On the Nonholonomic Congruence of the Riemannian Manifold

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리만-多様体上의 非-호로노미 Cougruence 에 관하여

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Summary

This paper, as the application of orthogonal nonholonomic frames, gives the some results with respect to its. In particular, it have the some properties of curvature of nonholonomic congruence, geodesic nonholonomic congruence and condition that nonholonomic congruence be normal on the n-dimensional Riemannian manifold.

1. Introduction

The concept of the nonholonomic frames introduced by V. Hlavaty 1957 with a set of 4 linearly independent basic null vectors and know that used it successfulfy as a tool to develop the algebra of the unifled field theory in the space-time X_4.

In our previous paper Chun, K.T. & Hyun, J.O. 1976 and Hyun, J.O. 1976, we introduced the concept of the general nonholonomic frames and orthogonal nonholonomic frames to an n-dimensional Riemannian space V_n, and constructed the characteristic orthogonal nonholonomic frames of V_n determined by a symmetric tensor $a_{\lambda\mu}$, composed of n different eigenvectors of $a_{\lambda\mu}$, and to derive its particular properties.

The purpose of the present paper, as the application of orthogonal nonholonomic frames, is to find the some results for the geodesic congruence and condition that a congruence be normal on the n-dimensional Riemannian manifold.

2. Preliminiary Results

In our present section, for our further discussion, results obtained in our previous paper Chung, K.T. & Hyun, J.O. 1976 and Hyun, J.O. 1976 will be introduced without proof.

Let $h_{\lambda\mu}$ be the fundamental metric tensor and let $\underset{i}{e}{}^{\nu}$ $(i=1,2,\cdots,n)$ be a set of n linearly independent unit vectors, when

$$(2.1)\qquad h_{\lambda\mu}\ h^{\lambda\nu} \overset{\text{def}}{=} \delta_\mu^\nu$$

and there is a unique reciprocal set of n linearly independent covariant vector $\underset{i}{e}{}_\lambda$ $(i=1,2,\cdots,n)$, satisfying

$$(2.2)^*\qquad \underset{i}{e}{}^{\nu}\ \overset{i}{e}{}_\lambda = \delta_\lambda^\nu \quad \underset{j}{e}{}^{\lambda}\ \overset{i}{e}{}_\lambda = \delta_j^i$$

Within the vectors $\underset{i}{e}{}^{\nu}$ and $\overset{i}{e}{}_\lambda$ a nonholonomic frame of V_n defined in the following way; If $T_{\lambda\cdots}^{\nu\cdots}$ are holonomic components of a tensor density of weight p, then its nonholonomic component are defined by

$$(2.3)\text{a}\qquad T_{j\cdots}^{i\cdots} = A^p T_{\lambda\cdots}^{\nu\cdots} \overset{i}{e}{}_\nu\ \underset{j}{e}{}^{\lambda}\cdots,\ A \overset{\text{def}}{=} \operatorname{Det}((\overset{i}{e})).$$

From (2.2) and (2.3)a,

$$(2.3)\text{b}\qquad T_{\lambda\cdots}^{\nu\cdots} = A^p\ T_{j\cdots}^{i\cdots}\ \underset{i}{e}{}^{\nu}\ \overset{j}{e}{}_\lambda\cdots$$

The nonholonomic frame in V_n constructed by the unit vectors $\underset{i}{e}{}^{\nu}$, $(i=1,2,\cdots,n)$ tangent to the n congruences of an orthogonal ennuple, will be termed an orthogonal nonholonomic frame of V_n.

Theorem (2.1). We have

(2.4)a $\quad e_i^{\nu} = \overset{j}{e}_{\lambda} h_{ij} h^{\lambda\nu}$, $\overset{i}{e}_{\lambda} = e_i^{\nu} h^{ij} h_{\lambda\nu}$,

(2.4)b $\quad h_{ij} = \delta_{ij}$, $h^{ij} = \delta^{ij}$, $e_i^{\nu} = \overset{i}{e}^{\nu}$, $\overset{j}{e}_{\lambda} = e_{j\lambda}$.

Theorem (2.2). The tensors $h_{\lambda\mu}$, $h^{\lambda\mu}$ and δ_λ^ν may be expressed in terms of e_j, as follows;

(2.5)
$$h_{\lambda\mu} = \sum_i e_{i\lambda}\, e_{i\mu},$$

$$h^{\lambda\mu} = \sum_i e_i^{\lambda}\, e_i^{\mu},$$

$$\delta_\lambda^\nu = \sum_i e_{i\lambda}\, e_i^{\nu}.$$

And let e_i^λ be unit eigenvectors determined by a symmetric covariant tensors $a_{\lambda\mu}$. Then they satisfy,

(2.6) $\quad (a_{\lambda\mu} - M_i h_{\lambda\mu})\, e_i^{\lambda} = 0$, ($M_i$: scalar)

(2.7) $\quad {}^{(1)}a_{\lambda\mu} \overset{\text{def}}{=} a_{\lambda\mu}$,

$$ {}^{(p)}a_{\lambda\mu} = {}^{(p-1)}a_{\lambda\kappa}\, a_\mu^\kappa \quad (p = 2, 3, \cdots).$$

Lemma (2.3). Every eigenvector e_i^λ of $a_{\lambda\mu}$ is also an eigenvector of the tensor ${}^{(p)}a_{\lambda\mu}$ $(p=2,3,\cdots)$.

Theorem (2.4). The nonholonomic components of ${}^{(p)}a_{\lambda\mu}$ are

(2.8) $\quad {}^{(p)}\overset{i}{a}_x = M_x^p\, \delta_x^i$ or $\underset{x}{a}_i = M_x^p\, \delta_{xi}$.

Theorem (2.5). The tensor $a_{\lambda\mu}$ may be expressed in terms of e_i^λ, as follows;

(2.9) $\quad {}^{(p)}a_{\lambda\mu} = \sum_i M_i^p\, e_{i\lambda}\, e_{i\mu}$ $(p = 1, 2, \cdots)$.

3. Curvature of Nonholonomic Congruence and Geodesic Nonholonomic Congruences

Let e_i be the unit tangents to the n congruences of m orthogonal ennuple in Riemannian manifold. Suppose, the derived vector of e_i in the direction of e_k has components $e_{j\,\lambda,\mu}\, e_k^{\mu}$ and the projection of this vector on e_i is a scalar invariant, denoted by $\mathcal{K}_{jik}$ so that

(3.1) $\quad \mathcal{K}_{jik} = e_{j\,\lambda,\mu}\, e_i^{\lambda}\, e_k^{\mu}$.

Since the derived vector of e_j for any direction is orthogonal to e_j, we have

Lemma (3.1). The nonholonomic invariants

(3.2) $\quad \mathcal{K}_{jjk} = 0$, for all values of j, k.

Proof. From (2.2) and (2.4)b, if $i \neq j$, then $e_{i\lambda}\, e_j^{\lambda} = 0$,

(3.3) $\quad e_{i\,\lambda,\mu}\, e_j^{\lambda} + e_{j\,\lambda,\mu}\, e_i^{\lambda} = 0.$

multiplying by e_k^μ and summing with respect to from 1 to n, we obtain

(3.4) $\quad e_{i\,\lambda,\mu}\, e_j^{\lambda}\, e_k^{\mu} + e_{j\,\lambda,\mu}\, e_i^{\lambda}\, e_k^{\mu} = 0$. i.e.

(3.5) $\quad \mathcal{K}_{ijk} + \mathcal{K}_{jik} = 0.$

Put $i = j$, we obtain the result.

Theorem (3.2). If we expressed the derived vector of e_j for the direction e_k in terms of the orthogonal nonholonomic components in the direction of the n congruences, then

(3.6) $\quad e_k \nabla e_j = \sum_i \mathcal{K}_{jik}\, e_i$.

Proof. By means of (2.4)a, (2.4)b and (3.1),

$$\sum_i \mathcal{K}_{jik}\, e_i^{\sigma} = \sum_i e_{j\,\lambda,\mu}\, e_i^{\lambda}\, e_k^{\mu}\, \overset{i}{e}_\nu h_{ij}\, h^{\nu\sigma} = e_{j\,\lambda,\mu}\, e_k^{\mu}.$$

Let p_j be the first curvature vector of a curve of the congruence, whose unit tangent is e_j, then it is wellknown results that p_j is the derived of e_j in its own direction. Consider the first curvature k of the vector p_j with respect to an orthogonal nonholonomic frame of the Riemannian manifold, we have

Theorem (3.3). Necessary and sufficient conditions that the curves of the congruence, whose unit tangent is e_j, be geodesics are expressed by the equations with respect to nonholonomic frame

(3.7) $\quad \mathcal{K}_{jij} = 0$, $(i = 1, 2, \cdots, n)$.

Proof. By using the (3.6),

$$p_j = e_j \nabla e_j = \sum_i \mathcal{K}_{jij}\, e_i.$$

Hence

$$k^2=\sum_i (\mathcal{H}_{jij})^2=\sum_i (\mathcal{H}_{iji})^2 .$$

4. Condition that a Nonholonomic Congruence be Normal

Let t be the unit tangent to the congruence consider. In order that the congruence may be normal to a family of hypersufaces, there exist a function whose gradiant at each point has the direction of t.

Hence $\psi_\lambda = c t_\lambda$ (c : constant).

Lemma (4.1). The given congruence be normal if and only if

$$t_\lambda(t_{\mu,\nu}-t_{\nu,\mu})+t_\mu(t_{\nu,\lambda}-t_{\lambda,\nu})+t_\nu(t_{\lambda,\mu}-t_{\mu,\lambda})$$

$$=0,\quad (\lambda,\mu,\nu=1,2,\cdots,n).$$

Proof. By **,

$$(4.1)\qquad t_\lambda\left(\frac{\partial t_\mu}{\partial x^\nu}-\frac{\partial t_\nu}{\partial x^\mu}\right)+t_\mu\left(\frac{\partial t_\nu}{\partial x^\lambda}-\frac{\partial t_\lambda}{\partial x^\nu}\right)$$

$$+\ t_\nu\left(\frac{\partial t_\lambda}{\partial x^\mu}-\frac{\partial t_\mu}{\partial x^\lambda}\right)=0.$$

Suppose the congruence is one of an orthogonal ennuple. Let it be taken as that whose unit tangent is $\underset{i}{e}$, then we have

Theorem (4.2). The nonholonomic congruence $\underset{i}{e}$ of an orthogonal ennuple be normal if and only if

$$(4.2)\qquad \mathcal{H}_{ijk}=\mathcal{H}_{ikj}\ (j,k=1,2,\cdots,n\text{-}1).$$

Proof. From (4.1)replacing t by $\underset{i}{e}$,

$$(4.3)\qquad \underset{i}{e}_\lambda(\underset{i}{e}_{\mu,\nu}-\underset{i}{e}_{\nu,\mu})+\underset{i}{e}_\mu(\underset{i}{e}_{\nu,\lambda}-\underset{i}{e}_{\lambda,\nu})$$

$$+\underset{i}{e}_\nu(\underset{i}{e}_{\lambda,\mu}-\underset{i}{e}_{\mu,\lambda})=0.$$

Multiplying both sides of (4.3) by $\underset{j}{e}^\lambda\underset{k}{e}^\nu$ $(j,k=1,2,\ldots,n\text{-}1)$, since $\underset{j}{e}$ and $\underset{k}{e}$ are orthogonal to $\underset{i}{e}$,

$$\underset{i}{e}_\mu(\underset{i}{e}_{\nu,\lambda}\underset{j}{e}^\lambda\underset{k}{e}^\nu-\underset{i}{e}_{\lambda,\nu}\underset{j}{e}^\lambda\underset{k}{e}^\nu)=0.$$

By (3.1)

$$\underset{i}{e}_\mu(\mathcal{H}_{ikj}-\mathcal{H}_{ijk})=0.$$

Since it must hold for all values i and vice versa.

We have (4.2).

Theorem (4.3). Necessary and sufficient conditions that all the nonholonomic congruences of an orthognal ennuple normal are expressed by

$$\mathcal{H}_{ikj}=0\ \ (i,j,k=1,2,\cdots,n\,;\,i\neq j\neq k).$$

Proof. By means of (3.5) and (4.2),

$$\mathcal{H}_{ikj}=\mathcal{H}_{ijk}=-\mathcal{H}_{jik}=-\mathcal{H}_{jki}=\mathcal{H}_{kji}=\mathcal{H}_{kij}=-\mathcal{H}_{ikj}\ .$$

Reference

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* Throughout the present paper, Greek indices take values 1,2,...,n unless explicitly stated otherwise and follow the summation convention, while Roman indices are used for the nonholonomic components of a tensor and run from 1 to n. Roman indices also follow the summation convention.

** Levi-Civita, 1927, 1, pp. 26-29: or Forsyth, 1903, 2, pp. 298-299.

論 文 抄 錄

本 論文에서는 nonholonomic 구조를 이용하여 리만-多樣体上의 nonholonomic congruence 의 곡률과 geodesic nonholonomic congruence 에 대한 몇가지 성질을 알아보고 nonholonomic congruence 가 normal 이 될 수 있는 조건을 재구성했다.