Fréchet derivative of the rational functions in a Banach space

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BANACH 空間에서의 유리함수의 FRÉCHET 미분

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SUMARY

In this paper we find to the quotient rule for the Fréchet derivative and to the general formular of the Fréchet derivative of the rational function in a Banach space over a complex field C.

INTRODUCTION

We shall be concerned with a generalization, due to M. Fréchet(1925) of the classical differential calculus of real-valued functions of a real-variable. Fréchet's generalization of the differential calculus applies to mappings of a normed linear space X into a normed linear space Y. f Let f be such a mapping. The derivative of at a point a of X will be defind to be a linear transformation T of X into Y which satisfies the following condition; for each $\epsilon>0$, there exists $\delta>0$ such that $\|f(x)-f(a)-T(x-a)\|\leq x-a$ whenever $\|x-a\|<\delta$.

In (4), K. Chang and K. Hong find the product rule for Fréchet derivative and obtain the formula for the differentiation of polynomials in a Banach algebra.

The purpose of the present paper is to find the formula of Fréchet derivative of the rational functions in a Banach algebra.

1. PRELIMINARY

This section is a collection of basic definitions and properties. We omit the prcofs of them, which are well known(3).

DEFINITION (1.1) Let A and B be Banach algebras over complex field $\mathcal{C}$, let S be a non-empty open subset of A. Let f be a mapping of S into B. Then the mapping f is said to be differentiable at a point a in S if and only if there is a linear transformation T of A into B which satifies the following condition; for each $\epsilon>0$ there exists $\delta>0$ such that

(*) $\|f(x)-f(a)-T(x-a)\|\leq x-a\|$ for all x in A with $\|x-a\|<\delta$.

The mapping f is said to be differentiable on S if and only if it is differentiable at each point of S.

By the above definition, we obtain the following property.

PROPOSITION (1.1) If f is differentiable at a point a in S then there is a unique linear transformation of A into B wich satisfies the condition (*).

DEFINITION (1.2) Let f be differentiale at a in S. The unique linear transformation of A

into B which satisfies condition(*) is called the derivative of f at a and is denoted by $(Df)_a$.

The following properties of Frechet derivative which are essential in section 2 are taken from (4). We will state them without proofs.

PROPOSITION (1.2) (LINEARLITY) Let f and g be mappings of a nonemoty open subset S of A into B that are differentiable at a in S and let α, β in ϕ. Then $h=\alpha f+\beta g$ is differentiable at a in S and $(Dh)_a= \alpha(Df)_a+\beta(Dg)_a$.

PROPOSITION (1.3) (CHAIN RULE) Let f be a mapping of non-empty open subset S of a Banach algebra A into an open subset V of a Banach algebra B and let g be a mapping of V into a Banach algebra C. Suppose that f is continuous and differentiable at a in S and that g is continuous and differentiable at $b=f(a)$ in B. Then $g \cdot f$ is continous and differentiable at a and $(D(g^\circ f))_a=(Dg)_b{}^\circ(Df)_a$.

DEFINITION (1.3) (1) L_x denotes the left-multiplication mappings in a Banach algebra A such that $L_x(z)=xz$ for all z in A

(2) R_x denotes the right-multiplication mappings in a Banach algebra A such that $R_x(_z)=$for all z in A.

(3) M_x denotes the multiplication mapping in a commutative Banach algebra. Note that if A is a commutative Banach algebra then $L_x=R_x=M_x$.

PROPOSITION (1.4) (PRODUCT RULE) Let Abe a Banach algebra (we do not require that A be commutative) and let f and g be mappings A into A itself. Suppose that f and g are continuous and differentiable at a in A. Then $k(x)=f(x)\cdot g(x)$ is continuous and differentiable at a in A and $(Dk)_a =L_{f(a)}\cdot(Dg)_a+R_{g(a)}\cdot(Df)_a$.

PROPOSITION (1.5) Let A be a Banach algebra. Suppose that f is a mapping of A into A itself such that $f(x)=x^n$(n is a natural number). Then $f(x)$ is differentiable at a in A and$(Df)_a=\sum_{j=0}^{n-1} L_a n-j-1 \cdot R_a j$.

In particular, if A is a commutative Banach algebra, then $(Df)_a=nM_a n-1$.

PROPOSITION (1.6) Let A be a Banach algebra. Suppose that f is mapping of A into A such that $f(x)=a_0e+a_1x+\cdots+a_nx^n$ for all x in A and for all a_i in $\mathbb{C}$ $(i=0,1,\cdots,n)$.

Then $f(x)$ is differentiable at a in A and
$(Df)_a=\sum_{n=1}^{m} a\sum_{j=0}^{n-1}(L_a n-j-1^\circ R_a j)$.

2. THE FORMULA FOR THE DIFFERENTIATION OF THE RATIONAL FUNCTIONS

The purpose of this section is to find the formula for the Frechet derivative of rational functions.

The following lemma is an easy consequence of the general theory of Banach algebra.(2)

LEMMA (2.1) The group U of invertible elements in a Banach algebra A is an open subset. Specifically, if x in U then $\{y \mid \|y-x\|<\|x^{-1}\|^{-1}\} \subset U$.

LEMMA (2.2) If z is in A, a is in $\mathbb{C}$ and $\|az\|<1$ then $e-az$ is invertible. Clearly, the sequence $y_n=e+az+(az)^2+\cdots+(az)^{n-1}$ converges to a limit y—usually, one writes $y=\sum_{n=0}(az)^n$(the convention is that $z^0=e$)—and one has $(e-az)y=y(e-az)=e$. Thus $e-az$ is invertible, with inverse $(e-az)^{-1}= \sum_{n=0} (az)^n$.

Moreover $\|(e-az)^{-1}\|\leq 1/(1-\|az\|)$.

Now, we introduce one of the main results of this section.

THEOREM (2.1) Les A be a Banach algebra and let U be the set of all invertible elements in A. Suppose that f is a mapping of U into A

such that $f(x)-x^{-1}$. Then $f(x)$ is differentiable at a in U and $(Df)_a=R_{-a}-1\cdot L_a-1$.

PROOF) Let $x\in\{t \mid \|t-a\|\leq\|a^{-1}\|^{-1}/2\}$ and $a\in U$. Then $a\neq 0, a^{-1}\neq 0$, that is, $\|a\|>0$, $\|a^{-1}\|>0$. And, by lemma (2.1), x is invertible.

Hence $\|e-a^{-1}x\|=\|a-x)\|\leq\|a^{-1}\| \ \|a-x\|\leq 1/2$. By lemma (2.2), $\|(a^{-1}x)^{-1}\|\leq 1/(1-\|e-a^{-1}x\|)\leq 2$, and therefore

$$\|x^{-1}\|=\|x^{-1}a)a^{-1}\|\leq\|x^{-1}a\| \ \|a^{-1}\|\leq 2\|a^{-1}\|.$$

Thus $\|x^{-1}\|\leq 2\|a^{-1}\|$ whenever $\|x-a\|\leq\|a^{-1}\|^{-1}/2$. Let$\in<0$ be given. Then there exists $\delta=\min\{\|a^{-1}\|^{-1}/2, \|a^{-1}\|^{-3}/2\}$ such that $\|x^{-1}-a^{-1}+R_{-a}-1\cdot L_a-1(x-a)\|\leq\|a^{-1}\| \ \|(a-x)x^{-1}+(x-a)a^{-1}\| \ \leq\|a^{-1}\| \ \|x-a\| \ \|x^{-1}(a-x)a^{-1}\|\leq 2\|a^{-1}\|^3\|x-a\|^2\leq\in\|x-a\|$ Whenever $\|x-a\|<\delta$ and $\{x \mid \|x-a\|<\delta\}\subset\{x \mid \|x-a\|\leq\|a^{-1}\|^{-1}\}$.

Hence, we obtain that $f(x)$ is differentiable at a in U and $(Df)_a=R_{-a}-1\cdot L_a-1$

REMARK; If h is a non-zero complex variable, then the limit

$\lim_{n\to 0}((x+he)^{-1}-x^{-1})/h$ exists and is equal to $-(x^{-1})^2$.

By THEOREM(2.1), we have the following result.

SHEOREM(2.2) Let A be a Banach algebra and let U be the set of all invertiable elements in A. Supppse that f is a mapping of U into A such that $f(x)^{-n}$(n is an arbitrary natural number). Then $f(x)$ is differentiable at a in U and $(Df)_a=\sum_{j=0}^{n-1}(R_{-a}-n+j\cdot L_a-j-1)$.

(PROOF) It follows from PROPOSITION 1.4 that $f(x)=x^{-n}$ is differentiable at a in U. We shall prove that $(Df)_a=\sum_{j=0}^{n-1}(R_{-a}-n+j\cdot L_a-j-1)$ by mathematical induction on n. Let $n=1$. Then the theorem is obvious by THEOREM(2.1). Assume that the theorem holds for $n=k$. For $n=k+1$, let $f(x)=x^{k-1}=g(x)h(x)$ where $g(x)=x^{-k}$ and $h(x)=x^{-1}$. Then, by product rule, $(Df)_a=(D(gh))_a=L_{g(a)}\cdot(Dh)_a+R_{h(a)}\cdot(Dg)_a=L_a-k\cdot(R_{-a}-1\cdot L_a-1)+R_a-1\cdot\sum_{j=0}^{k-1}L_a-j-1\cdot R_a-k+j)=\sum_{j=0}^{k}L_a-j-1\cdot R_a-k+j+1$ which completes the proof or this theorem.

COROLLARY(2.3) Let A be a Banach algebra and let U be the set of all invertible elements in A. Suppose that f is a mapping of U into A such that $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0e+b_1x^{-1}+\cdots b_mx^{-m}$($m$, n are arbitrary natural numbers). Then $f(x)$ is differentiable at a in U and $(Df)_a=\sum_{i=1}^{n}a_i(\sum_{j=0}^{i-1}L_ai-j-1\cdot R_aj)+\sum_{k=1}^{m}b_k(\sum_{l=0}^{k-1}L_a-k+1\cdot R_a-1-1)$.

COROLLARY (2.4) Let A be a commutative Banach algebra and let U be the set of all invertible elements in A. Suppose that f is a mapping of U into A such that $f(x)=a_ax^n+a_{n-1}x^{n-1}+\cdots+a_0e+b_1x^{-1}+\cdots+b_mx^m$ (m, n are arbitrary natural numbers). Then$(Df)_a=\sum_{i=1}^{n}ia_iM_ai-1+\sum_{k=1}^{m}kb_kM_{-a}-k-1$.

Using the above results, we obtain easily the following theorem.

THEOREM(2.5) (QUOTIENT RULE) Let A be a Banach algebra and let U be the set of all invertible elements in A. Suppose that f is continuous and differentiable mapping of A into U and g is continuous and differentiable mapping of U into A such that $g(x)=x^{-1}$. Then$(D(1/f))_a=(D(g\cdot f))_a=L_{-f(a)}-1\cdot R_{f(a)}-1\cdot(Df)_a$.

COROLLARY(2.6) Suppose that f is continuous and differentiable mapping of A into B and g is continuous and differentiable mapping of A into U. Then $(D(fg^{-1}))_a=L_{-f(a)g(a)}-1\cdot R_{g(a)}-1\cdot(Dg)_a+R_{g(a)}-1\cdot(Df)_a$.

COROLLARY (2.7) Suppose that $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0e$ and $g(x)=b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0e$ are mappings of A into A and $g(a)$ is invertiable. Then $(D(fg^{-1}))_a = L(a_n a^n + \cdots + a_0 e)(b_m a^m + \cdots + b_0 e)^{-1} \circ$

$R_{-(b_m a^m + \cdots + b_0 e)} - 1 \circ \sum_{i=1}^{m} b_i \left(\sum_{j=0}^{i-1} L_a i - j - 1 \cdot R_a j \right) +$

$R_{(b_m a^m + \cdots + b_0 e)} - 1 \cdot \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} L_a i - j - 1 \cdot R_a j.$

In fact, we have well known the fundamental theorem of algebra, that is, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 e$ has a factorization $f(x) = a_n (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$ with α_i in $\mathbb{C}$ where the factors $x - \alpha_i$ are uniquely determined up to a permutation.

Using this property, we obtain the following results.

LEMMA (2.3) Let A be a Banach algebra and let x be in A. Then $(x - \alpha e)(x - \beta e) = (x - \beta e)(x - \alpha e)$ for all α, β in $\mathbb{C}$.

PROOF) For all α, β in $\mathbb{C}$, $(x - \alpha e)(x - \beta e) = x^2 - \alpha x - \beta x + \alpha \beta e x^2 - \beta x - \alpha x + \beta \alpha e = (x - \beta e)(x - \alpha e)$.

LEMMA (2.4) Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 e$. Then there exist complex numbers $\alpha_1, \alpha_2, \cdots \alpha_n$ such that $f(x) = a_n (x - \alpha_1 e)(x - \alpha_2 e) \cdots (x - \alpha_n e)$

By the above LEMMA, we obtain the following theorem.

THEOREM (2.8) If $f(x) = a_\phi (x - \alpha_1 e)(x - \alpha_2 e) \cdots (x - \alpha_n e)$ then

$(Df)_a = \sum_{j=1}^{n} a_\phi M_{T(n,j)}$ where we define $T(n,j) = \prod_{\substack{i=1 \\ i \neq j}}^{n} (a - \alpha_i e)$ and $\prod_{i \in \phi} (a - \alpha_i e) = 1 \cdot$

PROOF) We prove this by induction. It is obvious for the case $n = 1$ by the $(Df)_a = \sum_{j=1}^{1} a_\phi M_{T(j)} = a_\phi \cdot 1 = a_\phi$. We assume that it holds for the case $n - 1$. Let $f(x) = a_\phi (x - \alpha_1 e)(x - \alpha_2 e) \cdots (x - \alpha_n e) = g(x) h(x)$ where $g(x) = a_\phi T(n, n-1)$ and $h(x) = (x - \alpha_n e)$. Then $(Df) = (D(gh))_a = M_{g(a)} \cdot (Dh)_a + M_{h(a)} \cdot (Dg)_a = a_\phi M_{T(n-1,0)} + M_{(a - a_n e)} \cdot \sum_{j=1}^{n-1} a_\phi M_{T(n-1,j)}$, which completes the proof of this theorem.

On the other hand, we have well known the following property; if $_{1, 2}, \cdots n$, are distinct elements of $\mathbb{C}$, then there exist $a_1, a_2, \cdots, a_n$ in $\mathbb{C}$ such that $1/(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n) = a_1/(x - \alpha_1) + a_2/(x - \alpha_2) + \cdots + a_n/(x - \alpha_n)$. Using the same method, we have the following lemma.

LEMMA(2.5) Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be distinet elements of $\mathbb{C}$ and let $(x - \alpha_1 e), (x - \alpha_2 e), \cdots, (x - \alpha_n e)$ be invertible elements in Banach algebra A. Then there exist elements $a_1, a_2, \cdots, a_n$ in $\mathbb{C}$ such that $1/[(x - \alpha_1 e)(x - \alpha_2 e) \cdots (x - \alpha_n e)] = a_1/x - \alpha_1 e) + a_2/(x - \alpha_2 e) + a_n/(x - \alpha_n e)$.

By LEMMA(2.2), we obtain that $x - \alpha e (\neq 0)$ is invertible in $\|x\| < \alpha$ and $(x - \alpha e)^{-1} = -\alpha \sum_{n=0} (\alpha^{-1} x)^n$. Therefore, we have

COROLLARY (2.9) If $f(x) = \prod_{i=1}^{n} (x - \alpha_i e)$ has n distinct roots, α_i's are non-zero comlex numbers, let $\alpha = \min\{|\alpha_1|, |\alpha_2|, \cdots, |\alpha_n|\}$ and $\|x\| < \alpha$, then there exist $a_1, a_2, \cdots, \cdots, a_n$ in $\mathbb{C}$ such that $f(x)^{-1} = \sum_{i=1}^{n} a_i (x - \alpha_i e)^{-1} = \sum_{j=0}^{\infty} \sum_{i=1}^{n} (-a_i \alpha_i)(\alpha_i^{-1} x)^j$.

THEOREM (2.10) Let $f(x)^{-1} = \sum_{j=0}^{\infty} \sum_{i=1}^{n} (-a_i \alpha_i)(\alpha_i^{-1} x)^j$. Then $(Df(x)^{-1})_a = \sum_{j=0}^{\infty} \left(\sum_{i=1}^{n} (-a_i \alpha_i) \sum_{k=0}^{i-1} L_a i - k - 1 \cdot R_a k \right)$.

In the preceding discussions, we have derived the formula of the Frechet derivative for rational functions.

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〈국문초록〉

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本 論文에서는 有理函數의 Fréchet微分에 關한 一般的인 公式을 求하였다.