction* . We denote by $\mathcal{A} = \mathcal{A}(\mathcal{H})$ the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the Sz.-Nagy-Foias functional calculus $\Phi_T : \mathcal{H}^\infty \to \mathcal{A}_T$ is an isometry ([2, Theorem 4.1]).

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If $\mathcal{A}$ is a dual algebra and m, n are cardinal numbers such that $1 \leq m, n \leq \aleph_0$, then $\mathcal{A}$ is said to have property $(\mathbb{A}_{m,n})$ if each $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], \quad 0 \leq i < m, 0 \leq j < n,$$

where $\{[L_{ij}]\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from $Q_{\mathcal{A}}$, has a solution $\{x_i\}_{0 \leq i < m}, \{y_j\}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$.

Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra and let $\theta \in [0, 1)$. Then $\mathcal{E}_\theta^r(\mathcal{A})$ denotes the set of all $[L]$ in $Q_{\mathcal{A}}$ such that there exist sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ of vectors from $\mathcal{H}$ satisfying

(a) $\limsup_{i \to \infty} \|[x_i \otimes y_i] - [L]\| \leq \theta$,
(b) $\|x_i\| \leq 1, \|y_i\| \leq 1, 1 \leq i < \infty$,
(c) $\|[x_i \otimes z]\| \to 0, \forall z \in \mathcal{H}$ and
(d) $\{y_i\}$ converges weakly to zero.

If $0 \leq \theta < \gamma < +\infty$, then a dual algebra $\mathcal{A}$ is said have property $E_{\theta,\gamma}^r$ if the closed absolutely convex hull of the set $\mathcal{E}_\theta^r(\mathcal{A})$ contains the closed ball $B_{0,\gamma}$ of radius γ centered at the origin in $Q_{\mathcal{A}}$:

$$\overline{aco}(\mathcal{E}_\theta^r(\mathcal{A})) \supset \{[L] \in Q_{\mathcal{A}} : \|[L]\| \leq \gamma\} = B_{0,\gamma}.$$

We shall employ the notation $C_{\cdot 0} = C_{\cdot 0}(\mathcal{H})$ for the class of all (completely nonunitary) contraction T in $\mathcal{L}(\mathcal{H})$ such that the sequence $\{T^{*n}\}$ converges to zero in the strong operator topology and is denoted by, as usual, $C_{0\cdot} = (C_{\cdot 0})^*$, and $C_{00} = C_{0\cdot} \cap C_{\cdot 0}$. Recall that every contraction T in $\mathcal{L}(\mathcal{H})$ has a minimal coisometric extension $B = B_T$ that is unique up to unitary equivalence. Given such T and B, one knows that there exists a canonical decomposition of the isometry B^* as

$$B^* = S \oplus R^*,$$

corresponding to a decomposition of the space

$$\mathcal{K} = \mathcal{S} \oplus \mathcal{R},$$

where, if $\mathcal{S} \neq (0), S$ is a unilateral shift operator of some multiplicity in $\mathcal{L}(\mathcal{S})$, and, if $\mathcal{R} \neq (0)$, R is a unitary operator in $\mathcal{L}(\mathcal{R})$. Of course, either $\mathcal{S}$ or $\mathcal{R}$ may be (0) ([4]).

The following proposition gives a sufficient condition for a dual algebra to have property $\mathbb{E}_{\theta,1}^r$.

Lemma 1.1. *([4]) Suppose $T \in \mathbb{A}(\mathcal{H}), 0 \le \theta < 1$, and $\Lambda \subset \mathbb{D}$ is dominating for $\mathbb{T}$. If for each $\lambda \in \Lambda$ there exists a sequence $\{x_n = x_n(\lambda)\}$ in the closed unit ball of $\mathcal{H}$ such that*

(a) $\overline{\lim} \, \|[C_\lambda] - [x_n \otimes x_n]\| \ge \theta$, *and*

(b) $\|[x_n \otimes z]\| \to 0, z \in \mathcal{H}$,

then $\mathcal{A}_T$ has property $\mathbb{E}^r_{\theta,1}$.

Definition 1.2. *([5]) Let m, n and l be any cardinal numbers such that $1 \le m, n, l \le \aleph_0$. We denote by $\mathbb{A}^l_{m,n}(\mathcal{H})$ the class of all sets $\{T_k\}^l_{k=1}$ such that T_k belongs to $\mathbb{A}(\mathcal{H})$ for all $k = 1, \cdots, l$ and that every $m \times n \times l$ system of simultaneous equations of the form*

$$[x_i \otimes y_j^{(k)}]_{T_k} = [L_{ij}^{(k)}]_{T_k},$$

where $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0 \le i < m \\ 0 \le j < n}}$ is an arbitrary $m \times n$ array from Q_{T_k} for each $k = 1, \cdots, l$, has a solution consisting of a pair of sequences $\{x_i\}_{0 \le i < m}$ and $\{y_j^{(k)}\}_{\substack{0 \le j < n \\ 1 \le k \le l}}$ of vectors from $\mathcal{H}$. Furthermore, if for every doubly indexed family $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0 \le i < m \\ 0 \le j < n}}$ of elements of Q_{T_k} for each $k = 1, \cdots, l$, such that the rows and columns of the matrix $(\|[L_{ij}^{(k)}]_{T_k}\|)$ are summable and r is a fixed real number satisfying $r \ge 1$, then we denoted by $\mathbb{A}^l_{m,n}(r)$ the class of all sets $\{T_k\}^l_{k=1}$ such that T_k belongs to $\mathbb{A}(\mathcal{H})$ for all $k = 1, \cdots, l$ and that every $m \times n \times l$ system of simultaneous equations of the form

$$[x_i \otimes y_j^{(k)}]_{T_k} = [L_{ij}^{(k)}]_{T_k},$$

has a solution consisting of a pair of sequences $\{x_i\}_{0 \le i < m}, \{y_j^{(k)}\}_{\substack{0 \le j < n \\ 1 \le k \le l}}$ of vectors from $\mathcal{H}$ and also satisfy the following conditions, for every $s > r$,

$$\|x_i\|^2 \le s \sum_{0 \le j < n} \|[L_{ij}^{(k)}]_{T_k}\|, \quad 0 \le i < m, 1 \le k \le l$$

and

$$\|y_j^{(k)}\|^2 \le s \sum_{0 \le i < m} \|[L_{ij}^{(k)}]_{T_k}\|, 0 \le j < n, 1 \le k \le l.$$

The theory of dual algebras is applied to the study of invariant subspaces, dilation theory, and reflexivity. The classes $\mathbb{A}_{m,n}$ were defined by H. Bercovici, C. Foias and C. Pearcy in [1]. Also these classes are closely related to the study of the theory of dual algebras. Especially, B. Chevreau and C. Pearcy [4] established property $E^r_{\theta,\gamma}$, and researched a relationship with the class $\mathbb{A}_{1,\aleph_0}$.

In Theorem 3.2.2 of [5] and Theorem 2.2 of [6], Hae Gyu Kim got a sufficient condition for the membership in the classes $\mathbb{A}^2_{1,1}(\rho)$ and $\mathbb{A}^2_{1,\aleph_0}(\rho)$, respectively. In a sequel to this study([5, Theorem 3.2.2] and [6, Theorem 2.2]), in this paper we obtain a spectral sufficient condition for membership in $\mathbb{A}^2_{1,1}(r)$ by using techniques in [3], [5] and [7].

2. Main Results

Convention. *In this paper we assume that $\mathcal{R}_1$ and $\mathcal{R}_2$ are either simultaneously (0) or simultaneously not (0) and so are $\mathcal{S}_1$ and $\mathcal{S}_2$.*

Lemma 2.1. ([6, Theorem 2.2]) *For $k = 1, 2$, suppose $T_k (\in \mathbb{A}(\mathcal{H}))$ with minimal coisometric extension $B_k (\in \mathcal{L}(\mathcal{S}_k \oplus \mathcal{R}_k)), B_k \in C_{\cdot 0}(\mathcal{K})$, and for some $0 \leq \theta < \gamma \leq 1, \mathcal{A}_{T_k}$ has property $E^r_{\theta,\gamma}$. Suppose also that we are given scalar $\beta > 1, \delta > 0, [L_k] \in Q_{T_k}$ are given such that*

$$\|[L_k]_{T_k}\| < \delta, k = 1, 2,$$

Moreover if we set

$$\alpha = \frac{1}{\gamma^{\frac{1}{2}} - \theta^{\frac{1}{2}}},$$

Then there exist $\hat{a} \in \mathcal{H}, \hat{w}_k \in \int_k$ and $\hat{b}_k \in \mathcal{R}_k$ such that

$$[L_k]_{T_k} = [\hat{a} \otimes P(\hat{w}_k + \hat{b}_k)]_{T_k}, \quad k = 1, 2,$$

$$\|\hat{a}\| < 6\alpha\delta^{\frac{1}{2}},$$

$$\|\hat{w}_k\| < \alpha\delta^{\frac{1}{2}} \quad and \quad k = 1, 2,$$

and

$$\|\hat{b}_k\| < \alpha\beta\delta^{\frac{1}{2}} \quad and \quad k = 1, 2,$$

where P is the projection of $\mathcal{K}$ onto the subspace $\mathcal{H}$.
Moreover, the set $\{T_1, T_2\}$ is in the class $\mathbb{A}^2_{1,1}(6\sqrt{2}\alpha^2)$. Inparticular, if $\mathcal{A}_{T_k}$ has property $\mathbb{E}^r_{0,1}$, then the set $\{T_1, T_2\} \in \mathbb{A}^2_{1,1}(6\sqrt{2})$.

The following theorem is immediate from Lemma 1.1 and Lemma 2.1.

Theorem 2.2. *For $k = 1, 2$, suppose $T_k (\in \mathbb{A}(\mathcal{H}))$ with minimal coisometric extension $B_k (\in \mathcal{L}(\mathcal{S}_k \oplus \mathcal{R}_k)), B_k \in C_{\cdot 0}(\mathcal{K})$, and $\Lambda_k \subset \mathbb{D}$ is dominating for $\mathbb{T}$. If for each $\lambda_k \in \Lambda_k$ there exists a sequence $\{x_n{}^{\lambda_k}\}_{n=1,k=1}^{\infty,\ 2}$ in the closed unit ball of $\mathcal{H}$ such that*

(a) $\overline{\lim} \|[C_{\lambda_k}] - [x_n{}^{\lambda_k} \otimes x_n{}^{\lambda_k}]\| \geq \theta$, *and*
(b) $\|[x_n{}^{\lambda_k} \otimes z_k]\| \to 0, \quad z_k \in \mathcal{H}$.

Then, the set

$$\{T_1, T_2\} \in \mathbb{A}^2_{1,1}\left(\frac{6\sqrt{2}}{(1-\theta^{\frac{1}{2}})^2}\right).$$

Recall that if $T \in \mathcal{L}(\mathcal{H})$ and H is a hole in $\sigma_e(T)$ (i.e, H is a bounded component of $\mathbb{C} \setminus \sigma_e(T)$), then H is associated with a unique finite Fredholm index $i(H)$, defined by choosing any λ in H and seting $i(H) = I(T - \lambda)$. If H is a hole in $\sigma_e(T)$ such that $i(H) \neq 0$, then $H \subset \sigma(T)$. On the other hand, if H is a hole in $\sigma_e(T)$ with $i(H) = 0$, then either $H \subset \sigma(T)$ or $H \cap \sigma(T)$ consists of a countable (possibly empty) set of isolated points. For each T in $\mathcal{L}(\mathcal{H})$ we write $\mathcal{F}_-(T)$ [resp. $\mathcal{F}_+(T)$] for the (possibly empty) union of all holes H in $\sigma_e(T)$ such that $i(H) \leq 0$ [resp. $i(H) \geq 0$] and $H \subset \sigma(T)$. Moreover, we write $\mathcal{F}'_-(T)$ [resp. $\mathcal{F}'_+(T)$] for the union of all holes H in $\sigma_e(T)$ such that $i(H) < 0$ [resp. $i(H) > 0$], and $\mathcal{F}(T) = \mathcal{F}_-(T) \cup \mathcal{F}_+(T)$. ([4])

We now present a spectral sufficient condition for membership in $\mathbb{A}^2_{1,1}(r)$.

Theorem 2.3. *For $k = 1, 2$, suppose T_k is absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, $B_k \in \mathcal{L}(\mathcal{S}_k \oplus \mathcal{R}_k)$ is a minimal coisometric extension of $T_k, B_k \in C_{\cdot 0}(\mathcal{K})$, and $\Lambda_k = (\sigma_e(T_k) \cap \mathbb{D}) \cup \mathcal{F}_-(T_k)$ is dominating for $\mathbb{T}$. Then, the set*

$$\{T_1, T_2\} \in \mathbb{A}^2_{1,1}(6\sqrt{2}).$$

Proof. That $T_k \in \mathbb{A}$ is from [2, Proposition 4.6], and we will prove the theorem by proving that Theorem 2.2 can be applied with $\theta = 0$. But by using the properties in the proof of Theorem 5.3 of [4] and Theorem 2.2, the proof of the Theorem 2.3 is clearly completed.

The following corollary is immediate from Theorem 2.3.

Corollary 2.4. *For $k = 1, 2$, suppose T_k is absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, $B_k \in \mathcal{L}(\mathcal{S}_k \oplus \mathcal{R}_k)$ is a minimal coisometric extension of $T_k, B_k \in C_{\cdot 0}(\mathcal{K})$, and there exist dominating set $\Lambda_k \subset \mathbb{D}, k = 1, 2$, such that for all $\lambda_k \in \Lambda_k, T_k - \lambda_k$ is a Fredholm operator such that $i(T_k - \lambda_k) < 0$. Then, the set*

$$\{T_1, T_2\} \in \mathbb{A}^2_{1,1}(6\sqrt{2}).$$

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