On the Orthogonal Nonholonomic Frames with Application to Vn(1)

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Vn에 適用한 垂直 Nonholonomic Frame에 관하여

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ABSTRACT

The purpose of the present paper, as the application of orthogonal nonholonomic frames, is to reconstruct the some results of Riemannian Geometry determined by a symmetric tensor $a_{\lambda\mu}$ Composed of n-different eigenvectors of $a_{\lambda\mu}$.

1. INTRODUCTION.

V. Hlavaty 1957) introduced the concept of the nonholonomic frames and used it sucoessfully as a tool to develop the algebra of the unified field theory in the space-time X_4. In our previous paper Chung K. T. & Hyun J.O. 1976 and Hyun J.O. 1976, we introduced the concept of the general nonholonomic frames and orthogonal nonholonomic frames to an n-dimensional Riemannian space V_n, and constructed the characteristic orthogonal nonholonomic frames of V_n determined by a symmetric tensor $a_{\lambda\mu}$, composed of n different eigenvectors of $a_{\lambda\mu}$, and to derive its particular properties.

This paper is a continuation of Chung K. T. & Hyun J. O. 1976. and Hyun J. O. 1976 The purpose of the present paper, as the application of orthogonal nonholonomlc frames, is to reconstruct the some resultse of Riemannian Geometry determined by a symmertic tensor $a_{\lambda\mu}$, Composed of n different eigenvectors of $a_{\lambda\mu}$.

2. PREL IMINARY RESULTS.

In the present section, for our further discussions, results obtained in our previous paper Chung K. T. & Hgun J. O. 1976 and Hyun J. O. 1976 will be introduced without proof.

Let V_n be a n-dimensional Riemannian space referred to a real coordinate system X^ν and defined by a fundamental metric tensor $h_{\lambda\mu}$, whose determinant

$$h \overset{\text{def}}{=} \mathrm{Det}((h_{\lambda\mu}))\neq 0. \tag{2.1}$$

According to (2.1) there is a unique tensor $h^{\lambda\nu}=h^{\nu\lambda}$ defined by

$$h_{\lambda\mu}\, h^{\lambda\nu} \overset{\text{def}}{=} \delta^\nu_\mu \tag{2.2}$$

The tensor $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and lowering indices of tensor quantities in V_n in the usual manner.

If $\underset{i}{e}^\nu$, $(i=1, 2, \cdots n)$, are a set of n linearly independent unit vectors, then there is a unique reciprocal set of n linearly independent covariant vectors $\overset{i}{e}_\lambda$, $(i=1,2,\cdots n)$, satisfying

(2.3) $\quad e^{\nu}_{i}\overset{i}{e}_\lambda=\delta^{\nu}_{\lambda}$ (*) $\quad e^{\lambda}_{j}\overset{i}{e}_\lambda=\delta^{i}_{j}$.

with the vectors $\overset{i}{e}_{\nu}$ and e_{λ} a nonholonomic frames of V_n is defined in the following way: If $T^{\nu\cdots}_{\lambda\cdots}$ are holonomic components of a tensor density of weight p, then its nonholonomic components are defined by

(2.4)a $\quad T^{i\cdots}_{j\cdots} \overset{\text{def}}{=} A^{-p}\, T^{\nu\cdots}_{\lambda\cdots}\, \overset{i}{e}_\nu e^\lambda_j$

$$A \overset{\text{def}}{=} \text{Det}\,(\overset{i}{e}_\lambda),$$

From (2.3) and (2.4)

(2.4)b $\quad T^{\nu\cdots}_{\lambda\cdots}=A^{p}\, T^{i\cdots}_{j\cdots}\, e^{\nu}_{i}\, \overset{j}{e}_{\lambda}\cdots$

The nonholonomic frame in V_n constructed by the unit vectors e^{ν}_{i}, $(i=1,\cdots\cdots,n)$, thangent to the n congruences of an orthogonal ennuple, will be termed an orthogonal nonholonomic frame of V_n.

Theorem(2.1). We have

(2.5) $\quad h_{ij}=\delta_{ij},\ h^{ij}=\delta_{ij},\ \overset{\nu i}{e}=e^{\nu}_{i},\ \overset{i}{e}_\lambda=e_{\lambda j}$.

Theorem (2.2). The tensors $h_{\lambda\mu}$, $h^{\lambda\mu}$, and δ^{ν}_{λ} may be experessed in terms of $\underset{i}{e}$, as follows;

(2.6) $\quad h_{\lambda\mu}=\sum_i \underset{i}{e}_\lambda \underset{i}{e}_\mu,\quad h^{\lambda\mu}=\sum_i \underset{i}{e}^\lambda \underset{i}{e}^\mu,\quad \delta^{\nu}_{\lambda}=\sum_i \underset{i}{e}_\lambda \underset{i}{e}^\nu$

And let e^{λ}_{i} be unit eigenvectors determined by a symmetric covariant tensor $a_{\lambda\mu}$, Then they satisfy

(2.7) $\quad (a_{\lambda\mu}-\underset{i}{M}\, h_{\lambda\mu})\, \underset{i}{\overset{\lambda}{e}}=0$ ($\underset{i}{M}$: scalar)

For our further discussion, we need the tensors ${}^{(p)}a_{\lambda\mu}$, defined as

(2.8) $\quad {}^{(1)}a_{\lambda\mu} \overset{\text{def}}{=} a_{\lambda\mu}$

$\quad {}^{(p)}a_{\lambda\mu}={}^{(p-1)}a_{\lambda\alpha}a_{\mu}{}^{\alpha},\quad p=2,3\cdots\cdots.$

A simple inspection shows that ${}^{(p)}a_{\lambda\mu}$ is symmetric.

Lemma (2.3). Every eigenvector $\underset{i}{e}^{\lambda}$ of $a_{\lambda\mu}$ is also an eigenvector of the tensor ${}^{(p)}a_{\lambda\mu}$, $(p=2,3\cdots)$

Theorem (2.4). The nonholonomic components of ${}^{(p)}a_{\lambda\mu}$ are

(2.9) $\quad \overset{(p)\,i}{\underset{x}{a}}=\underset{x}{\overset{p}{M}}\,\underset{x}{\overset{i}{\delta}}$ or $\underset{xi}{a}=\underset{x}{\overset{p}{M}}\,\delta_{xi}$, $(p=1,2,\cdots)$

Theorem(2.5). The tensor $a_{\lambda\mu}$ may be expressed in terms of $\underset{i}{e}^{\lambda}$, as follows:

(2.10) $\quad {}^{(p)}a_{\lambda\mu}=\sum_i \underset{i}{\overset{p}{M}}\,\underset{i}{e}_\lambda \underset{i}{e}_\mu\ (p=1,2,\cdots,)$

3. MAIN RESULTS.

In this section, our main results will be proved as application of orthogonal nonholonomic frames.

Lemma(3.1). The nonholonomic component of tensor δ^{ν}_{μ} are

(3.1) $\quad \delta^{k}_{j}=h_{ij}\, h^{ik}$

Proof. From the results of (2.2), (2.3) and (2.4)

$$h_{ij}\, h^{ki}=h_{\lambda\mu}\, \underset{i}{\overset{\lambda}{e}}\, \underset{j}{\overset{\mu}{e}}\, h^{\omega\nu}\, \underset{\omega}{\overset{i}{e}}\, \underset{\nu}{\overset{k}{e}}=h_{\omega\mu}h^{\omega\nu}\, \underset{j}{\overset{\mu}{e}}\, \underset{\nu}{\overset{k}{e}}=\delta^{k}_{j}$$

(*) throughout the present paper, Greek indices take values 1, 2, n unless eplicity statedotherwise and follow the summation convention, while Roman nindices are used for the nonhlonomic component of a tensor and run from 1 to n. Roman indices also follow the summation convention.

Theorem(3.2). We have

(3.2)a $(a_{hj}a_{jk}-a_{hk}a_{ij})a^{hj}=(n-1)a_{ik}.$

(3.2)b $\frac{\partial k}{\partial x^j}(a_{hk}a_{il}-a_{hl}a_{ik})\,a^{hj}$

$=\frac{\partial k}{\partial x^k}a_{il}-\frac{\partial k}{\partial x^l}a_{ik}$

Proof. By means of (3.1)

(3.2)a $(a_{hj}a_{ik}-a_{hk}a_{ij})\,a^{hj}=\delta^j_j a_{jk}-a_{hk}\,\delta^h_i$

$=(n-1)a_{ik}.$

(3.2) b may by proved as follows:

$\frac{\partial k}{\partial x^j}(a_{hk}a_{ij}-a_{hl}a_{ik})a^{hj}$

$=\frac{\partial k}{\partial x^j}\delta^j_k a_{il}-\frac{\partial k}{\partial x^j}\delta^j_i a_{ik}=\frac{\partial k}{\partial x^k}a_{il}$

$-\frac{\partial k}{\partial x^l}a_{ik}.$

Lemma (3.3). We have

(3.3) a $a_{ij}\,\overset{i}{e}_\lambda\,\overset{j}{e}_\lambda=1$

(3.3) b $a_{ij}\,\overset{i}{e}_\lambda\,\overset{j}{e}_\mu=0$

Proof. Accoording to the results (2.5),

$a_{\lambda\mu}\,\underset{i}{e^\lambda}\,\underset{j}{e^\mu}=\delta_{ij}$

(3.3) a $a_{ij}\,\overset{i}{e}_\lambda\,\overset{j}{e}_\lambda=\delta_{ij}\,\overset{i}{e}_\lambda\,\overset{j}{e}_\lambda=\underset{j}{e}_\lambda\,\underset{\lambda}{e}^j=1$

Similarly,

(3.3) b $a_{ij}\,\overset{i}{e}_\lambda\,\overset{j}{e}_\mu=\delta_{ij}\,\underset{j}{e}_\lambda\,\underset{j}{e}_\mu=0$

Theorem(3.4). If $\underset{\lambda}{\overset{i}{e}}$ are orthogonal unit eigenvectors of $a_{\lambda\mu}$, then

(3.4) $(a_{hj}a_{ik}-a_{hk}a_{ij})\,\overset{h}{e}_\lambda\,\overset{i}{e}_\mu\,\overset{j}{e}_\lambda\,\overset{k}{e}_\mu=1$

Proof. By means of (3.3)

$(a_{hj}a_{ik}-a_{hk}a_{ij})\,\overset{h}{e}_\lambda\,\overset{i}{e}_\mu\,\overset{j}{e}_\lambda\,\overset{k}{e}_\mu$

$=(a_{hj}\overset{h}{e}_\lambda\,\overset{j}{e}_\lambda)\;(a_{ik}\overset{i}{e}_\mu\,\overset{k}{e}_\mu)$

$-(a_{hk}\overset{i}{e}_\lambda\,\overset{k}{e}_\mu)\;(a_{ij}\overset{i}{e}_\lambda\,\overset{j}{e}_\mu)=1.$

Referances

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要 略

본 論文에서는, n개의 다른 Eigen-Vector들에 의하여 形成된 Tensor $a_{\lambda\mu}$에 의하여 結定 되어지는 Riemann幾何學의 몇가지 結果를 Orthogonal Nonholonomic Frame을 適用 하므로써 再構成 하고저 한다.