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Parametric operations for 2-dimensional fuzzy sets

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Parametric operations for 2-dimensional fuzzy sets

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Contents

⟨**Abstract**⟩

Parametric operations for 2-dimensional fuzzy sets

We generalize trapezoidal fuzzy numbers on $\mathbb{R}$ to $\mathbb{R}^2$ and compute the parameter calculations between two-dimensional triangular fuzzy numbers and trapezoidal fuzzy sets. In addition, we prove that the result for the parametric operation for two 2-dimensional quadratic fuzzy numbers are the generalization of algebraic operations for two quadratic fuzzy numbers on $\mathbb{R}$. We give examples to support our assertions.

1 Introduction

In fuzzy set theory, various types of operations between two fuzzy sets have been defined and studied. The operations of two fuzzy numbers (A, μ_A) and (B, μ_B) are based on the Zadeh's extension principle ([15], [16], [17]). The results of extended algebraic operations between two triangular fuzzy numbers for the four operations—addition $A(+)B$, subtraction $A(-)B$, multiplication $A(\cdot)B$ and division $A(/)B$ described in Definition 2.6.—are well known ([1], [2]).

In Chapter 2, Zadeh et al. calculated many results and examples of extended algebraic operations between two quadratic fuzzy numbers and trapezoidal fuzzy numbers ([10]).

In Chapter 3, Yun et al. introduced a generalized triangular fuzzy set and calculated extended algebraic operations between two generalized triangular fuzzy sets in Section 3.1 ([12]). In Section 3.2 and 3.3, Song et al. introduced the generalized quadratic and trapezoidal fuzzy sets and calculated an extended algebraic operations between two generalized quadratic and trapezoidal fuzzy sets, respectively ([9], [11], [12]).

In Chapter 4, Kim et al. generalized extended algebraic operations on $\mathbb{R}$ to $\mathbb{R}^2$. For this, Zadeh defined the parametric operations for two fuzzy numbers defined on $\mathbb{R}$ in Definition 2.6 and the results for parametric operations turned out to be as same as those for the extended operations in Theorem 4.4 ([6]). Using parametric operations, Kim et al. generalized the extended algebraic operations on $\mathbb{R}$ to $\mathbb{R}^2$ in Definiton 4.8 ([6]). In Section 4.1, Kim and Yun generalized the triangular fuzzy numbers on $\mathbb{R}$ to $\mathbb{R}^2$. By defining parametric operations between two regions valued α-cuts, Kim et al. calculated the parametric operations for two triangular fuzzy numbers defined on $\mathbb{R}^2$

([6]). In Section 4.2, Kim and Yun defined generalized triangular fuzzy number and further calculated the parametric operations for two generalized 2-dimensional triangular fuzzy sets defined on $\mathbb{R}^2$ ([5]). In Section 4.3, Kang and Yun also generalized the quadratic fuzzy numbers on $\mathbb{R}$ to $\mathbb{R}^2$. Kang et al. calculated the parametric operations for two 2-dimensional quadratic fuzzy numbers ([2]).

Based on these results, in chapter 5, we generalize fuzzy numbers and parametric operation. In Section 5.1, we generalize the trapezoidal fuzzy number on $\mathbb{R}$ to $\mathbb{R}^2$ and calculate the parametric operation between the two-dimensional triangular fuzzy number and the trapezoidal fuzzy set ([7]). Lastly, in Section 5.2, we prove that the results for the parametric operations for two 2-dimensional quadratic fuzzy numbers are the generalization of algebraic operations for two quadratic fuzzy numbers on $\mathbb{R}$ ([8]) and give examples to support our assertions.

2 Preliminaries

Let X be a set. A classical subset A of X is often viewed as a characteristic function μ_A from X to $\{0,1\}$ such that $\mu_A(x) = 1$ if $x \in A$, and $\mu_A(x) = 0$ if $x \notin A$. $\{0,1\}$ is called a valuation set. The following definition is a generalization of this notion.

Definition 2.1. A fuzzy set A on X is a function from X to the interval $[0,1]$. The function is called the *membership function* of A.

Let A be a fuzzy set on X with a membership function μ_A. A is completely characterized by the set of pairs $A = \{(x, \mu_A(x)) \mid x \in X\}$ elements with a zero degree of membership are normally not listed.

Definition 2.2. A α-*cut* of a fuzzy number A is defined by $A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}$ if $\alpha \in (0,1]$ and $A_\alpha = \text{cl}\ \{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$ if $\alpha = 0$.

Definition 2.3. ([19]) A fuzzy set A on $\mathbb{R}$ is *convex* if

$$\mu_A(\lambda x_1 + (1-\lambda) x_2) \geq min(\mu_A(x_1), \mu_A(x_2)), \quad \forall x_1, x_2 \in \mathbb{R}, \quad \forall \lambda \in [0,1].$$

Definition 2.4. ([19]) A convex fuzzy set A on $\mathbb{R}$ is called a *fuzzy number* if

(1) There exists exactly one $x \in \mathbb{R}$ such that $\mu_A(x) = 1$,

(2) $\mu_A(x)$ is piecewise continuous.

Definition 2.5. ([12]) A triangular fuzzy number on $\mathbb{R}$ is a fuzzy number A which has a membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \ \ a_3 \leq x, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x < a_3. \end{cases}$$

The above triangular fuzzy number is denoted by $A = (a_1, a_2, a_3)$.

Definition 2.6. ([19]) The addition, subtraction, multiplication, and division of two fuzzy numbers are defined as

1. Addition $A(+)B$:

$$\mu_{A(+)B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

2. Subtraction $A(-)B$:

$$\mu_{A(-)B}(z) = \sup_{z=x-y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

3. Multiplication $A(\cdot)B$:

$$\mu_{A(\cdot)B}(z) = \sup_{z=x\cdot y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

4. Division $A(/)B$:

$$\mu_{A(/)B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

Remark 2.7. Let A and B be fuzzy sets. $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. Then the α-cuts of $A(+)B, A(-)B$, $A(\cdot)B$ and $A(/)B$ can be calculated as the followings.

(1) $(A(+)B)_\alpha = A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}]$.

(2) $(A(-)B)_\alpha = A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}]$.

(3) $$(A(\cdot)B)_\alpha = A_\alpha(\cdot)B_\alpha = [\min(a_1^{(\alpha)}b_1^{(\alpha)}, a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}),$$
$$\max(a_1^{(\alpha)}b_1^{(\alpha)}, a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)})].$$

(4) $$(A(/)B)_\alpha = A_\alpha(/)B_\alpha = [\min(a_1^{(\alpha)}/b_1^{(\alpha)}, a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}, a_2^{(\alpha)}/b_2^{(\alpha)}),$$
$$\max(a_1^{(\alpha)}/b_1^{(\alpha)}, a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}, a_2^{(\alpha)}/b_2^{(\alpha)})].$$

Example 2.8. ([12]) For two triangular fuzzy numbers $A = (1, 2, 4)$ and $B = (2, 4, 5)$, we have

1. Addition : $A(+)B = (3, 6, 9)$.

2. Subtraction : $A(-)B = (-4, -2, 2)$.

3. Multiplication :

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < 2, \ \ 20 \le x, \\ \frac{-2+\sqrt{2x}}{2}, & 2 \le x < 8, \\ \frac{7-\sqrt{9+2x}}{2}, & 8 \le x < 20. \end{cases}$$

Note that $A(\cdot)B$ is not a triangular fuzzy number.

4. Division :

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{1}{5}, \ \ 2 \le x, \\ \frac{5x-1}{x+1}, & \frac{1}{5} \le x < \frac{1}{2}, \\ \frac{-x+2}{x+1}, & \frac{1}{2} \le x < 2. \end{cases}$$

Note that $A(/)B$ is not a triangular fuzzy number.

Definition 2.9. ([12]) A fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \ \ a_4 \le x, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \le x < a_2, \\ 1, & a_2 \le x < a_3, \\ \frac{a_4-x}{a_4-a_3}, & a_3 \le x < a_4. \end{cases}$$

is called a *trapezoidal fuzzy set*.

Denotes the trapezoidal fuzzy set above $A = (a_1, a_2, a_3, a_4)$.

Theorem 2.10. ([10]) For two trapezoidal fuzzy sets $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$, we have

1. $A(+)B = (a_1 + b_1, \ a_2 + b_2, \ a_3 + b_3, \ a_4 + b_4)$.

2. $A(-)B = (a_1 - b_4, \ a_2 - b_3, \ a_3 - b_2, \ a_4 - b_1)$.

3. $A(\cdot)B$ and $A(/)B$ are not trapezoidal fuzzy sets.

Example 2.11. ([10]) Let $A = (1,5,6,9)$ and $B = (2,3,5,8)$ be trapezoidal fuzzy sets, i.e.,

$$\mu_A(x) = \begin{cases} 0, & x < 1,\ 9 \le x, \\ \frac{x-1}{4}, & 1 \le x < 5, \\ 1, & 5 \le x < 6, \\ \frac{-x+9}{3}, & 6 \le x < 9, \end{cases} \quad \text{and} \quad \mu_B(x) = \begin{cases} 0, & x < 2,\ 8 \le x, \\ x-2, & 2 \le x < 3, \\ 1, & 3 \le x < 5, \\ \frac{-x+8}{3}, & 5 \le x < 8, \end{cases}$$

we calculate exactly the above four operations using $\alpha-$ cuts.

Let A_α and B_α be the α-cuts of A and B, respectively. Put $A_\alpha = [a_1^{(\alpha)},\ a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)},\ b_2^{(\alpha)}]$. Since $\alpha = \frac{a_1^{(\alpha)}-1}{4}$ and $\alpha = \frac{-a_2^{(\alpha)}+9}{3}$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [4\alpha+1, -3\alpha+9]$. Similarly, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha+2, -3\alpha+8]$.

1. Addition : By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)}+b_1^{(\alpha)}, a_2^{(\alpha)}+b_2^{(\alpha)}] = [5\alpha+3, -6\alpha+17]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[3,17]^c$ and $\mu_{A(+)B}(x) = 1$ on the interval $[8,11]$. By the routine calculation, we have

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < 3,\ 17 \le x, \\ \frac{x-3}{5}, & 3 \le x < 8, \\ 1, & 8 \le x < 11, \\ \frac{-x+17}{6}, & 11 \le x < 17, \end{cases}$$

i.e., $A(+)B = (3,8,11,17)$.

2. Subtraction : Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)}-b_2^{(\alpha)}, a_2^{(\alpha)}-b_1^{(\alpha)}] = [7\alpha-7, -4\alpha+7]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-7,7]^c$ and $\mu_{A(-)B}(x) = 1$ on the interval $[0,3]$. By the

routine calculation, we have

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -7,\ \ 7 \le x, \\ \frac{x+7}{7}, & -7 \le x < 0, \\ 1, & 0 \le x < 3, \\ \frac{-x+7}{4}, & 3 \le x < 7, \end{cases}$$

i.e., $A(-)B = (-7, 0, 3, 7)$.

3. Multiplication : Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_1^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] = [4\alpha^2+9\alpha+2, 9\alpha^2-51\alpha+72]$, $\mu_{A(\cdot)B}(x) = 0$ on the interval $[2, 72]^c$ and $\mu_{A(\cdot)B}(x) = 1$ on the interval $[15, 30]$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < 2,\ \ 72 \le x, \\ \frac{-9+\sqrt{49+16x}}{8}, & 2 \le x < 15, \\ 1, & 15 \le x < 30, \\ \frac{17-\sqrt{1+4x}}{6}, & 30 \le x < 72. \end{cases}$$

Thus $A(\cdot)B$ is not a trapezoidal fuzzy set.

4. Division : Since $A_\alpha(/)B_\alpha = [\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}] = [\frac{4\alpha+1}{-3\alpha+8}, \frac{-3\alpha+9}{\alpha+2}]$, $\mu_{A(/)B}(x) = 0$ on the interval $[\frac{1}{8}, \frac{9}{2}]^c$ and $\mu_{A(/)B}(x) = 1$ on the interval $[1, 2]$. By the routine calculation, we have

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{1}{8},\ \ \frac{9}{2} \le x, \\ \frac{8x-1}{3x+4}, & \frac{1}{8} \le x < 1, \\ 1, & 1 \le x < 2, \\ \frac{-2x+9}{x+3}, & 2 \le x < \frac{9}{2}. \end{cases}$$

Thus $A(/)B$ is not a trapezoidal fuzzy set.

Similar to a triangular fuzzy number, the quadratic fuzzy number is defined by a quadratic curve.

Definition 2.12. ([10]) A *quadratic fuzzy number* is a fuzzy number A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < \alpha, \ \ \beta \leq x, \\ -a(x-\alpha)(x-\beta) = -a(x-k)^2 + 1, & \alpha \leq x < \beta, \end{cases}$$

where $a > 0$.

The above quadratic fuzzy number is denoted by $A = [\alpha, k, \beta]$.

Theorem 2.13. ([10]) For two quadratic fuzzy numbers $A = [x_1, k, x_2]$ and $B = [x_3, m, x_4]$, we have

1. $A(+)B = [x_1 + x_3, k + m, x_2 + x_4]$.

2. $A(-)B = [x_1 - x_4, k - m, x_2 - x_3]$.

3. $\mu_{A(\cdot)B}(x) = 0$ on the interval $[x_1x_3, x_2x_4]^c$ and $\mu_{A(\cdot)B}(x) = 1$ at $x = km$. Note that $A(\cdot)B$ are not a quadratic fuzzy number.

4. $\mu_{A(/)B}(x) = 0$ on the interval $[\frac{x_1}{x_4}, \frac{x_2}{x_3}]^c$ and $\mu_{A(/)B}(x) = 1$ at $x = \frac{k}{m}$. Note that $A(/)B$ are not a quadratic fuzzy number.

Proof. Note that

$$\mu_A(x) = \begin{cases} 0, & x < x_1, \ \ x_2 \leq x, \\ -a(x-k)^2 + 1 = -a(x-x_1)(x-x_2), & x_1 \leq x < x_2, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < x_3, \ \ x_4 \leq x, \\ -b(x-m)^2 + 1 = -b(x-x_3)(x-x_4), & x_3 \leq x < x_4. \end{cases}$$

We calculate exactly four operations using α-cuts. Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ are the α-cuts of A and B, respectively. Since $\alpha = -a(a_1^{(\alpha)} - k)^2 + 1$ and

$\alpha = -a(a_2^{(\alpha)} - k)^2 + 1$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = \Big[k - \sqrt{\frac{1-\alpha}{a}},\ \ k + \sqrt{\frac{1-\alpha}{a}}\ \Big].$$

Similarly, we have

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = \Big[m - \sqrt{\frac{1-\alpha}{b}},\ \ m + \sqrt{\frac{1-\alpha}{b}}\ \Big].$$

1. Addition : By the above facts,

$$\begin{aligned} A_\alpha(+)B_\alpha &= [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] \\ &= \Big[k + m - \sqrt{\frac{1-\alpha}{a}} - \sqrt{\frac{1-\alpha}{b}},\ \ k + m + \sqrt{\frac{1-\alpha}{a}} + \sqrt{\frac{1-\alpha}{b}}\ \Big]. \end{aligned}$$

Thus $\mu_{A(+)B}(x) = 0$ on the interval $[k+m-\frac{1}{\sqrt{a}}-\frac{1}{\sqrt{b}}, k+m+\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}]^c = [x_1+x_3, x_2+x_4]^c$ and $\mu_{A(+)B}(x) = 1$ at $x = k + m$. Therefore

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < x_1 + x_3,\ \ x_2 + x_4 \le x, \\ -\frac{ab}{(\sqrt{a}+\sqrt{b})^2}\{x - (k+m)\}^2 + 1, & x_1 + x_3 \le x < x_2 + x_4, \end{cases}$$

i.e., $A(+)B = [x_1 + x_3, k + m, x_2 + x_4]$.

2. Subtraction : Since

$$\begin{aligned} A_\alpha(-)B_\alpha &= [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] \\ &= \Big[k - m - \sqrt{\frac{1-\alpha}{a}} - \sqrt{\frac{1-\alpha}{b}},\ \ k - m + \sqrt{\frac{1-\alpha}{a}} + \sqrt{\frac{1-\alpha}{b}}\ \Big], \end{aligned}$$

we have $\mu_{A(-)B}(x) = 0$ on the interval $[k - m - (\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}), k - m + (\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}})]^c = [x_1 - x_4, x_2 - x_3]$ and $\mu_{A(-)B}(x) = 1$ at $x = k - m$. Therefore

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < x_1 - x_4,\ \ x_2 - x_3 \le x, \\ -\frac{ab}{(\sqrt{a}+\sqrt{b})^2}\{x - (k-m)\}^2 + 1, & x_1 - x_4 \le x < x_2 - x_3, \end{cases}$$

i.e., $A(-)B = [x_1 - x_4, k - m, x_2 - x_3]$.

3. Multiplication : Since

$$A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}]$$
$$= \Big[\Big(k-\sqrt{\frac{1-\alpha}{a}}\Big)\Big(m-\sqrt{\frac{1-\alpha}{b}}\Big),\ \Big(k+\sqrt{\frac{1-\alpha}{a}}\Big)\Big(m+\sqrt{\frac{1-\alpha}{b}}\Big)\Big],$$

$\mu_{A(\cdot)B}(x) = 0$ on the interval

$$\Big[\Big(k-\frac{1}{\sqrt{a}}\Big)\Big(m-\frac{1}{\sqrt{b}}\Big),\ \Big(k+\frac{1}{\sqrt{a}}\Big)\Big(m+\frac{1}{\sqrt{b}}\Big)\Big]^c = [x_1x_3,\ x_2x_4]^c$$

and $\mu_{A(\cdot)B}(x) = 1$ at $x = km$. Therefore

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < x_1x_3,\ \ x_2x_4 \le x, \\ \frac{1}{4}\Big(4-2k^2a-2m^2b-4\sqrt{ab}x+2(k\sqrt{a}+m\sqrt{b})\sqrt{(k\sqrt{a}-m\sqrt{b})^2+4\sqrt{ab}x}\Big), & \\ & x_1x_3 \le x < x_2x_4. \end{cases}$$

4. Division : Since

$$A_\alpha(/)B_\alpha = \left[\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\right] = \left[\frac{k-\sqrt{\frac{1-\alpha}{a}}}{m+\sqrt{\frac{1-\alpha}{b}}},\ \frac{k+\sqrt{\frac{1-\alpha}{a}}}{m-\sqrt{\frac{1-\alpha}{b}}}\right],$$

$\mu_{A(/)B}(x) = 0$ on the interval

$$\Big[\frac{\sqrt{b}(k\sqrt{a}-1)}{\sqrt{a}(m\sqrt{b}+1)},\ \frac{\sqrt{b}(k\sqrt{a}+1)}{\sqrt{a}(m\sqrt{b}-1)}\Big]^c = \Big[\frac{x_1}{x_4},\ \frac{x_2}{x_3}\Big]^c$$

and $\mu_{A(/)B}(x) = 1$ at $x = \frac{k}{m}$. Therefore

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{x_1}{x_4},\ \ \frac{x_2}{x_3} \le x, \\ \frac{a(1-bm^2)x^2+2\sqrt{ab}(1+\sqrt{ab}km)x+b(1-ak^2)}{(\sqrt{a}x+\sqrt{b})^2}, & \frac{x_1}{x_4} \le x < \frac{x_2}{x_3}. \end{cases}$$

□

Example 2.14. ([10]) Let $A = [1,2,3]$ and $B = [2,5,8]$. Then

1. Addition :

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < 3,\ \ 11 \le x, \\ -\frac{1}{16}(x-7)^2+1, & 3 \le x < 11, \end{cases}$$

i.e., $A(+)B = [3, 7, 11]$.

2. Subtraction :

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -7, \;\; 1 \le x, \\ -\frac{1}{16}(x+3)^2 + 1, & -7 \le x < 1, \end{cases}$$

i.e., $A(-)B = [-7, -3, 1]$.

3. Multiplication :

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < 2, \;\; 24 \le x, \\ -\frac{1}{18}(6x + 43 - 11\sqrt{12x+1}), & 2 \le x < 24. \end{cases}$$

4. Division :

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{1}{8}, \;\; \frac{3}{2} \le x, \\ \dfrac{-(8x-1)(2x-3)}{(3x+1)^2}, & \frac{1}{8} \le x < \frac{3}{2}. \end{cases}$$

3 Generalized fuzzy set

3.1. Generalized triangular fuzzy set

Yun et al. generalized the triangular fuzzy number. A generalized triangular fuzzy set is symmetric and may not have value 1.

Definition 3.1. ([12]) A generalized triangular fuzzy set is a symmetric fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1,\ \ a_2 \leq x, \\ \frac{2c(x-a_1)}{a_2-a_1}, & a_1 \leq x < \frac{a_1+a_2}{2}, \\ \frac{-2c(x-a_2)}{a_2-a_1}, & \frac{a_1+a_2}{2} \leq x < a_2, \end{cases}$$

where $a_1, a_2 \in \mathbb{R}$ and $0 < c \leq 1$.

The above generalized triangular fuzzy set is denoted by $A = ((a_1, c, a_2))$.

Theorem 3.2. ([12]) For two generalized triangular fuzzy sets $A = ((a_1, c_1, a_2))$ and $B = ((b_1, c_2, b_2))$, if $c_1 \leq c_2$ and $\mu_B(x) \geq c_1$ in $[k_1, k_2]$, we have the followings.

1. $A(+)B = (a_1 + b_1,\ \frac{1}{2}(a_1 + a_2) + k_1,\ c_1,\ \frac{1}{2}(a_1 + a_2) + k_2,\ a_2 + b_2)$, i.e., $A(+)B$ is a generalized trapezoidal fuzzy set.

2. $A(-)B = (a_1 - b_2,\ \frac{1}{2}(a_1 + a_2) - k_2,\ c_1,\ \frac{1}{2}(a_1 + a_2) - k_1,\ a_2 - b_1))$, i.e., $A(-)B$ is a generalized trapezoidal fuzzy set.

3. $A(\cdot)B$ is a fuzzy set on (a_1b_1, a_2b_2), but are not a generalized triangular fuzzy set or a generalized trapezoidal fuzzy set. The membership function of $A(\cdot)B$ is

$\mu_{A(\cdot)B}(x)$

$$= \begin{cases} 0, & x < a_1b_1, \ \ a_2b_2 \le x, \\ \frac{1}{2pq}\left(-pb_1 - qa_1 + \sqrt{(pb_1+qa_1)^2 - 4pq(a_1b_1 - x)}\right), & a_1b_1 \le x < a_1b_1 + \frac{1}{2}(a_1+a_2)k_1, \\ \frac{1}{2}, & a_1b_1 + \frac{1}{2}(a_1+a_2)k_1 \le x < a_1b_1 + \frac{1}{2}(a_1+a_2)k_2, \\ \frac{1}{2pq}\left(pb_2 + qa_2 - \sqrt{(pb_2+qa_2)^2 - 4pq(a_2b_2 - x)}\right), & a_1b_1 + \frac{1}{2}(a_1+a_2)k_2 \le x < a_2b_2, \end{cases}$$

where $p = \frac{a_2-a_1}{2c_1}$ and $q = \frac{b_2-b_1}{2c_2}$.

4. $A(/)B$ is a fuzzy set on $(\frac{a_1}{b_2}, \frac{a_2}{b_1})$, but are not a generalized triangular fuzzy set or a generalized trapezoidal fuzzy set. The membership function of $A(/)B$ is

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{a_1}{b_2}, \ \ \frac{a_2}{b_1} \le x, \\ \frac{2c_1c_2(b_2x-a_1)}{c_2(a_2-a_1)+c_1(b_2-b_1)x}, & \frac{a_1}{b_2} \le x < \frac{a_1+a_2}{2k_2}, \\ \frac{1}{2}, & \frac{a_1+a_2}{2k_2} \le x < \frac{a_1+a_2}{2k_1}, \\ \frac{-2c_1c_2(b_1x-a_2)}{c_2(a_2-a_1)+c_1(b_2-b_1)x}, & \frac{a_1+a_2}{2k_1} \le x < \frac{a_2}{b_1}. \end{cases}$$

Example 3.3. ([12]) Let $A = ((2, \frac{1}{2}, 8))$ and $B = ((1, \frac{4}{5}, 5))$. Then

1. Addition :

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < 3, \ \ 13 \le x, \\ \frac{2}{17}(x-3), & 3 \le x < \frac{29}{4}, \\ \frac{1}{2}, & \frac{29}{4} \le x < \frac{35}{4}, \\ \frac{-2}{17}(x-13), & \frac{35}{4} \le x < 13, \end{cases}$$

i.e., $A(+)B = (3, \frac{29}{4}, \frac{1}{2}, \frac{35}{4}, 13)$.

2. Subtraction :

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -3, \ \ 7 \le x, \\ \frac{2}{17}(x+3), & -3 \le x < \frac{5}{4}, \\ \frac{1}{2}, & \frac{5}{4} \le x < \frac{11}{4}, \\ \frac{-2}{17}(x-7), & \frac{11}{4} \le x < 7, \end{cases}$$

i.e., $A(-)B = (-3, \frac{5}{4}, \frac{1}{2}, \frac{11}{4}, 7)$.

3. Multiplication :

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < 2, \;\; 40 \le x, \\ \frac{1}{30}(-11 + \sqrt{121 - 60(2-x)}), & 2 \le x < \frac{45}{4}, \\ \frac{1}{2}, & \frac{45}{4} \le x < \frac{75}{4}, \\ \frac{1}{30}(50 - \sqrt{2500 - 60(40-x)}), & \frac{75}{4} \le x < 40. \end{cases}$$

4. Division :

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{2}{5}, \;\; 8 \le x, \\ \frac{10x-4}{5x+12}, & \frac{2}{5} \le x < \frac{4}{3}, \\ \frac{1}{2}, & \frac{4}{3} \le x < \frac{20}{9}, \\ \frac{-2(x-8)}{5x+12}, & \frac{20}{9} \le x < 8. \end{cases}$$

3.2. Generalized quadratic fuzzy set

Yun and Park generalized the quadratic fuzzy number. A generalized quadratic fuzzy set is symmetric and may not have value 1.

Definition 3.4. ([11]) A fuzzy set A with a membership function

$$\mu_A(x) = \begin{cases} 0, & x < x_1, \;\; x_2 \le x, \\ -a(x-x_1)(x-x_2) = -a(x-m)^2 + p, & x_1 \le x < x_2. \end{cases}$$

where $0 < a$ and $0 < p \le 1$ is called a *generalized quadratic fuzzy set* and denoted by $[[x_1, p, x_2]]$ or $[[a, m, p]]_+$.

Theorem 3.5. ([11]) Let $A = [[x_1, p, x_2]] = [[a, m, p]]_+$ and $B = [[x_3, q, x_4]] = [[b, n, q]]_+$ be generalized quadratic fuzzy sets. Suppose $p \le q$ and $\mu_B(x) \ge p$ on $[k_1, k_2]$. Then we have the followings.

(1) $A(+)B$ is a fuzzy set with a membership function.

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < x_1 + x_3,\ x_2 + x_4 \le x, \\ f_1(x), & x_1 + x_3 \le x < m + k_1, \\ p, & m + k_1 \le x < m + k_2, \\ f_2(x), & m + k_2 \le x < x_2 + x_4, \end{cases}$$

where

$$f_1(x) = \frac{1}{a^2 - 2ab + b^2}\Big(-abm(a + b + an + bn) - abn(am + bm + an + bn) - ab(p + q)p + a^2q + b^2 + 2ab(am + bm + an + bn)x - ab(a + b)x^2 + 2ab(m + n - x) \cdot \sqrt{g(x)}\Big)$$

and

$$f_2(x) = \frac{1}{a^2 - 2ab + b^2}\Big(-abm(a + b + an + bn) - abn(am + bm + an + bn) - ab(p + q)p + a^2q + b^2 + 2ab(am + bm + an + bn)x - ab(a + b)x^2 - 2ab(m + n - x) \cdot \sqrt{g(x)}\Big),$$

where $g(x) = ab(m + n)^2 + (a - b)(p - q) - 2ab(m + n)x + abx^2$.

(2) $A(-)B$ is a fuzzy set with a membership function

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < x_1 - x_4,\ x_2 - x_3 \le x, \\ f_1(x), & x_1 - x_4 \le x < m - k_2, \\ p, & m - k_2 \le x < m - k_1, \\ f_2(x), & m - k_1 \le x < x_2 - x_3, \end{cases}$$

where

$$f_1(x) = \frac{1}{a^2 - 2ab + b^2}\Big(-abm(am + bm - an - bn) - abn(an + bn - am - bm) - ab(p + q) + a^2q + b^2p + 2ab(am + bm - an - bn)x - ab^2x^2 + 2ab(m - n - x) \cdot \sqrt{g(x)}\Big)$$

and

$$f_2(x) = \frac{1}{a^2 - 2ab + b^2}\Big(-abm(am + bm - an - bn) - abn(an + bn - am - bm) - ab(p + q) + a^2q + b^2p + 2ab(am + bm - an - bn)x - ab^2x^2 - 2ab(m - n - x) \cdot \sqrt{g(x)}\Big),$$

where $g(x) = ab(m-n)^2 + (a-b)(p-q) - 2ab(m-n)x + abx^2$.

(3) If $p = q$, $A(\cdot)B$ is a fuzzy set with a membership function

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < x_1x_3,\ \ x_2x_4 \le x, \\ f(x), & x_1x_3 \le x < x_2x_4, \end{cases}$$

where

$$f(x) = \frac{1}{2}(-am^2 - bn^2 + 2p) - \sqrt{ab}x + \frac{1}{2}\sqrt{g(x)},$$

and

$$g(x) = -am^2(am^2 + 3bn^2) - bn^2(bn^2 + 3am^2) + 2(am^2 + bn^2 - 2p)^2 + 8p(am^2 + bn^2 - p)$$
$$+ 8abmnx - \frac{1}{8\sqrt{ab}x}\Big\{-8(am^2 + bn^2 - 2p)^3 + 8(am^2 + bn^2 - 2p)h_1(x) - 16h_2(x)\Big\},$$

and where

$$h_1(x) = am^2(am^2 + 2bn^2) + bn^2(bn^2 + 2am^2) - 6p(am^2 + bn^2 - p) - 4abmnx - 2abx^2$$

and

$$h_2(x) = abm^2n^2(am^2 + bn^2 - 4p) - am^2p(am^2 - 3p) - bn^2p(bn^2 - 3p)$$
$$- 2p^3 - 2abmn(am^2 + bn^2 - 2p)x + ab(am^2 + bn^2 + 2p)x^2.$$

(4) $A(/)B$ is a fuzzy set with a membership function

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{x_1}{x_4},\ \ \frac{x_2}{x_3} \le x, \\ f_1(x), & \frac{x_1}{x_4} \le x < \frac{m}{k_2}, \\ p, & \frac{m}{k_2} \le \frac{m}{k_1}, \\ f_2(x), & \frac{m}{k_1} \le x < \frac{x_2}{x_3}, \end{cases}$$

where

$$f_1(x) = \frac{1}{b^2 - 2abx^2 + a^2x^4}\Big(-b^2(am^2 + p) + 2ab^2mnx - ab(am^2 + bn^2 + p + q)x^2$$
$$+ 2a^2bmnx^3 - a^2(bn^2 - q)x^4 + 2abx(m - nx) \cdot \sqrt{g(x)}\Big)$$

and

$$f_2(x) = \frac{1}{b^2 - 2abx^2 + a^2x^4}\Big(-b^2(am^2+p) + 2ab^2mnx - ab(am^2+bn^2+p+q)x^2 + 2a^2bmnx^3 - a^2(bn^2-q)x^4 - 2abx(m-nx)\cdot\sqrt{g(x)}\Big),$$

and where $g(x) = b(am^2 - p + q) - 2abmnx + a(bn^2 + p - q)x^2$.

Remark 3.6. ([11]) In the case of extended multiplication, if $p \neq q$, the membership function of $A(\cdot)B$ contains so many terms and so the explicit form was not written down.

Example 3.7. ([11]) Let $A = [[2, \frac{2}{3}, 8]]$ and $B = [[3, \frac{3}{4}, 11]]$. Then we have the followings.

(1) The extended addition reduces to

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < 5, \ \ 19 \le x, \\ f_1(x), & 5 \le x < \frac{32}{3}, \\ 2/3, & \frac{32}{3} \le x < \frac{40}{3}, \\ f_2(x), & \frac{40}{3} \le x < 19, \end{cases}$$

where

$$f_1(x) = \frac{1}{4418}\Big(-357204 + 60192x - 2508x^2 + 288(x-12)\sqrt{10321 - 1728x + 72x^2}\Big)$$

and

$$f_2(x) = \frac{1}{4418}\Big(-357204 + 60192x - 2508x^2 - 288(x-12)\sqrt{10321 - 1728x + 72x^2}\Big).$$

(2) The extended subtraction reduces to

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -9,\ \ 5 \le x, \\ f_1(x), & -9 \le x < \frac{-10}{3}, \\ 2/3, & \frac{-10}{3} \le x < \frac{-2}{3}, \\ f_2(x), & \frac{-2}{3} \le x < 5, \end{cases}$$

where

$$f_1(x) = -\frac{1}{4418}\Big(6084 + 10032x + 2508x^2 + 288(x+2)\sqrt{241 + 288x + 72x^2}\Big)$$

and

$$f_2(x) = -\frac{1}{4418}\Big(6084 + 10032x + 2508x^2 - 288(x+2)\sqrt{241 + 288x + 72x^2}\Big).$$

Example 3.8. ([11]) Let $A = [[2, \frac{2}{3}, 8]]$ and $B = [[3, \frac{2}{3}, 11]]$. Then we have the followings.

(1) The extended multiplication reduces to

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < 6,\ \ 88 \le x, \\ f(x), & 6 \le x < 88, \end{cases}$$

where $f(x) = \frac{1}{432}(-553 - 24x + 41\sqrt{1+48x})$.

(2) The extended division reduces to

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{2}{11},\ \ \frac{8}{3} \le x, \\ f_1(x), & \frac{2}{11} \le x < \frac{3}{5}, \\ 2/3, & \frac{3}{5} \le x < \frac{15}{17}, \\ f_2(x), & \frac{15}{17} \le x < \frac{8}{3}, \end{cases}$$

where

$$f_1(x) = -\frac{1}{6561 - 20736x^2 + 16384x^4}\Big(7776 - 34020x + 57702x^2 - 53760x^3 + 25344x^4 + 144x(7x-5)\sqrt{1881 - 5040x + 3400x^2}\Big)$$

and

$$f_2(x) = -\frac{1}{6561 - 20736x^2 + 16384x^4}\Big(7776 - 34020x + 57702x^2 - 53760x^3 + 25344x^4 - 144x(7x-5)\sqrt{1881 - 5040x + 3400x^2}\Big).$$

3.3. Generalized trapezoidal fuzzy set

Lee and Yun generalized the trapezoidal fuzzy number. A generalized trapezoidal fuzzy set is symmetric and may not have value 1.

Definition 3.9. ([9]) A fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_4 \le x, \\ \frac{c(x-a_1)}{a_2-a_1}, & a_1 \le x < a_2, \\ c, & a_2 \le x < a_3, \\ \frac{c(a_4-x)}{a_4-a_3}, & a_3 \le x < a_4. \end{cases}$$

where $a_i \in \mathbb{R}, i = 1,2,3,4$ and $0 < c < 1$, is called *a generalized trapezoidal fuzzy set* and will be denoted by $A = (a_1, a_2, c, a_3, a_4)$.

Remark 3.10. ([9]) A triangular fuzzy number $A = (a_1, a_2, a_3)$ is just a special case of a generalized trapezoidal fuzzy set. In fact, $(a_1, a_2, a_3) = (a_1, a_2, 1, a_2, a_3)$.

Remark 3.11. ([9]) A generalized triangular fuzzy set is also a special case of a generalized trapezoidal fuzzy set. In fact,

$$A = \big((a_1, c_1, a_2)\big) = \Big(a_1, \frac{a_1 + a_2}{2}, c_1, \frac{a_1 + a_2}{2}, a_2\Big)$$

Theorem 3.12. ([9]) Let $A = (a_1, a_2, m_1, a_3, a_4)$ and $B = (b_1, b_2, m_2, b_3, b_4)$, where $a_i, b_i \in \mathbb{R}, i = 1,2,3,4, 0 < m_1 \le m_2 < 1$ and $\mu_B(x) \ge m_1$ in $[p, r]$. Then

1. Addition

$$\mu_{A(+)B}(z)=\begin{cases}0, & z<a_1+b_1, a_4+b_4\le z,\\ \frac{m_1m_2(z-a_1-b_1)}{m_2(a_2-a_1)+m_1(b_2-b_1)}, & a_1+b_1\le z<a_2+b_1+(b_2-b_1)\cdot\frac{m_1}{m_2},\\ m_1, & a_2+b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}\le z<a_3+b_4-(b_4-b_3)\cdot\frac{m_1}{m_2},\\ \frac{m_1m_2(a_4+b_4-z)}{m_2(a_4-a_3)+m_1(b_4-b_3)}, & a_3+b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}\le z<a_4+b_4.\end{cases}$$

2. Subtraction

$$\mu_{A(-)B}(z)=\begin{cases}0, & z<a_1-b_4, a_4-b_1\le z,\\ \frac{m_1m_2(z+b_4-a_1)}{m_2(a_2-a_1)+m_1(b_4-b_3)}, & a_1-b_4\le z<a_2-\left(b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}\right),\\ m_1, & a_2-\left(b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}\right)\le z<a_3-\left(b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}\right),\\ \frac{m_1m_2(a_4-b_1-z)}{m_2(a_4-a_3)+m_1(b_2-b_1)}, & a_3-\left(b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}\right)\le z<a_4-b_1.\end{cases}$$

3. Multiplication

$$\mu_{A(\cdot)B}(z)$$

$$=\begin{cases}0, & z<a_1b_1, a_4b_4\le z,\\ \frac{-D_1+\sqrt{D^2+4m_1m_2(b_2-b_1)(a_2-a_1)z}}{2(b_2-b_1)(a_2-a_1)}, & a_1b_1\le z<a_2\left(b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}\right),\\ m_1, & a_2\left(b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}\right)\le z<a_3\left(b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}\right),\\ \frac{\widetilde{D}_1-\sqrt{\widetilde{D}^2+4m_1m_2(b_4-b_3)(a_4-a_3)z}}{2m_1(b_4-b_3)}, & a_3\left(b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}\right)\le z<a_4b_4,\end{cases}$$

where

$$D=b_1m_2(a_2-a_1)-a_1m_1(b_2-b_1),$$

$$D_1=b_1m_2(a_2-a_1)+a_1m_1(b_2-b_1),$$

$$\widetilde{D}=a_4m_1(b_4-b_3)-b_4m_2(a_4-a_3),$$

$$\widetilde{D}_1=a_4m_1(b_4-b_3)+b_4m_2(a_4-a_3).$$

4. Division

$$\mu_{A(/)B}(z)=\begin{cases} 0, & z<\frac{a_1}{b_4},\ \frac{a_4}{b_1}\le z \\ \frac{m_1m_2(b_4z-a_1)}{m_1(b_4-b_3)z+m_2(a_2-a_1)}, & \frac{a_1}{b_4}\le z<\frac{a_2}{b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}} \\ m_1, & \frac{a_2}{b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}}\le z<\frac{a_3}{b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}} \\ \frac{m_1m_2(a_4-b_1z)}{m_1(b_2-b_1)z+m_2(a_4-a_3)}, & \frac{a_3}{b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}}\le z<\frac{a_4}{b_1}. \end{cases}$$

Example 3.13. ([9]) For two generalized trapezoidal sets, $A=(1,2,\frac{1}{2},3,6)$ and $B=(2,4,\frac{7}{10},5,8)$, we have the followings.

$$\mu_{A(+)B}(z)=\begin{cases} 0, & z<3,14\le z, \\ \frac{7(z-3)}{34}, & 3\le z<\frac{38}{7}, \\ \frac{1}{2}, & \frac{38}{7}\le z<\frac{62}{7}, \\ \frac{7(14-z)}{72}, & \frac{62}{7}\le z<14, \end{cases}$$

$$\mu_{A(-)B}(z)=\begin{cases} 0, & z<-7,4\le z, \\ \frac{7(7+z)}{44}, & -7\le z<-\frac{27}{7}, \\ \frac{1}{2}, & -\frac{27}{7}\le z<-\frac{3}{7}, \\ \frac{7(4-z)}{62}, & -\frac{3}{7}\le z<4, \end{cases}$$

$$\mu_{A(\cdot)B}(z)=\begin{cases} 0, & z<2,48\le z, \\ \frac{-12+\sqrt{4+70z}}{20}, & 2\le z<\frac{48}{7}, \\ \frac{1}{2}, & \frac{48}{7}\le z<\frac{123}{7}, \\ \frac{43-\sqrt{169+35z}}{30}, & \frac{123}{7}\le z<48, \end{cases}$$

$$\mu_{A(/)B}(z)=\begin{cases} 0, & z<\frac{1}{8},3\le z, \\ \frac{7(-1+8z)}{2(7+15z)}, & \frac{1}{8}\le z<\frac{14}{41}, \\ \frac{1}{2}, & \frac{4}{3}\le z<\frac{20}{9}, \\ \frac{7(3-z)}{21+10z}, & \frac{7}{8}\le z<3. \end{cases}$$

4 2-dimensional fuzzy sets

Yun et al. studied a fuzzy number defined on $\mathbb{R}$. If A is a fuzzy number on $\mathbb{R}$, the membership functions $\mu_A(x)$ is piecewise continuous. Byun and Yun found some piecewise continuous function $f_\alpha(t)$ such that the α-cut A_α of A equals to $\{f_\alpha(t) \mid t \in [0,1]\}$ in Theorem 4.1. Using $f_\alpha(t)$, we define parametric operations. Then Byun and Yun had the same results in Theorem 4.4 as the extended operations.

Note that a piecewise continuous function f on $[a,b] \in \mathbb{R}$ means that the function f is continuous on $[a,b]$ except on finitely many points(it may contains a or b) in $[a,b]$.

Theorem 4.1. ([1]) Let A be a fuzzy number defined on $\mathbb{R}$ and $A_\alpha = \{x \in A \mid \mu_A(x) \geq \alpha\}$ be a α-cut of A. Then for all $\alpha \in [0,1]$, there exists a piecewise continuous function $f_\alpha(t)$ defined on $[0,1]$ such that $A_\alpha = \{f_\alpha(t) \mid t \in [0,1]\}$.

Proof. Since A is a fuzzy number defined on $\mathbb{R}$, the membership function $\mu_A(x)$ is piecewise continuous. Let $A_0 = [a,b]$ be the 0-cut of A. Then $\mu_A(x)$ is continuous on $[a,b]$ except on finitely many points $x_1 < x_2 < \cdots < x_n$. Note that x_1 and x_n may be equal to the end points a and b, respectively. Let $\alpha \in [0,1]$ be fixed. Let $a_1^{(\alpha)}$ and $a_2^{(\alpha)}$ be the left and right end points of A_α, respectively. Assume that $x_1 < \cdots < x_i < a_1^{(\alpha)} < x_{i+1} < \cdots < x_{i+m} < a_2^{(\alpha)} < x_{i+m+1} < \cdots < x_n$. If the end points $a_1^{(\alpha)}$ and $a_2^{(\alpha)}$(or one of them) are equal to some x_i, it can be proved similarly. Define

$$f_\alpha(t) = (a_2^{(\alpha)} - a_1^{(\alpha)})t + a_1^{(\alpha)} \quad if \;\; t \in [0,1]$$

except the points

$$t = \frac{x_{i+j} - a_1^{(\alpha)}}{a_2^{(\alpha)} - a_1^{(\alpha)}}, \;\; j = 1, 2, \cdots, m.$$

Then $f_\alpha(t)$ is piecewise continuous on $[0,1]$ and $A_\alpha = \{f_\alpha(t) \mid t \in [0,1]\}$. In fact, if $x \in A_\alpha, \mu_A(x) \geq \alpha$ and $x \neq x_i$ $(i = 1, 2, \cdots, n)$. Thus $a_1^{(\alpha)} \leq x \leq a_2^{(\alpha)}$. If $x = a_1^{(\alpha)}$ or $x = a_2^{(\alpha)}$, $f_\alpha(0) = a_1^{(\alpha)}$ or $f_\alpha(1) = a_2^{(\alpha)}$. If $a_1^{(\alpha)} < x < a_2^{(\alpha)}$, we have

$$0 < \frac{x - a_1^{(\alpha)}}{a_2^{(\alpha)} - a_1^{(\alpha)}} < 1.$$

Let

$$t = \frac{x - a_1^{(\alpha)}}{a_2^{(\alpha)} - a_1^{(\alpha)}}.$$

Then $t \in (0,1)$ and $f_\alpha(t) = x$. Thus $x \in \{f_\alpha(t) \mid t \in [0,1]\}$. This proves that $A_\alpha \subset \{f_\alpha(t) \mid t \in [0,1]\}$. Let $x = f_\alpha(t) = (a_2^{(\alpha)} - a_1^{(\alpha)})t + a_1^{(\alpha)}$ for some $t \in [0,1]$ except $t = \frac{x_{i+j} - a_1^{(\alpha)}}{a_2^{(\alpha)} - a_1^{(\alpha)}}$, $j = 1, 2, \cdots, m$. Then $a_1^{(\alpha)} \leq x \leq a_2^{(\alpha)}$ and $x \neq x_i$ $(i = 1, 2, \cdots, n)$. Thus $\mu_A(x) \geq \alpha$ and $x \in A_\alpha$. The proof is complete. □

We call a fuzzy number A is *continuous* if the membership function $\mu_A(x)$ of A is continuous. If A is a continuous fuzzy number, then the α-cut A_α of A is a closed interval in $\mathbb{R}$.

Corollary 4.2. ([1]) Let A be a continuous fuzzy number defined on $\mathbb{R}$. Then the α-cut $A_\alpha = \{x \in A \mid \mu_A(x) \geq \alpha\}$ becomes a closed interval $[a_1^{(\alpha)}, a_2^{(\alpha)}]$ on $\mathbb{R}$. And for all $\alpha \in [0,1]$, there exists a continuous function $f_\alpha(t)$ defined on $[0,1]$ such that $[a_1^{(\alpha)}, a_2^{(\alpha)}] = \{f_\alpha(t) \mid t \in [0,1]\}$.

The above corresponding function $f_\alpha(t)$ is said to be the *parametric α-function* of A. And the parametric α-function of A is denoted by $f_\alpha(t)$ or $f_A(t)$.

Definition 4.3. ([1]) Let A and B be two continuous fuzzy numbers defined on $\mathbb{R}$ and A_α, B_α, $f_A(t)$, $f_B(t)$ be the α-cuts and parametric α-functions of A and B, respectively. The parametric addition, parametric subtraction, parametric multiplication and parametric division are fuzzy numbers which have their α-cuts as the followings.

(1) parametric addition $A(+)_p B$:

$$(A(+)_p B)_\alpha = \{f_A(t) + f_B(t) \mid t \in [0,1]\}.$$

(2) parametric subtraction $A(-)_p B$:

$$(A(-)_p B)_\alpha = \{f_A(t) - f_B(1-t) \mid t \in [0,1]\}.$$

(3) parametric multiplication $A(\cdot)_p B$:

$$(A(\cdot)_p B)_\alpha = \{f_A(t) \cdot f_B(t) \mid t \in [0,1]\}.$$

(4) parametric division $A(/)_p B$:

$$(A(/)_p B)_\alpha = \{f_A(t)/f_B(1-t) \mid t \in [0,1]\}.$$

Theorem 4.4. ([1]) Let A and B be two continuous fuzzy numbers defined on $\mathbb{R}$. Then we have the followings.

(1) $A(+)_p B = A(+)B$.

(2) $A(-)_p B = A(-)B$.

(3) $A(\cdot)_p B = A(\cdot)B$.

(4) $A(/)_p B = A(/)B$.

Corollary 4.5. ([1]) Let A and B be two triangular fuzzy numbers defined on $\mathbb{R}$. Then we have the followings.

(1) $A(+)_p B = A(+)B$.

(2) $A(-)_p B = A(-)B$.

(3) $A(\cdot)_p B = A(\cdot)B$.

(4) $A(/)_p B = A(/)B$.

Theorem 4.6. ([6]) Let A be a convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_A(x,y) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0,1)$, there exist piecewise continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0,2\pi]$ such that

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}.$$

Proof. Let $\alpha \in (0,1)$ be fixed. Since A is a convex fuzzy number defined on $\mathbb{R}^2$, the α-cut A_α is convex subset in $\mathbb{R}^2$. Let

$$l = inf\{x \mid \mu_A(x,y) = \alpha\} \quad \text{and} \quad m = sup\{x \mid \mu_A(x,y) = \alpha\}$$

The upper boundary of A_α is the graph of a piecewise continuous concave function $h_1(x)$ and the lower boundary of A_α is also the graph of a piecewise continuous convex function $h_2(x)$ defined on $[l,m]$. Since $h_1(x)$ is piecewise continuous, $h_1(x)$ is continuous on $[l,m]$ except finitely many points $l < x_n < x_{n-1} < \cdots < x_1 < m$. Note that x_1 and x_n may equal to the end points m and l, respectively. Similarly, since $h_2(x)$ is also piecewise continuous, $h_2(x)$ is continuous on $[l,m]$ except finitely many points $l < x_{n+1} < x_{n+2} < \cdots < x_{n+m} < m$. Note that x_{n+1} and x_{n+m} may equal to the end points l and m , respectively. If the end points l and m (or one of them) equal to some x_i, we can prove the above facts similarly. Define

$$f_1^\alpha(t) = \frac{1}{2}(m-l)(\cos t - 1) + m, \quad \text{if} \quad t \in [0,\pi]$$

except the points

$$t_i = \cos^{-1}\Big(\frac{2(x_i - m)}{m-l} + 1\Big), \quad i = 1, 2, \cdots, n.$$

Then $f_1^\alpha(t)$ is piecewise continuous on $[0,\pi]$ and

$$\{l \le x \le m \mid x \ne x_i,\ i = 1,2,\cdots,n\} = \{f_1^\alpha(t) \mid t \in [0,\pi],\ t \ne t_i,\ i = 1,2,\cdots,n\}.$$

Define

$$f_1^\alpha(t) = \frac{1}{2}(m-l)(\cos t - 1) + m, \quad \text{if} \ \ t \in [\pi, 2\pi]$$

except the points

$$t_j = \cos^{-1}\Big(\frac{2(x_{n+j} - m)}{m-l} + 1\Big), \quad j = 1,2,\cdots,m.$$

Then $f_1^\alpha(t)$ is piecewise continuous on $[\pi,2\pi]$ and

$$\{l \le x \le m \mid x \ne x_{n+j},\ j = 1,2,\cdots,m\} = \{f_1^\alpha(t) \mid t \in [\pi,2\pi],\ t \ne t_{n+j},\ j = 1,2,\cdots,m\}.$$

The explicit proof for piecewise continuity can be proved by the same way in the proof of Theorem 3.2 ([1]). Focussing the construction of functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$, we outline our proof. Define $f_1^\alpha(t)$ and $f_2^\alpha(t)$ by

$$f_1^\alpha(t) = \frac{1}{2}(m-l)(\cos t - 1) + m, \quad \text{if} \ \ t \in [0, 2\pi]$$

and

$$f_2^\alpha(t) = \begin{cases} h_1(f_1^\alpha(t)), & 0 \le t \le \pi, \\ h_2(f_1^\alpha(t)), & \pi \le t \le 2\pi. \end{cases}$$

Then we have $A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$. The proof is complete. □

If A is a continuous convex fuzzy number defined on $\mathbb{R}^2$, then the α-set A^α is a closed circular convex subset in $\mathbb{R}^2$.

Corollary 4.7. ([6]) Let A be a continuous convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_A(x,y) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0,2\pi]$ such that

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}.$$

Definition 4.8. ([6]) Let A and B be convex fuzzy numbers defined on $\mathbb{R}^2$ and

$$A^\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_A(x,y) = \alpha\} = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},$$

$$B^\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_B(x,y) = \alpha\} = \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, the parametric operations defined by parametric addition, parametric subtraction, parametric multiplication and parametric division are fuzzy numbers that have their α-sets as the followings.

(1) parametric addition $A(+)_pB$:

$$(A(+)_pB)^\alpha = \{(f_1^\alpha(t) + g_1^\alpha(t), f_2^\alpha(t) + g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}.$$

(2) parametric subtraction $A(-)_pB$:

$$(A(-)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \begin{cases} f_1^\alpha(t) - g_1^\alpha(t+\pi), & \text{if } 0 \le t \le \pi, \\ f_1^\alpha(t) - g_1^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi, \end{cases}$$

and

$$y_\alpha(t) = \begin{cases} f_2^\alpha(t) - g_2^\alpha(t+\pi), & \text{if } 0 \le t \le \pi, \\ f_2^\alpha(t) - g_2^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi. \end{cases}$$

(3) parametric multiplication $A(\cdot)_pB$:

$$(A(\cdot)_pB)^\alpha = \{(f_1^\alpha(t) \cdot g_1^\alpha(t), f_2^\alpha(t) \cdot g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}.$$

(4) parametric division $A(/)_pB$:

$$(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t+\pi)} \quad (0 \le t \le \pi), \quad x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi)$$

and

$$y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t+\pi)} \quad (0 \le t \le \pi), \quad y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi).$$

For $\alpha = 0$ and $\alpha = 1$, define

$$(A(*)_p B)^0 = \lim_{\alpha \to 0^+} (A(*)_p B)^\alpha \quad \text{and} \quad (A(*)_p B)^1 = \lim_{\alpha \to 1^-} (A(*)_p B)^\alpha,$$

where $* = +, -, \cdot, /$.

4.1. 2-dimensional triangular fuzzy number

In this section, Kim and Yun defined the 2-dimensional triangular fuzzy numbers on $\mathbb{R}^2$ as a generalization of triangular fuzzy numbers on $\mathbb{R}$. Then Kim and Yun want to defined the parametric operations between two 2-dimensional triangular fuzzy numbers. For that, Kim and Yun had to calculate operations between α-cuts in $\mathbb{R}^2$. The α-cuts are intervals in $\mathbb{R}$ but in $\mathbb{R}^2$ the α-cuts are regions, which makes the existing method of calculations between α-cuts unusable. We interpret the existing method from a different perspective and apply the method to the region valued α-cuts on $\mathbb{R}^2$.

Definition 4.9. ([6]) A fuzzy set A with a membership function

$$\mu_A(x,y) = \begin{cases} 1 - \sqrt{\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}}, & b^2(x-x_1)^2 + a^2(y-y_1)^2 \le a^2 b^2, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b > 0$ is called the *2-dimensional triangular fuzzy number* and denoted by $(a,\ x_1,\ b,\ y_1)^2$.

Note that $\mu_A(x,y)$ is a cone. The intersections of $\mu_A(x,y)$ and the horizontal planes $z = \alpha \ \ (0 < \alpha < 1)$ are ellipses. The intersections of $\mu_A(x,y)$ and the vertical planes $y - y_1 = k(x - x_1) \ \ (k \in \mathbb{R})$ are symmetric triangular fuzzy numbers in those planes. If

$a = b$, ellipses become circles. The α-cut A_α of a 2-dimensional triangular fuzzy number $A = (a, x_1, b, y_1)^2$ is an interior of ellipse in an xy-plane including the boundary

$$A_\alpha = \Big\{(x,y) \in \mathbb{R}^2 \;\Big|\; b^2(x - x_1)^2 + a^2(y - y_1)^2 \le a^2b^2(1-\alpha)^2\Big\}$$
$$= \Big\{(x,y) \in \mathbb{R}^2 \;\Big|\; \Big(\frac{x - x_1}{a(1-\alpha)}\Big)^2 + \Big(\frac{y - y_1}{b(1-\alpha)}\Big)^2 \le 1\Big\}.$$

In Remark 2.7, if $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ is the α-cut of $A = (a_1, a_2, a_3)$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ is the α-cut of $B = (b_1, b_2, b_3)$, then $(A(+)B)_\alpha = A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}]$. However in a 2-dimensional case, $A_\alpha(+)B_\alpha$ cannot be calculated by the same way since α-cuts are not intervals but subsets of $\mathbb{R}^2$. For the calculation in a 2-dimensional case, we consider the operations of α-cuts on $\mathbb{R}$ by using a parameter as in Definition 4.3.

Theorem 4.10. ([6]) Let $A = (a_1,\ x_1,\ b_1,\ y_1)^2$ and $B = (a_2,\ x_2,\ b_2,\ y_2)^2$ be two 2-dimensional triangular fuzzy numbers. Then we have the following.

(1) $A(+)_pB = \Big(a_1 + a_2,\ x_1 + x_2,\ b_1 + b_2,\ y_1 + y_2\Big)^2$.

(2) $A(-)_pB = \Big(a_1 + a_2,\ x_1 - x_2,\ b_1 + b_2,\ y_1 - y_2\Big)^2$.

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = x_1x_2 + (x_1a_2 + x_2a_1)(1-\alpha)\cos t + a_1a_2(1-\alpha)^2\cos^2 t,$$
$$y_\alpha(t) = y_1y_2 + (y_1b_2 + y_2b_1)(1-\alpha)\sin t + b_1b_2(1-\alpha)^2\sin^2 t.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{x_1 + a_1(1-\alpha)\cos t}{x_2 - a_2(1-\alpha)\cos t}\,, \quad y_\alpha(t) = \frac{y_1 + b_1(1-\alpha)\sin t}{y_2 - b_2(1-\alpha)\sin t}.$$

Thus $A(+)_pB$ and $A(-)_pB$ become 2-dimensional triangular fuzzy numbers, but $A(\cdot)_pB$ and $A(/)_pB$ are not 2-dimensional triangular fuzzy numbers.

Proof. Since A and B are convex fuzzy numbers defined on $\mathbb{R}^2$, by Theorem 4.6, there exists $f_i^\alpha(t), g_i^\alpha(t)$ $(i=1,2)$ such that

$$A^\alpha = \{(x,y)\in\mathbb{R}^2 \mid \mu_A(x,y)=\alpha\} = \{(f_1^\alpha(t), f_2^\alpha(t))\in\mathbb{R}^2 \mid 0\le t\le 2\pi\},$$

$$B^\alpha = \{(x,y)\in\mathbb{R}^2 \mid \mu_B(x,y)=\alpha\} = \{(g_1^\alpha(t), g_2^\alpha(t))\in\mathbb{R}^2 \mid 0\le t\le 2\pi\}.$$

Since $A=(a_1,x_1,b_1,y_1)^2$ and $B=(a_2,x_2,b_2,y_2)^2$, we have

$$f_1^\alpha(t) = x_1 + a_1(1-\alpha)\cos t,\ \ f_2^\alpha(t) = y_1 + b_1(1-\alpha)\sin t,$$

$$g_1^\alpha(t) = x_2 + a_2(1-\alpha)\cos t,\ \ g_2^\alpha(t) = y_2 + b_2(1-\alpha)\sin t.$$

(1) Since

$$f_1^\alpha(t)+g_1^\alpha(t) = x_1+x_2+(a_1+a_2)(1-\alpha)\cos t,$$

$$f_2^\alpha(t)+g_2^\alpha(t) = y_1+y_2+(b_1+b_2)(1-\alpha)\sin t,$$

we have

$$(A(+)_pB)^\alpha = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, \left(\frac{x-x_1-x_2}{(a_1+a_2)(1-\alpha)}\right)^2+\left(\frac{y-y_1-y_2}{(b_1+b_2)(1-\alpha)}\right)^2=1\right\}.$$

Thus

$$A(+)_pB = \Big(a_1+a_2,\ x_1+x_2,\ b_1+b_2,\ y_1+y_2\Big)^2.$$

(2) If $0\le t\le\pi$,

$$f_1^\alpha(t)-g_1^\alpha(t+\pi) = x_1-x_2+(a_1+a_2)(1-\alpha)\cos t,$$

$$f_2^\alpha(t)-g_2^\alpha(t+\pi) = y_1-y_2+(b_1+b_2)(1-\alpha)\sin t.$$

In the case of $\pi\le t\le 2\pi$, we have

$$f_1^\alpha(t)-g_1^\alpha(t-\pi) = f_1^\alpha(t)-g_1^\alpha(t+\pi),$$

$$f_2^\alpha(t)-g_2^\alpha(t-\pi) = f_2^\alpha(t)-g_2^\alpha(t+\pi).$$

Thus

$$(A(-)_pB)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \,\Big|\, \Big(\frac{x - x_1 + x_2}{(a_1 + a_2)(1-\alpha)}\Big)^2 + \Big(\frac{y - y_1 + y_2}{(b_1 + b_2)(1-\alpha)}\Big)^2 = 1\Big\},$$

i.e.,

$$A(-)_pB = \Big(a_1 + a_2,\ x_1 - x_2,\ b_1 + b_2,\ y_1 - y_2\Big)^2.$$

(3) Let $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$. Since

$$f_1^\alpha(t) = x_1 + a_1(1-\alpha)\cos t,\ \ f_2^\alpha(t) = y_1 + b_1(1-\alpha)\sin t,$$

$$g_1^\alpha(t) = x_2 + a_2(1-\alpha)\cos t,\ \ g_2^\alpha(t) = y_2 + b_2(1-\alpha)\sin t,$$

we have

$$x_\alpha(t) = f_1^\alpha(t)\cdot g_1^\alpha(t) = x_1x_2 + (x_1a_2 + x_2a_1)(1-\alpha)\cos t + a_1a_2(1-\alpha)^2\cos^2 t,$$

$$y_\alpha(t) = f_2^\alpha(t)\cdot g_2^\alpha(t) = y_1y_2 + (y_1b_2 + y_2b_1)(1-\alpha)\sin t + b_1b_2(1-\alpha)^2\sin^2 t.$$

(4) Let $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$. Similarly, we have

$$x_\alpha(t) = \frac{x_1 + a_1(1-\alpha)\cos t}{x_2 - a_2(1-\alpha)\cos t}\ , \quad y_\alpha(t) = \frac{y_1 + b_1(1-\alpha)\sin t}{y_2 - b_2(1-\alpha)\sin t}.$$

The proof is complete. □

Example 4.11. ([6]) Let $A = (6,\ 3,\ 8,\ 5)^2$ and $B = (4,\ 2,\ 5,\ 3)^2$. Then by Theorem 4.10, we have the following.

(1) $A(+)_pB = (10,\ 5,\ 13,\ 8)^2$.

(2) $A(-)_pB = (10,\ 1,\ 13,\ 2)^2$.

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = 6 + 24(1-\alpha)\cos t + 24(1-\alpha)^2\cos^2 t,$$

$$y_\alpha(t) = 15 + 49(1-\alpha)\sin t + 40(1-\alpha)^2 \sin^2 t.$$

(4) $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{3 + 6(1-\alpha)\cos t}{2 - 4(1-\alpha)\cos t}, \quad y_\alpha(t) = \frac{5 + 8(1-\alpha)\sin t}{3 - 5(1-\alpha)\sin t}.$$

Thus $A(+)_p B$ and $A(-)_p B$ become 2-dimensional triangular fuzzy numbers, but $A(\cdot)_p B$ and $A(/)_p B$ are not 2-dimensional triangular fuzzy numbers.

4.2. Generalized 2-dimensional triangular fuzzy set

Kim and Yun defined the generalized 2-dimensional triangular fuzzy numbers on $\mathbb{R}^2$ as a generalization of generalized triangular fuzzy numbers on $\mathbb{R}$. Then Kim and Yun want to defined the parametric operations between two generalized 2-dimensional triangular fuzzy numbers.

Definition 4.12. ([5]) A fuzzy set A with a membership function

$$\mu_A(x,y) = \begin{cases} h - \sqrt{\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}}, & b^2(x-x_1)^2 + a^2(y-y_1)^2 \le a^2\, b^2\, h^2, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b > 0$ and $0 < h < 1$ is called *the generalized 2-dimensional triangular fuzzy set* and denoted by $((a,\ x_1,\ h,\ b,\ y_1))^2$.

Note that $\mu_A(x,y)$ is a cone. The intersections of $\mu_A(x,y)$ and the horizontal planes $z = \alpha \ \ (0 < \alpha < h)$ are ellipses. The intersections of $\mu_A(x,y)$ and the vertical planes $y - y_1 = k(x - x_1) \ \ (k \in \mathbb{R})$ are symmetric triangular fuzzy numbers in those planes. If $a = b$, ellipses become circles. The α-cut A_α of a generalized 2-dimensional triangular fuzzy number $A = (a, x_1, h, b, y_1)^2$ is an interior of ellipse in an xy-plane including the boundary

$$A_\alpha = \Big\{(x,y) \in \mathbb{R}^2 \,\Big|\, b^2(x-x_1)^2 + a^2(y-y_1)^2 \le a^2b^2(h-\alpha)^2\Big\}$$
$$= \Big\{(x,y) \in \mathbb{R}^2 \,\Big|\, \Big(\frac{x-x_1}{a(h-\alpha)}\Big)^2 + \Big(\frac{y-y_1}{b(h-\alpha)}\Big)^2 \le 1\Big\}.$$

Theorem 4.13. ([5]) Let $A = ((a_1,\ x_1,\ h_1,\ b_1,\ y_1))^2$ and $B = ((a_2,\ x_2,\ h_2,\ b_2,\ y_2))^2$ be two generalized 2-dimensional triangular fuzzy sets. If $0 < h_1 < h_2 \le 1$, then we have the following.

(1) For $0 < \alpha < h_1$, the α-set of $A(+)_pB$ is

$$(A(+)_pB)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \,\Big|\, \Big(\frac{x-x_1-x_2}{a_1(h_1-\alpha)+a_2(h_2-\alpha)}\Big)^2 + \Big(\frac{y-y_1-y_2}{b_1(h_1-\alpha)+b_2(h_2-\alpha)}\Big)^2 = 1\Big\}.$$

(2) For $0 < \alpha < h_1$, the α-set of $A(-)_pB$ is

$$(A(-)_pB)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \,\Big|\, \Big(\frac{x-x_1+x_2}{a_1(h_1-\alpha)+a_2(h_2-\alpha)}\Big)^2 + \Big(\frac{y-y_1+y_2}{b_1(h_1-\alpha)+b_2(h_2-\alpha)}\Big)^2 = 1\Big\}.$$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = x_1x_2 + (x_1a_2(h_2-\alpha) + x_2a_1(h_1-\alpha))\cos t + a_1a_2(h_1-\alpha)(h_2-\alpha)\cos^2 t,\ \ 0 < \alpha < h_1,$$

$$y_\alpha(t) = y_1y_2 + (y_1b_2(h_2-\alpha) + y_2b_1(h_1-\alpha))\sin t + b_1b_2(h_1-\alpha)(h_2-\alpha)\sin^2 t,\ \ 0 < \alpha < h_1.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{x_1 + a_1(h_1-\alpha)\cos t}{x_2 - a_2(h_2-\alpha)\cos t},\quad y_\alpha(t) = \frac{y_1 + b_1(h_1-\alpha)\sin t}{y_2 - b_2(h_2-\alpha)\sin t},\quad 0 < \alpha < h_1.$$

Proof. Since A and B are convex fuzzy sets defined on $\mathbb{R}^2$, by Theorem 4.6, there exists $f_i^\alpha(t), g_i^\alpha(t)$ $(i = 1,2)$ such that

$$A^\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_A(x,y) = \alpha\} = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},\ \ 0 \le \alpha \le h_1$$

and

$$B^\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_B(x,y) = \alpha\} = \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},\ \ 0 \le \alpha \le h_2.$$

Since $A = ((a_1, x_1, h_1, b_1, y_1))^2$ and $B = ((a_2, x_2, h_2, b_2, y_2))^2$, we have

$$f_1^{\alpha}(t) = x_1 + a_1(h_1 - \alpha)\cos t, \;\; f_2^{\alpha}(t) = y_1 + b_1(h_1 - \alpha)\sin t, \;\; 0 \le \alpha \le h_1$$

and

$$g_1^{\alpha}(t) = x_2 + a_2(h_2 - \alpha)\cos t, \;\; g_2^{\alpha}(t) = y_2 + b_2(h_2 - \alpha)\sin t, \;\; 0 \le \alpha \le h_2.$$

(1) If $0 < \alpha < h_1$, since

$$f_1^{\alpha}(t) + g_1^{\alpha}(t) = x_1 + x_2 + (a_1(h_1 - \alpha) + a_2(h_2 - \alpha))\cos t$$

and

$$f_2^{\alpha}(t) + g_2^{\alpha}(t) = y_1 + y_2 + (b_1(h_1 - \alpha) + b_2(h_2 - \alpha))\sin t,$$

we have

$$(A(+)_p B)^{\alpha} = \Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; \Big(\frac{x - x_1 - x_2}{a_1(h_1 - \alpha) + a_2(h_2 - \alpha)}\Big)^2 + \Big(\frac{y - y_1 - y_2}{b_1(h_1 - \alpha) + b_2(h_2 - \alpha)}\Big)^2 = 1\Big\}.$$

Furthermore, we have

$$(A(+)_p B)^0 = \lim_{\alpha \to 0^+}(A(+)_p B)^{\alpha} = \Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; \Big(\frac{x - x_1 - x_2}{a_1 h_1 + a_2 h_2}\Big)^2 + \Big(\frac{y - y_1 - y_2}{b_1 h_1 + b_2 h_2}\Big)^2 = 1\Big\},$$

$$(A(+)_p B)^{h_1} = \lim_{\alpha \to h_1^-}(A(+)_p B)^{\alpha} = \Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; \Big(\frac{x - x_1 - x_2}{a_2(h_2 - h_1)}\Big)^2 + \Big(\frac{y - y_1 - y_2}{b_2(h_2 - h_1)}\Big)^2 = 1\Big\},$$

and

$$(A(+)_p B)^{\alpha} = \varnothing, \quad h_1 < \alpha \le h_2.$$

(2) If $0 \le t \le \pi$ and $0 < \alpha < h_1$,

$$f_1^{\alpha}(t) - g_1^{\alpha}(t + \pi) = x_1 - x_2 + (a_1(h_1 - \alpha) + a_2(h_2 - \alpha))\cos t$$

$$f_2^{\alpha}(t) - g_2^{\alpha}(t + \pi) = y_1 - y_2 + (b_1(h_1 - \alpha) + b_2(h_2 - \alpha))\sin t.$$

In the case of $\pi \le t \le 2\pi$, we have

$$f_1^\alpha(t) - g_1^\alpha(t-\pi) = f_1^\alpha(t) - g_1^\alpha(t+\pi)$$

and

$$f_2^\alpha(t) - g_2^\alpha(t-\pi) = f_2^\alpha(t) - g_2^\alpha(t+\pi).$$

Thus

$$(A(-)_pB)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \;\Big|\; \Big(\frac{x - x_1 + x_2}{a_1(h_1-\alpha) + a_2(h_2-\alpha)}\Big)^2 + \Big(\frac{y - y_1 + y_2}{b_1(h_1-\alpha) + b_2(h_2-\alpha)}\Big)^2 = 1\Big\}.$$

Furthermore, we have

$$(A(-)_pB)^0 = \lim_{\alpha \to 0^+}(A(-)_pB)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \;\Big|\; \Big(\frac{x - x_1 + x_2}{a_1h_1 + a_2h_2}\Big)^2 + \Big(\frac{y - y_1 + y_2}{b_1h_1 + b_2h_2}\Big)^2 = 1\Big\},$$

$$(A(-)_pB)^{h_1} = \lim_{\alpha \to h_1^-}(A(-)_pB)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \;\Big|\; \Big(\frac{x - x_1 + x_2}{a_2(h_2 - h_1)}\Big)^2 + \Big(\frac{y - y_1 + y_2}{b_2(h_2 - h_1)}\Big)^2 = 1\Big\},$$

and

$$(A(-)_pB)^\alpha = \varnothing, \quad h_1 < \alpha \le h_2.$$

(3) Let $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$. Since

$$f_1^\alpha(t) = x_1 + a_1(h_1-\alpha)\cos t,\;\; f_2^\alpha(t) = y_1 + b_1(h_1-\alpha)\sin t$$

and

$$g_1^\alpha(t) = x_2 + a_2(h_2-\alpha)\cos t,\;\; g_2^\alpha(t) = y_2 + b_2(h_2-\alpha)\sin t,$$

we have

$$\begin{aligned} x_\alpha(t) &= f_1^\alpha(t)\cdot g_1^\alpha(t) \\ &= x_1x_2 + (x_1a_2(h_2-\alpha) + x_2a_1(h_1-\alpha))\cos t + a_1a_2(h_1-\alpha)(h_2-\alpha)\cos^2 t, \quad 0 < \alpha < h_1 \end{aligned}$$

and

$$y_\alpha(t) = f_2^\alpha(t)\cdot g_2^\alpha(t)$$

$$= y_1y_2 + (y_1b_2(h_2-\alpha) + y_2b_1(h_1-\alpha))\sin t + b_1b_2(h_1-\alpha)(h_2-\alpha)\sin^2 t, \quad 0<\alpha<h_1.$$

Furthermore, we have

$$x_0(t) = \lim_{\alpha\to 0^+} x_\alpha(t) = x_1x_2 + (x_1a_2h_2 + x_2a_1h_1)\cos t + a_1a_2h_1h_2\cos^2 t,$$

$$y_0(t) = \lim_{\alpha\to 0^+} y_\alpha(t) = y_1y_2 + (y_1b_2h_2 + y_2b_1h_1)\sin t + b_1b_2h_1h_2\sin^2 t,$$

$$x_{h_1}(t) = \lim_{\alpha\to h_1^-} x_\alpha(t) = x_1x_2 + x_1a_2(h_2-h_1)\cos t,$$

$$y_{h_1}(t) = \lim_{\alpha\to h_1^-} y_\alpha(t) = y_1y_2 + y_1b_2(h_2-h_1)\sin t,$$

and

$$(A(\cdot)_pB)^\alpha = \varnothing, \quad h_1<\alpha\le h_2.$$

(4) Let $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0\le t\le 2\pi\}$. Similarly, we have

$$x_\alpha(t) = \frac{x_1 + a_1(h_1-\alpha)\cos t}{x_2 - a_2(h_2-\alpha)\cos t}, \quad y_\alpha(t) = \frac{y_1 + b_1(h_1-\alpha)\sin t}{y_2 - b_2(h_2-\alpha)\sin t}, \quad 0<\alpha<h_1.$$

Furthermore, we have

$$x_0(t) = \lim_{\alpha\to 0^+} x_\alpha(t) = \frac{x_1 + a_1h_1\cos t}{x_2 - a_2h_2\cos t}, \quad y_0(t) = \lim_{\alpha\to 0^+} y_\alpha(t) = \frac{y_1 + b_1h_1\sin t}{y_2 - b_2h_2\sin t},$$

$$x_{h_1}(t) = \lim_{\alpha\to h_1^-} x_\alpha(t) = \frac{x_1}{x_2 - a_2(h_2-h_1)\cos t}, \quad y_{h_1}(t) = \lim_{\alpha\to h_1^-} y_\alpha(t) = \frac{y_1}{y_2 - b_2(h_2-h_1)\sin t}$$

and

$$(A(/)_pB)^\alpha = \varnothing, \quad h_1<\alpha\le h_2.$$

The proof is complete. □

Example 4.14. ([5]) Let $A = ((6,\ 3,\ \frac{1}{2},\ 8,\ 5))^2$ and $B = ((4,\ 2,\ \frac{2}{3},\ 5,\ 3))^2$. Then by Theorem 4.13, we have the following.

(1) For $0 < \alpha < \frac{1}{2}$, the α-set of $A(+)_pB$ is

$$(A(+)_pB)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \Big| \Big(\frac{3x-15}{17-30\alpha}\Big)^2 + \Big(\frac{3y-24}{22-39\alpha}\Big)^2 = 1\Big\}.$$

(2) For $0 < \alpha < \frac{1}{2}$, the α-set of $A(-)_pB$ is

$$(A(-)_pB)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \Big| \Big(\frac{3x-3}{17-30\alpha}\Big)^2 + \Big(\frac{3y-6}{22-39\alpha}\Big)^2 = 1\Big\}.$$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = 6 + (14-24\alpha)\cos t + 4(1-2\alpha)(2-3\alpha)\cos^2 t, \quad 0 < \alpha < \frac{1}{2},$$
$$y_\alpha(t) = 15 + \Big(\frac{86}{3} - 49\alpha\Big)\sin t + 20(1-2\alpha)\Big(\frac{2}{3} - \alpha\Big)\sin^2 t, \quad 0 < \alpha < \frac{1}{2}.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{9+9(1-2\alpha)\cos t}{6-4(2-3\alpha)\cos t}, \quad y_\alpha(t) = \frac{15+12(1-2\alpha)\sin t}{9-15(2-3\alpha)\sin t}, \quad 0 < \alpha < \frac{1}{2}.$$

Remark 4.15. ([5]) $A(+)_pB$ and $A(-)_pB$ become truncated cones, $A(\cdot)_pB$ becomes a twisted truncated cone and $A(/)_pB$ becomes a more complicated type that cannot be explained.

4.3. 2-dimensional quadratic fuzzy number

Kang and Yun defined the 2-dimensional quadratic fuzzy numbers on $\mathbb{R}^2$ as a generalization of quadratic fuzzy numbers on $\mathbb{R}$. Then Kang and Yun want to defined the parametric operations between two 2-dimensional quadratic fuzzy numbers.

Definition 4.16. ([2]) A fuzzy set A with a membership function

$$\mu_A(x,y) = \begin{cases} 1 - \big(\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}\big), & b^2(x-x_1)^2 + a^2(y-y_1)^2 \le a^2b^2, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b > 0$ is called the *2-dimensional quadratic fuzzy number* and denoted by $[a,\ x_1,\ b,\ y_1]^2$.

Note that $\mu_A(x,y)$ is a cone. The intersections of $\mu_A(x,y)$ and the horizontal planes $z = \alpha$ $(0 < \alpha < 1)$ are ellipses. The intersections of $\mu_A(x,y)$ and the vertical planes $y - y_1 = k(x - x_1)$ $(k \in \mathbb{R})$ are symmetric quadratic fuzzy numbers in those planes. If $a = b$, ellipses become circles. The α-cut A_α of a 2-dimensional quadratic fuzzy number $A = [a, x_1, b, y_1]^2$ is an interior of ellipse in an xy-plane including the boundary

$$\begin{aligned} A_\alpha &= \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, b^2(x-x_1)^2 + a^2(y-y_1)^2 \le a^2b^2(1-\alpha)\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \frac{(x-x_1)^2}{a^2(1-\alpha)} + \frac{(y-y_1)^2}{b^2(1-\alpha)} \le 1\right\}. \end{aligned}$$

Theorem 4.17. ([2]) Let $A = [a_1,\ x_1,\ b_1,\ y_1]^2$ and $B = [a_2,\ x_2,\ b_2,\ y_2]^2$ be two 2-dimensional quadratic fuzzy numbers. Then we have the following.

(1) $A(+)_pB = \Big[a_1 + a_2,\ x_1 + x_2,\ b_1 + b_2,\ y_1 + y_2\Big]^2$.

(2) $A(-)_pB = \Big[a_1 + a_2,\ x_1 - x_2,\ b_1 + b_2,\ y_1 - y_2\Big]^2$.

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = x_1x_2 + (x_1a_2 + x_2a_1)\sqrt{1-\alpha}\cos t + a_1a_2(1-\alpha)\cos^2 t$$

and

$$y_\alpha(t) = y_1y_2 + (y_1b_2 + y_2b_1)\sqrt{1-\alpha}\sin t + b_1b_2(1-\alpha)\sin^2 t.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{x_1 + a_1\sqrt{1-\alpha}\cos t}{x_2 - a_2\sqrt{1-\alpha}\cos t} \quad \text{and} \quad y_\alpha(t) = \frac{y_1 + b_1\sqrt{1-\alpha}\sin t}{y_2 - b_2\sqrt{1-\alpha}\sin t}.$$

Thus $A(+)_pB$ and $A(-)_pB$ become 2-dimensional quadratic fuzzy numbers, but $A(\cdot)_pB$ and $A(/)_pB$ are not 2-dimensional quadratic fuzzy numbers.

Proof. Since A and B are convex fuzzy numbers defined on $\mathbb{R}^2$, by Theorem 4.6, there exists $f_i^\alpha(t)$, $g_i^\alpha(t)$ $(i = 1,2)$ such that

$$A^\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_A(x,y) = \alpha\} = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$$

and

$$B^{\alpha} = \{(x,y) \in \mathbb{R}^2 \mid \mu_B(x,y) = \alpha\} = \{(g_1^{\alpha}(t), g_2^{\alpha}(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}.$$

Since $A = [a_1, x_1, b_1, y_1]^2$ and $B = [a_2, x_2, b_2, y_2]^2$, we have

$$f_1^{\alpha}(t) = x_1 + a_1\sqrt{1-\alpha}\cos t, \;\; f_2^{\alpha}(t) = y_1 + b_1\sqrt{1-\alpha}\sin t$$

and

$$g_1^{\alpha}(t) = x_2 + a_2\sqrt{1-\alpha}\cos t, \;\; g_2^{\alpha}(t) = y_2 + b_2\sqrt{1-\alpha}\sin t.$$

(1) Since

$$f_1^{\alpha}(t) + g_1^{\alpha}(t) = x_1 + x_2 + (a_1 + a_2)\sqrt{1-\alpha}\cos t$$

and

$$f_2^{\alpha}(t) + g_2^{\alpha}(t) = y_1 + y_2 + (b_1 + b_2)\sqrt{1-\alpha}\sin t,$$

we have

$$(A(+)_p B)^{\alpha} = \Big\{(x,y) \in \mathbb{R}^2 \;\Big|\; \frac{(x - x_1 - x_2)^2}{(a_1 + a_2)^2(1-\alpha)} + \frac{(y - y_1 - y_2)^2}{(b_1 + b_2)^2(1-\alpha)} = 1\Big\}.$$

Thus $A(+)_p B = \Big[a_1 + a_2,\ x_1 + x_2,\ b_1 + b_2,\ y_1 + y_2\Big]^2$.

(2) If $0 \le t \le \pi$,

$$f_1^{\alpha}(t) - g_1^{\alpha}(t+\pi) = x_1 - x_2 + (a_1 + a_2)\sqrt{1-\alpha}\cos t$$

and

$$f_2^{\alpha}(t) - g_2^{\alpha}(t+\pi) = y_1 - y_2 + (b_1 + b_2)\sqrt{1-\alpha}\sin t.$$

In the case of $\pi \le t \le 2\pi$, we have

$$f_1^{\alpha}(t) - g_1^{\alpha}(t-\pi) = f_1^{\alpha}(t) - g_1^{\alpha}(t+\pi)$$

and

$$f_2^\alpha(t) - g_2^\alpha(t-\pi) = f_2^\alpha(t) - g_2^\alpha(t+\pi).$$

Thus

$$(A(-)_pB)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \;\Big|\; \frac{(x-x_1+x_2)^2}{(a_1+a_2)^2(1-\alpha)} + \frac{(y-y_1+y_2)^2}{(b_1+b_2)^2(1-\alpha)} = 1\Big\},$$

i.e.,

$$A(-)_pB = \Big[a_1+a_2,\ x_1-x_2,\ b_1+b_2,\ y_1-y_2\Big]^2.$$

(3) Let $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$. Since

$$f_1^\alpha(t) = x_1 + a_1\sqrt{1-\alpha}\cos t, f_2^\alpha(t) = y_1 + b_1\sqrt{1-\alpha}\sin t$$

and

$$g_1^\alpha(t) = x_2 + a_2\sqrt{1-\alpha}\cos t,\ \ g_2^\alpha(t) = y_2 + b_2\sqrt{1-\alpha}\sin t,$$

we have

$$x_\alpha(t) = f_1^\alpha(t)\cdot g_1^\alpha(t) = x_1x_2 + (x_1a_2 + x_2a_1)\sqrt{1-\alpha}\cos t + a_1a_2(1-\alpha)\cos^2 t$$

and

$$y_\alpha(t) = f_2^\alpha(t)\cdot g_2^\alpha(t) = y_1y_2 + (y_1b_2 + y_2b_1)\sqrt{1-\alpha}\sin t + b_1b_2(1-\alpha)\sin^2 t.$$

(4) Let $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$. Similarly, we have

$$x_\alpha(t) = \frac{x_1 + a_1\sqrt{1-\alpha}\cos t}{x_2 - a_2\sqrt{1-\alpha}\cos t} \quad \text{and} \quad y_\alpha(t) = \frac{y_1 + b_1\sqrt{1-\alpha}\sin t}{y_2 - b_2\sqrt{1-\alpha}\sin t}.$$

The proof is complete. □

Example 4.18. ([2]) Let $A = [6,\ 3,\ 8,\ 5]^2$ and $B = [4,\ 2,\ 5,\ 3]^2$. Then by Theorem 4.17, we have the following.

(1) $A(+)_pB = [10,\ 5,\ 13,\ 8]^2$.

(2) $A(-)_pB = [10,\ 1,\ 13,\ 2]^2$.

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = 6 + 24\sqrt{1-\alpha}\cos t + 24(1-\alpha)\cos^2 t$$

and

$$y_\alpha(t) = 15 + 49\sqrt{1-\alpha}\sin t + 40(1-\alpha)\sin^2 t.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{3 + 6\sqrt{1-\alpha}\cos t}{2 - 4\sqrt{1-\alpha}\cos t} \quad \text{and} \quad y_\alpha(t) = \frac{5 + 8\sqrt{1-\alpha}\sin t}{3 - 5\sqrt{1-\alpha}\sin t}.$$

Thus $A(+)_pB$ and $A(-)_pB$ become 2-dimensional quadratic fuzzy numbers, but $A(\cdot)_pB$ and $A(/)_pB$ are not 2-dimensional quadratic fuzzy numbers.

5 2-dimensional parametric operations

5.1. Parametric operations between 2-dimensional triangular fuzzy number and trapezoidal fuzzy set

We generalized the trapezoidal fuzzy numbers on $\mathbb{R}$ to $\mathbb{R}^2$ and calculate the parametric operations between 2-dimensional triangular fuzzy number and trapezoidal fuzzy set ([7]).

Definition 5.1. ([7]) A fuzzy set B with a membership function

$$\mu_B(x,y) = \begin{cases} h - \sqrt{\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}}, & h-1 \le \sqrt{\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}} \le h, \\ 1, & 0 \le \sqrt{\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}} \le h-1, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b > 0$ and $1 < h$ is called *the 2-dimensional trapezoidal fuzzy set* and denoted by $B = ((a,\ x_1,\ h,\ b,\ y_1))^2$.

$\mu_B(x,y)$ is a truncated cone. The intersections of $\mu_B(x,y)$ and the horizontal planes $z = \alpha \ \ (0 < \alpha < 1)$ are ellipses. The intersections of $\mu_B(x,y)$ and the vertical planes $y - y_1 = k(x - x_1) \ \ (k \in \mathbb{R})$ are symmetric trapezoidal fuzzy sets in those planes. If $a = b$, ellipses become circles. The α-cut B_α of a 2-dimensional trapezoidal fuzzy number $B = ((a,\ x_1,\ h,\ b,\ y_1))^2$ is the interior of an ellipse in the xy-plane including the boundary

$$\begin{aligned} B_\alpha &= \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, b^2(x-x_1)^2 + a^2(y-y_1)^2 \le a^2b^2(h-\alpha)^2\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \left(\frac{x-x_1}{a(h-\alpha)}\right)^2 + \left(\frac{y-y_1}{b(h-\alpha)}\right)^2 \le 1\right\}. \end{aligned}$$

Note that if $0 < h < 1$, $((a,\ x_1,\ h,\ b,\ y_1))^2$ becomes a generalized 2-dimensional triangular fuzzy number and if $1 < h$, $((a,\ x_1,\ h,\ b,\ y_1))^2$ becomes a 2-dimensional trapezoidal fuzzy set.

Theorem 5.2. ([7]) Let $A = (a_1,\ x_1,\ b_1,\ y_1)^2$ be a 2-dimensional triangular fuzzy number and $B = ((a_2,\ x_2,\ h,\ b_2,\ y_2))^2$ be a 2-dimensional trapezoidal fuzzy set. Then we have the followings.

(1) For $0 < \alpha < 1$, the α-set of $A(+)_p B$ is

$$(A(+)_p B)^\alpha = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \left(\frac{x - x_1 - x_2}{a_1(1-\alpha) + a_2(h-\alpha)}\right)^2 + \left(\frac{y - y_1 - y_2}{b_1(1-\alpha) + b_2(h-\alpha)}\right)^2 = 1\right\}.$$

(2) For $0 < \alpha < 1$, the α-set of $A(-)_p B$ is

$$(A(-)_p B)^\alpha = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \left(\frac{x - x_1 + x_2}{a_1(1-\alpha) + a_2(h-\alpha)}\right)^2 + \left(\frac{y - y_1 + y_2}{b_1(1-\alpha) + b_2(h-\alpha)}\right)^2 = 1\right\}.$$

(3) For $0 < \alpha < 1$, the α-set of $A(\cdot)_p B$ is

$$(A(\cdot)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = x_1x_2 + (x_1a_2(h-\alpha) + x_2a_1(1-\alpha))\cos t + a_1a_2(1-\alpha)(h-\alpha)\cos^2 t,$$

$$y_\alpha(t) = y_1y_2 + (y_1b_2(h-\alpha) + y_2b_1(1-\alpha))\sin t + b_1b_2(1-\alpha)(h-\alpha)\sin^2 t.$$

(4) For $0 < \alpha < 1$, the α-set of $A(/)_p B$ is

$$(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \frac{x_1 + a_1(1-\alpha)\cos t}{x_2 - a_2(h-\alpha)\cos t}, \quad y_\alpha(t) = \frac{y_1 + b_1(1-\alpha)\sin t}{y_2 - b_2(h-\alpha)\sin t}.$$

Proof. Since A and B are convex fuzzy numbers defined on $\mathbb{R}^2$, by Theorem 4.6, there exists $f_i^\alpha(t), g_i^\alpha(t)$ $(i = 1, 2)$ such that for $0 < \alpha < 1$,

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\},$$

$$B^\alpha = \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}.$$

Since $A = (a_1, x_1, b_1, y_1)^2$ and $B = ((a_2, x_2, h, b_2, y_2))^2$, we have

$$f_1^\alpha(t) = x_1 + a_1(1-\alpha)\cos t, \;\; f_2^\alpha(t) = y_1 + b_1(1-\alpha)\sin t, \;\; 0 < \alpha < 1,$$

$$g_1^\alpha(t) = x_2 + a_2(h-\alpha)\cos t, \;\; g_2^\alpha(t) = y_2 + b_2(h-\alpha)\sin t, \;\; 0 < \alpha < 1.$$

(1) If $0 < \alpha < 1$, since

$$f_1^\alpha(t) + g_1^\alpha(t) = x_1 + x_2 + (a_1(1-\alpha) + a_2(h-\alpha))\cos t,$$

$$f_2^\alpha(t) + g_2^\alpha(t) = y_1 + y_2 + (b_1(1-\alpha) + b_2(h-\alpha))\sin t,$$

we have

$$(A(+)_p B)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \;\Big|\; \Big(\frac{x - x_1 - x_2}{a_1(1-\alpha) + a_2(h-\alpha)}\Big)^2 + \Big(\frac{y - y_1 - y_2}{b_1(1-\alpha) + b_2(h-\alpha)}\Big)^2 = 1\Big\}.$$

Furthermore, we have

$$(A(+)_p B)^0 = \lim_{\alpha \to 0^+} (A(+)_p B)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \;\Big|\; \Big(\frac{x - x_1 - x_2}{a_1 + a_2 h}\Big)^2 + \Big(\frac{y - y_1 - y_2}{b_1 + b_2 h}\Big)^2 = 1\Big\},$$

$$(A(+)_p B)^1 = \lim_{\alpha \to 1^-} (A(+)_p B)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \Big| \Big(\frac{x - x_1 - x_2}{a_2(h-1)}\Big)^2 + \Big(\frac{y - y_1 - y_2}{b_2(h-1)}\Big)^2 = 1\Big\}.$$

(2) If $0 \le t \le \pi$ and $0 < \alpha < 1$, we have

$$f_1^\alpha(t) - g_1^\alpha(t+\pi) = x_1 - x_2 + (a_1(1-\alpha) + a_2(h-\alpha))\cos t,$$

$$f_2^\alpha(t) - g_2^\alpha(t+\pi) = y_1 - y_2 + (b_1(1-\alpha) + b_2(h-\alpha))\sin t.$$

In the case of $\pi \le t \le 2\pi$, we have

$$f_1^\alpha(t) - g_1^\alpha(t-\pi) = f_1^\alpha(t) - g_1^\alpha(t+\pi),$$

$$f_2^\alpha(t) - g_2^\alpha(t-\pi) = f_2^\alpha(t) - g_2^\alpha(t+\pi).$$

Thus

$$(A(-)_pB)^\alpha = \Big\{(x,y)\in\mathbb{R}^2 \,\Big|\, \Big(\frac{x-x_1+x_2}{a_1(1-\alpha)+a_2(h-\alpha)}\Big)^2 + \Big(\frac{y-y_1+y_2}{b_1(1-\alpha)+b_2(h-\alpha)}\Big)^2 = 1\Big\}.$$

Furthermore, we have

$$(A(-)_pB)^0 = \lim_{\alpha\to0^+}(A(-)_pB)^\alpha = \Big\{(x,y)\in\mathbb{R}^2 \,\Big|\, \Big(\frac{x-x_1+x_2}{a_1+a_2h}\Big)^2 + \Big(\frac{y-y_1+y_2}{b_1+b_2h}\Big)^2 = 1\Big\},$$

$$(A(-)_pB)^1 = \lim_{\alpha\to1^-}(A(-)_pB)^\alpha = \Big\{(x,y)\in\mathbb{R}^2 \,\Big|\, \Big(\frac{x-x_1+x_2}{a_2(h-1)}\Big)^2 + \Big(\frac{y-y_1+y_2}{b_2(h-1)}\Big)^2 = 1\Big\}.$$

(3) Let $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t),y_\alpha(t)) \mid 0\le t\le 2\pi\}$. Since

$$f_1^\alpha(t) = x_1 + a_1(1-\alpha)\cos t,\ \ f_2^\alpha(t) = y_1 + b_1(1-\alpha)\sin t,$$

$$g_1^\alpha(t) = x_2 + a_2(h-\alpha)\cos t,\ \ g_2^\alpha(t) = y_2 + b_2(h-\alpha)\sin t,$$

we have

$$\begin{aligned} x_\alpha(t) &= f_1^\alpha(t)\cdot g_1^\alpha(t) \\ &= x_1x_2 + (x_1a_2(h-\alpha)+x_2a_1(1-\alpha))\cos t + a_1a_2(1-\alpha)(h-\alpha)\cos^2 t, \quad 0<\alpha<1, \\ y_\alpha(t) &= f_2^\alpha(t)\cdot g_2^\alpha(t) \\ &= y_1y_2 + (y_1b_2(h-\alpha)+y_2b_1(1-\alpha))\sin t + b_1b_2(1-\alpha)(h-\alpha)\sin^2 t, \quad 0<\alpha<1. \end{aligned}$$

Furthermore, we have

$$x_0(t) = \lim_{\alpha\to0^+} x_\alpha(t) = x_1x_2 + (x_1a_2h + x_2a_1)\cos t + a_1a_2h\cos^2 t,$$

$$y_0(t) = \lim_{\alpha\to0^+} y_\alpha(t) = y_1y_2 + (y_1b_2h + y_2b_1)\sin t + b_1b_2h\sin^2 t,$$

$$x_1(t) = \lim_{\alpha\to1^-} x_\alpha(t) = x_1x_2 + x_1a_2(h-1)\cos t,$$

$$y_1(t) = \lim_{\alpha\to1^-} y_\alpha(t) = y_1y_2 + y_1b_2(h-1)\sin t.$$

(4) Let $(A(/)_pB)^\alpha = \{(x_\alpha(t),y_\alpha(t)) \mid 0\le t\le 2\pi\}$. Similarly, we have

$$x_\alpha(t) = \frac{x_1+a_1(1-\alpha)\cos t}{x_2-a_2(h-\alpha)\cos t},\quad y_\alpha(t) = \frac{y_1+b_1(1-\alpha)\sin t}{y_2-b_2(h-\alpha)\sin t},\quad 0<\alpha<1.$$

Furthermore, we have

$$x_0(t) = \frac{x_1 + a_1 \cos t}{x_2 - a_2 h \cos t}, \quad y_0(t) = \frac{y_1 + b_1 \sin t}{y_2 - b_2 h \sin t},$$

$$x_1(t) = \frac{x_1}{x_2 - a_2(h-1)\cos t}, \quad y_1(t) = \frac{y_1}{y_2 - b_2(h-1)\sin t}.$$

The proof is complete. □

Example 5.3. Let $A = (2,\ 7,\ 1,\ 5)^2$ and $B = ((6,\ 4,\ \frac{3}{2},\ 3,\ 8))^2$. Then by Theorem 5.2, we have the following.

(1) For $0 < \alpha < 1$, the α-set of $A(+)_pB$ is

$$(A(+)_pB)^\alpha = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \left(\frac{x-11}{8(\frac{11}{8}-\alpha)}\right)^2 + \left(\frac{y-13}{4(\frac{11}{8}-\alpha)}\right)^2 = 1\right\}.$$

(2) For $0 < \alpha < 1$, the α-set of $A(-)_pB$ is

$$(A(-)_pB)^\alpha = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \left(\frac{x-3}{8(\frac{11}{8}-\alpha)}\right)^2 + \left(\frac{y+3}{4(\frac{11}{8}-\alpha)}\right)^2 = 1\right\}.$$

(3) For $0 < \alpha < 1$, the α-set of $A(\cdot)_pB$ is

$$(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = 28 + (80 - 50\alpha)\cos t + 12(1-\alpha)\left(\frac{3}{2} - \alpha\right)\cos^2 t,$$

$$y_\alpha(t) = 40 + \left(\frac{61}{2} - 23\alpha\right)\sin t + 3(1-\alpha)\left(\frac{3}{2} - \alpha\right)\sin^2 t.$$

(4) For $0 < \alpha < 1$, the α-set of $A(/)_pB$ is

$$(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \frac{7 + 2(1-\alpha)\cos t}{4 - 6\left(\frac{3}{2} - \alpha\right)\cos t}, \quad y_\alpha(t) = \frac{5 + (1-\alpha)\sin t}{8 - 3\left(\frac{3}{2} - \alpha\right)\sin t}.$$

5.2. An extension of algebraic operations for 2-dimensional quadratic fuzzy number

By defining parametric operations between two regions valued α-cuts, we get the parametric operations for two quadratic fuzzy numbers defined on $\mathbb{R}^2$ in Section 4.3. In this section, we prove that the results for the parametric operations for two 2-dimensional quadratic fuzzy numbers are the generalization of algebraic operations for two quadratic fuzzy numbers on $\mathbb{R}$.

Theorem 5.4. Parametric operations for two 2-dimensional quadratic fuzzy numbers are the generalization of algebraic operation for two quadratic fuzzy numbers on $\mathbb{R}$

Proof. Consider two 2-dimensional quadratic fuzzy numbers $A = [a_1,\ x_1,\ b_1,\ 0]^2$ and $B = [a_2,\ x_2,\ b_2,\ 0]^2$. By Theorem 4.17,

(1) $A(+)_p B = [a_1 + a_2,\ x_1 + x_2,\ b_1 + b_2,\ 0]^2$.

(2) $A(-)_p B = [a_1 + a_2,\ x_1 - x_2,\ b_1 + b_2,\ 0]^2$.

(3) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = x_1 x_2 + (x_1 a_2 + x_2 a_1)\sqrt{1-\alpha}\cos t + a_1 a_2 (1-\alpha)\cos^2 t$$

and

$$y_\alpha(t) = b_1 b_2 (1-\alpha)\sin^2 t.$$

(4) $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{x_1 + a_1\sqrt{1-\alpha}\cos t}{x_2 - a_2\sqrt{1-\alpha}\cos t} \quad \text{and} \quad y_\alpha(t) = -\frac{b_1}{b_2}.$$

The intersections of these 2-dimensional quadratic fuzzy numbers and vertical xz-plane ($y = 0$) are as follows.

(1) $A(+)_p B$; Note that

$$\mu_{A(+)_p B}(x,y) = 1 - \Big(\big(\frac{x - x_1 - x_2}{a_1 + a_2} \big)^2 + \big(\frac{y}{b_1 + b_2} \big)^2 \Big).$$

If $y = 0$ and $\mu_{A(+)_p B}(x,y) = 0$,

$$x = x_1 + x_2 \pm (a_1 + a_2).$$

Thus the intersection is the symmetric quadratic fuzzy number C on the xz-plane with $\mu_C(x_1 + x_2) = 1$ and the zero cut

$$C_0 = [x_1 + x_2 - (a_1 + a_2),\ x_1 + x_2 + (a_1 + a_2)].$$

(2) $A(-)_p B$; Note that

$$\mu_{A(-)_p B}(x,y) = 1 - \Big(\big(\frac{x - x_1 + x_2}{a_1 + a_2} \big)^2 + \big(\frac{y}{b_1 + b_2} \big)^2 \Big).$$

If $y = 0$ and $\mu_{A(-)_p B}(x,y) = 0$,

$$x = x_1 - x_2 \pm (a_1 + a_2).$$

Thus the intersection is the symmetric quadratic fuzzy number D on the xz-plane with $\mu_D(x_1 - x_2) = 1$ and the zero cut

$$D_0 = [x_1 - x_2 - (a_1 + a_2),\ x_1 - x_2 + (a_1 + a_2)].$$

(3) $A(\cdot)_p B$; If $\alpha = 0$,

$$x_0(t) = x_1 x_2 + (x_1 a_2 + x_2 a_1)\cos t + a_1 a_2 \cos^2 t.$$

Since

$$x_0(0) = x_1 x_2 + x_1 a_2 + x_2 a_1 + a_1 a_2 \ \text{ and } \ x_0(\pi) = x_1 x_2 - (x_1 a_2 + x_2 a_1) + a_1 a_2,$$

the intersection is a fuzzy number E on the xz-plane with $\mu_E(x_1x_2) = 1$ and the zero cut

$$E_0 = [x_1x_2 - (x_1a_2 + x_2a_1) + a_1a_2,\ x_1x_2 + x_1a_2 + x_2a_1 + a_1a_2].$$

(4) $A(/)_pB$; If $\alpha = 0$,

$$x_0(t) = \frac{x_1 + a_1\cos t}{x_2 - a_2\cos t}.$$

Since

$$x_0(0) = \frac{x_1 + a_1}{x_2 - a_2} \ \text{ and } \ x_0(\pi) = \frac{x_1 - a_1}{x_2 + a_2},$$

the intersection is a fuzzy number F on the xz-plane with $\mu_F(\frac{x_1}{x_2}) = 1$ and the zero cut

$$F_0 = \left[\frac{x_1 - a_1}{x_2 + a_2},\ \frac{x_1 + a_1}{x_2 - a_2}\right].$$

On the other hand, the intersection of 2-dimensional quadratic fuzzy number $A = [a_1,\ x_1,\ b_1,\ 0]^2$ and vertical xz-plane ($y = 0$) is the symmetric quadratic fuzzy number G on the xz-plane with $\mu_G(x_1) = 1$ and the zero cut

$$G_0 = [x_1 - a_1,\ x_1 + a_1].$$

The intersection of 2-dimensional quadratic fuzzy number $B = [a_2,\ x_2,\ b_2,\ 0]^2$ and vertical xz-plane ($y = 0$) is the symmetric quadratic fuzzy number H on the xz-plane with $\mu_H(x_2) = 1$ and the zero cut

$$H_0 = [x_2 - a_2,\ x_2 + a_2].$$

For two quadratic fuzzy numbers G and H, we had proved the following result for Zadeh's extension principle ([11]).

$$G(+)H = C,\ \ G(-)H = D,\ \ G(\cdot)H = E \ \text{ and } \ 0G(/)H = F$$

The proof is complete. □

Example 5.5. Let $A = [2,\ 9,\ 6,\ 0]^2$ and $B = [4,\ 1,\ 7,\ 0]^2$. Then by Theorem 5.4, we have the following.

(1) $A(+)_p B = [6,\ 10,\ 13,\ 0]^2$

$$\mu_{A(+)_p B}(x,y) = 1 - \left(\left(\frac{x-10}{6}\right)^2 + \left(\frac{y}{13}\right)^2\right).$$

If $y = 0$ and $\mu_{A(+)_p B}(x,y) = 0$,

$$x = 4 \ \text{ or } \ x = 16.$$

Thus the intersection is the symmetric quadratic fuzzy number C on the xz-plane with $\mu_C(10) = 1$ and the zero cut

$$C_0 = [4,\ 16].$$

(2) $A(-)_p B = [6,\ 8,\ 13,\ 0]^2$

$$\mu_{A(-)_p B}(x,y) = 1 - \left(\left(\frac{x-8}{6}\right)^2 + \left(\frac{y}{13}\right)^2\right).$$

If $y = 0$ and $\mu_{A(+)_p B}(x,y) = 0$,

$$x = 2 \ \text{ or } \ x = 14.$$

Thus the intersection is the symmetric quadratic fuzzy number D on the xz-plane with $\mu_D(10) = 1$ and the zero cut

$$D_0 = [2,\ 14].$$

(3) $A(\cdot)_p B$; If $\alpha = 0$,

$$x_0(t) = 9 + 38\cos t + 8\cos^2 t.$$

Since

$$x_0(0) = 55 \ \text{ and } \ x_0(\pi) = -21,$$

the intersection is a fuzzy number E on the xz-plane with $\mu_E(9) = 1$ and the zero cut

$$E_0 = [-21,\ 55].$$

(4) $A(/)_p B$; If $\alpha = 0$,

$$x_0(t) = \frac{9 + 2\cos t}{1 - 4\cos t}.$$

Since

$$x_0(0) = -\frac{11}{3} \ \text{ and } \ x_0(\pi) = \frac{7}{5},$$

the intersection is a fuzzy number F on the xz-plane with $\mu_F(9) = 1$ and the zero cut

$$F_0 = \left[-\frac{11}{3},\ \frac{7}{5}\right].$$

On the other hand, the intersection of 2-dimensional quadratic fuzzy number $A = [2,\ 9,\ 6,\ 0]^2$ and vertical xz-plane ($y = 0$) is the symmetric quadratic fuzzy number G on the xz-plane with $\mu_G(9) = 1$ and the zero cut

$$G_0 = [7,\ 11].$$

The intersection of 2-dimensional quadratic fuzzy number $B = [4,\ 1,\ 7,\ 0]^2$ and vertical xz-plane ($y = 0$) is the symmetric quadratic fuzzy number H on the xz-plane with $\mu_H(1) = 1$ and the zero cut

$$H_0 = [-3,\ 5].$$

For two quadratic fuzzy numbers G and H, we had proved the following result for Zadeh's extension principle ([11]),

$$G(+)H = C,\ \ G(-)H = D,\ \ G(\cdot)H = E \ \text{ and } \ G(/)H = F.$$

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<국문초록>

2차원 퍼지집합에 대한 파라메트릭 연산

사다리꼴 퍼지수를 R에서 R^2로 일반화하고 2차원 삼각 퍼지수와 사다리꼴 퍼지집합 사이의 파라메트릭 연산을 계산하였다. 또한 두 개의 2차원 2차 퍼지수에 대한 파라메트릭 연산의 결과는 R에 있는 두 개의 2차 퍼지수에 대한 대수 연산의 일반화임을 증명하였다. 그리고, 각각에 대한 예제를 찾았다.