博士學位論文

Common Coupled Coincidence Point Theorems of Three Mappings satisfying Generalized Contractive Condition

濟州大學校 大學院

數 學 科

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2019年 2月

일반화 축약조건을 만족하는 세 함수의 공통 연계된 일치점 정리

指導教授 梁　永　五

宋　美　惠

이 論文을 理學 博士學位 論文으로 提出함

2018年　12月

宋美惠의 理學 博士學位 論文을 認准함

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濟州大學校 大學院

2018年　12月

Common Coupled Coincidence Point Theorems of Three Mappings satisfying Generalized Contractive Condition

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A thesis submitted in partial fulfillment of the requirement for the degree of Doctor of Science

2018. 12.

This thesis has been examined and approved.

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Date : ____________

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CONTENTS

<Abstract>

Common Coupled Coincidence Point Theorems of Three Mappings satisfying Generalized Contractive Condition

Mi Hye Song

Let X be an arbitrary nonempty set and $f: X \to X$ be a mapping. A fixed point for f is a point $x \in X$ such that $fx = x$. In 1922, Banach proved the fixed point theorem for a single-valued mapping in the setting of a complete metric space known as the Banach contraction principle. After that a considerable amount of research work for the development of fixed point theory have been executed by several authors. Fixed point theory is one of the most powerful and fruitful tools of modern mathematics and may be considered a core subject of nonlinear analysis started by Banach.

In 2013, Liu and Xu introduced the concept of cone metric space over Banach algebra and considered fixed point theorems. After that, some researchers started to study the existence problems of fixed points for some contractions in such cone metric spaces.

In this paper, using the properties of spectral radius, we obtain sufficient conditions for existence of some common coupled coincidence point results and coupled fixed point results for three mappings $S, T: X \times X \to X$ and $g: X \to X$ satisfying generalized contractive condition in a complete cone b-metric space (X, d) over Banach algebra. And by applying Theorem 3.2(not giving direct proof), we prove some common coupled coincidence point results and coupled fixed point results for three self-mappings $f, h, g: X \to X$ satisfying more general contractive condition in a cone b-metric space (X, d) over Banach algebra. Our results not only directly improve and expand several well-known comparable assertions in cone b-metric spaces over Banach algebra, but also unify and improve some previous results in cone metric spaces over Banach algebras.

As applications, we obtain the existence and uniqueness of solution for some equations by using our results.

1. **Introduction**

A mapping $T : X \to X$, where (X, d) is a metric space, is said to be a *contraction mapping* if, for all $x, y \in X$, there is a contractive constant $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y).$$

In 1922, Banach proved the fixed point theorem for a single-valued mapping in the setting of a complete metric space known as the Banach contraction principle. The famous Banach contraction principle states that if (X, d) is a complete metric space and $f : X \to X$ is a contraction mapping, then f has a unique fixed point. As a classical example, it is well known that every continuous function $f : [0, 1] \to [0, 1]$ has a fixed point and Brouwer generalized it like this: If $f : D^n \to D^n$ is continuous where $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, then f has a fixed point.

This Banach contraction principle has further several generalizations in metric spaces as well as in cone metric spaces. This principle has been generalized in different directions in all kinds of spaces by mathematicians over the years and is widely recognized as one of the most influential sources in pure and applied mathematics. Also, in the contemporary research, it remains a heavily investigated branch as a consequence of the strong applicability. After that a considerable amount of research work for the development of fixed point theory have been executed by several authors.

The interplay between the notion of a nearness among abstract objects of a set and fixed point theory is very strong and fruitful. This gives rise to an interesting branch of nonlinear functional analysis called metric fixed point theory. This theory is studied in the framework of a set equipped with some notion of a distance along with appropriate mappings satisfying certain contraction conditions and has many applications in economics, computer science and other related disciplines.

In 2007, Huang and Zhang [6] introduced the concept of cone metric space which is a meaningful generalization of metric space by replacing the set of real numbers with an ordering Banach space, and proved some fixed point theorems for contractive mappings on these spaces. Recently, in ([6], [10], [13], [14], [16], [17], [18]), some common fixed point theorems have been proved for contractive maps on cone metric spaces. Gnana Bhaskar and Lakshmikantham [3] introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered sets. Since then this new concept is used in various directions and also extended in various spaces like metric space, partially ordered metric space, fuzzy metric space, cone metric space, etc([9]).

Very recently, Liu and Xu [8] introduced the concept of cone metric space over Banach algebra and considered fixed point theorems in such spaces in a different way by replacing Banach space with Banach algebra and by restricting the contractive constants to be vectors and the relevant multiplications to be vector ones instead of usual real constants and scalar multiplications. And they proved some fixed point theorems of generalized Lipschitz mappings with weaker and natural conditions on generalized Lipschitz constant k by means of spectral radius and pointed out that it is significant to introduce this concept because it can be proved that cone metric spaces over Banach algebras are not equivalent to metric spaces in terms of the existence of the fixed points of the generalized Lipschitz mappings. That is, they provided an example to explain the non-equivalence of fixed point results between the vectorial versions and scalar versions. In the past three years, some researchers started to study the existence problems of (coupled) fixed points for some contractions in cone metric spaces over Banach algebras (see [7], [8], [12], [16], [17], [18], [19], [20]). As a result, there is still both interest and need for research in the field of studying fixed point theorems in the framework of cone metric or cone b-metric spaces.

This thesis consists of four sections as follows:

In section 2, we give well known properties of Banach algebra, cone metric spaces, cone b -metric spaces over Banach algebra, convergence and Cauchy sequence on these spaces, spectral radius, etc. In section 3, we prove some common coupled coincidence point results and coupled fixed point results for three mappings $S, T : X \times X \to X$ and $g : X \to X$ satisfying generalized contractive condition in a cone b-metric space (X, d) over Banach algebra without assumption of normality. The results not only directly improve and expand several well-known comparable assertions in cone b-metric spaces over Banach algebra([12], [19], [20]), but also unify and improve some previous results in cone metric spaces over Banach algebras in three senses. Firstly, the considered contractive conditions are more general than earlier ones. Secondly, noncommuting maps are considered. Finally, normality of the cone is not assumed. Furthermore, we give two examples to support our conclusions.

In section 4, we prove some common coupled coincidence point results and coupled fixed point results for three self-mappings $f, h, g : X \to X$ satisfying more generalized contractive condition with ten terms in a cone b-metric space (X, d) over Banach algebra, where the cone is not necessarily normal. This results extend and improve recent related results in the literature([8], [12], [14], [19], [20]).

In section 5, we obtain the existence and uniqueness of solution for some equations by using our results.

2. **Preliminaries**

Let A always be a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in A$, $\alpha \in \mathbb{R}$):

(1) $(xy)z = x(yz)$;
(2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
(4) $\|xy\| \le \|x\|\|y\|$.

In this paper, we shall assume that A is a real Banach algebra with a unit (i.e., a multiplicative identity) e. An element $x \in A$ is said to be *invertible* if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}.

Let A be a real Banach algebra with a unit e and θ the zero element of A. A nonempty closed subset P of Banach algebra A is called a *cone* if

(1) $\{\theta, e\} \subset P$;
(2) $\alpha P + \beta P \subset P$ for all nonnegative real numbers α, β ;
(3) $P^2 = PP \subset P$;
(4) $P \cap (-P) = \{\theta\}$ i.e, $x \in P$ and $-x \in P$ imply $x = \theta$.

For any cone $P \subseteq A$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ stands for $x \preceq y$ but $x \ne y$. Also, we use $x \ll y$ to indicate that $y - x \in \operatorname{int} P$ where $\operatorname{int} P$ denotes the interior of P. If $\operatorname{int} P \ne \emptyset$ then P is called a *solid cone*. A cone P is called *normal* if there exists a number K such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \le K\|y\|. \tag{2.1}$$

Equivalently, the cone P is normal if

$$x_n \preceq y_n \preceq z_n \text{ and } \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x \tag{2.2}$$

The least positive number K satisfying condition (2.1) is called the *normal constant* of P.

Example 2.1. ([5]) Let $E = C^1_{\mathbb{R}}[0,1]$ be the set of all real valued functions on X which also have continuous derivatives on $[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \ge 0\}$. This cone is non-normal. For example, consider $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$.

Then $\theta \preceq x_n \preceq y_n$ and $y_n \to \theta$ as $n \to \infty$. but

$$\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1.$$

Hence x_n does not converge to zero and hence P is a non-normal cone.

Definition 2.2. ([6]) Let X be a nonempty set and let E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone $P \subseteq E$. Suppose the mapping $d : X \times X \to E$ satisfies the following conditions:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X, and (X, d) is called a *cone metric space*.

Example 2.3. (1) Let $E = \mathbb{R}^2, P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$. Define d : $X \times X \to E$ by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is clearly a complete cone metric space.

(2) (The case of non-normal cone) Let $X = [0, \infty), E = C^1_{\mathbb{R}}[0, 1]$ and $P = \{\phi \in E : \phi(t) \geq 0,\ t \in [0, 1]\}$. The mapping $d : X \times X \to E$ is defined in the following way: $d(x, y) = |x - y|\phi$, where $\phi(t) = e^t$. Then (X, d) is clearly a complete cone metric space.

Definition 2.4. ([7]) Let X be a nonempty set, $s \geq 1$ be a constant and A be a real Banach algebra. Suppose the mapping $d : X \times X \to A$ satisfies the following conditions:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \preceq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then d is called a *cone b-metric* on X, and (X, d) is called a *cone b-metric space* over Banach algebra A.

In particular, if $s = 1$, then a cone b-metric space over Banach algebra A is a cone metric space over Banach algebra A.

Remark 2.5. The class of cone b-metric space over Banach algebra is larger than the class of cone metric space over Banach algebra since the latter must be the former, but the converse is not true. We can present many examples, as follows, which show that introducing a cone b-metric space over Banach algebra instead of a cone metric space over Banach algebra is very meaningful since there exist cone b-metric spaces over Banach algebras which are not cone metric spaces over Banach algebras.

Example 2.6. Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. We show that ρ is a b-metric with $s = 2^{p-1}$.

Proof. Obviously condition (1) and (2) of Definition 2.4 are satisfied.

If $1 < p < \infty$, then the convexity of the function $f(x) = x^p \quad (x > 0)$ implies

$$(\frac{a+c}{2})^p \leq \frac{1}{2}(a^p + c^p),$$

and hence $(a + c)^p \leq 2^{p-1}(a^p + c^p)$ holds. Thus for each $x, y, z \in X$ we obtain

$$\begin{aligned} \rho(x, y) = (d(x, y))^p &\leq (d(x, z) + d(z, y))^p \\ &\leq 2^{p-1}((d(x, z))^p + (d(z, y))^p) \\ &= 2^{p-1}(\rho(x, z) + \rho(z, y)). \end{aligned}$$

So condition (3) of Definition 2.4 holds and ρ is a b-metric. □

Remark 2.7. In the preceding example, it should be noted that if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space. For example, let $X = \mathbb{R}$ be the set of real numbers and $d(x, y) = |x - y|$ be the usual Euclidean metric then $\rho(x, y) = |x - y|^2$ is a cone b-metric on $\mathbb{R}$ with $s = 2$, but is not a metric on $\mathbb{R}$, because the triangle inequality does not hold.

In the following, we always assume that (X, d) is a cone b-metric space over Banach algebra A.

Definition 2.8. ([8]) Let $\{x_n\}$ be a sequence in (X, d) and $x \in X$.

(1) If for every $c \in A$ with $\theta \ll c$, there exists a natural number N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$ *converges to* x, and the point x is the *limit* of $\{x_n\}$. We denote this by

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \quad (n \to \infty).$$

(2) If for all $c \in A$ with $\theta \ll c$, there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $m, n > N$, then $\{x_n\}$ is called a *Cauchy sequence* in X.

(3) A cone metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent.

Definition 2.9. Let E be a real Banach space with a solid cone P. A sequens $\{x_n\} \subset P$ is called a $c-$*sequence* if for any $c \in A$ with $\theta \ll c$, there exists a positive integer N such that $x_n \ll c$ for all $n \geq N$.

Example 2.10. Let $A = \mathbb{R}^2$ with the norm $\|(u_1, u_2)\| = |u_1| + |u_2|$. (1) Define the multiplication by

$$uv = (u_1, u_2)(v_1, v_2) = (u_1v_1, u_2v_2).$$

Let $P = \{u = (u_1, u_2) \in A : u_1, u_2 \geq 0\}$. It is clear that P is a normal cone and A is a Banach algebra with a unit $e = (1, 1)$. Put $X = \mathbb{R}^2$ and define a mapping $d : X \times X \to A$ by

$$d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2|^2, |y_1 - y_2|^2).$$

It is easy to see that (X, d) is a complete cone b-metric space over Banach algebra A with the coefficient $s = 2$.

(2) Define the multiplication by

$$uv = (u_1, u_2)(v_1, v_2) = (u_1v_1, u_1v_2 + u_2v_1).$$

Let $P = \{u = (u_1, u_2) \in A : u_1, u_2 \geq 0\}$. It is clear that P is a normal cone and A is a Banach algebra with a unit $e = (1, 0)$. Put $X = \mathbb{R}^2$ and define a mapping $d : X \times X \to A$ by

$$d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2|, |y_1 - y_2|).$$

It is easy to see that (X, d) is a complete cone b-metric space over Banach algebra A with the coefficient $s = 1$.

Example 2.11. Let $A = C[a, b]$ be the set of continuous functions on $[a, b]$ with the supremum. Define multiplication in the usual way. Then A is a Banach algebra with a unit 1. Set $P = \{x \in A : x(t) \geq 0,\ t \in [a, b]\}$ and $X = \mathbb{R}$. We define a mapping $d : X \times X \to A$ by $d(x, y)(t) = |x - y|^p e^t$ for all $x, y \in X$ and for each $t \in [a, b]$, where $p > 1$ is a constant. This makes (X, d) into a cone b-metric space over Banach algebra A with the coefficient $s = 2^{p-1}$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality.

Example 2.12. Let $A = C^1_{\mathbb{R}}[0, 1]$ be the set of all real valued functions on X which also have continuous derivatives on $[0, 1]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$. Define multiplication in A as just pointwise multiplication. Then A is a real Banach algebra with a unit $e = 1$ ($e(t) = 1$ for all $t \in [0, 1]$). The set

$$P = \{x \in A : x(t) \geq 0, t \in X\}$$

is cone. Moreover P is a non-normal solid cone. Let $X = \{a, b, c\}$. Define a mapping $d : X \times X \to A$ by

$$\begin{aligned} d(a,b)(t) &= d(b,a)(t) = e^t,\ d(b,c)(t) = d(c,b)(t) = 2e^t, \\ d(c,a)(t) &= d(a,c)(t) = 3e^t \end{aligned}$$

and $d(x, x)(t) = \theta$ for all $t \in [0, 1]$ and each $x \in X$. Then (X, d) is a solid cone metric space over Banach algebra A.

Example 2.13. ([7]) (the case of a non-normal cone). Let $X = [0, 1]$ and let $A = C^1_{\mathbb{R}}[0, 1]$ be the set of all real valued functions on X which also have continuous derivatives on X with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and the usual multiplication. Let

$$P = \{x \in A : x(t) \geq 0, t \in X\}.$$

It is clear that P is a non-normal cone and A is a Banach algebra with a unit $e = 1$. Define a mapping $d : X \times X \to A$ by

$$d(x, y)(t) = |x - y|^2 e^t.$$

We make a conclusion that (X, d) is a complete cone b-metric space over Banach algebra A with the coefficient $s = 2$.

Example 2.14. ([7]) Let $A = \{a = (a_{ij})_{3\times 3} : a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 3\}$ be the set of 3×3 real matrices and

$$\|a\| = \frac{1}{3} \sum_{1 \leq i,j \leq 3} |a_{ij}|.$$

Take a cone

$$P = \{a \in A : a_{ij} \geq 0,\ \ 1 \leq i, j \leq 3\}$$

in A. Let $X = \{1, 2, 3\}$. Define a mapping $d : X \times X \to A$ by

$$d(1, 1) = d(2, 2) = d(3, 3) = \theta \quad \text{and}$$

$$d(1,2) = d(2,1) = \begin{pmatrix} 1 & 1 & 4 \\ 4 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad d(1,3) = d(3,1) = \begin{pmatrix} 4 & 1 & 4 \\ 4 & 3 & 5 \\ 2 & 3 & 1 \end{pmatrix},$$

$$d(2,3) = d(3,2) = \begin{pmatrix} 9 & 5 & 6 \\ 16 & 4 & 4 \\ 3 & 4 & 2 \end{pmatrix}.$$

It ensures us that (X, d) is a cone b-metric space over Banach algebra A with the coefficient $s = \frac{5}{2}$, but it is not a cone metric space over Banach algebra since the triangle inequality is lacked.

Example 2.15. ([7]) Let $X = l_p = \{x = (x_n)_{n\geq 1} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ $(0 < p < 1)$. Define a mapping $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$$

where $x = (x_n)_{n\geq 1}, y = (y_n)_{n\geq 1} \in l_p$. Clearly, (X, d) is a b-metric space. Put

$$A = l_1 = \{a = (a_n)_{n\geq 1} : \sum_{n=1}^{\infty} |a_n| < \infty\}$$

with convolution as multiplication:

$$ab = (a_n)_{n\geq 1}(b_n)_{n\geq 1} = (\sum_{i+j=n} a_i b_j)_{n\geq 1}.$$

It is valid that A is a Banach algebra with a unit $e = (1, 0, 0, \cdots)$. Choose a cone

$$P = \{a = (a_n)_{n\geq 1} \in A : a_n \geq 0 \quad \text{for all} \quad n \geq 1\}.$$

Define $\bar{d} : X \times X \to A$ by $\bar{d}(x, y) = (\frac{d(x,y)}{2^n})_{n\geq 1}$, it may be verified that $(X, \bar{d})$ is a cone b-metric space over Banach algebra A with the coefficient $s = 2^{\frac{1}{p}-1} > 1$, but it is not a cone metric space over Banach algebra since the triangle inequality does not hold.

Lemma 2.16. *([10]) Let E be a real Banach space with a cone P. Then*

(1) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*
(2) *If $a \preceq b$ and $b \ll c$, then $a \ll c$.*
(3) *If $a \preceq b + c$ for each $\theta \ll c$, then $a \preceq b$.*
(4) *If $\{x_n\}, \{y_n\}$ are sequences in E such that $x_n \to x$, $y_n \to y$ and $x_n \preceq y_n$ for all $n \geq 1$, then $x \preceq y$.*

Lemma 2.17. *([10]) If E is a real Banach space with cone P. Then*

(1) If $a \preceq \lambda a$ where $a \in P$ and $0 < \lambda < 1$ then $a = \theta$.

(2) If $c \in \operatorname{int} P$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.

Proof. (1) The condition $a \preceq \lambda a$ means that $\lambda a - a \in P$ that is, $-(1 - \lambda)a \in P$. Since $a \in P$ and $1 - \lambda > 0$, then also $(1 - \lambda)a \in P$. Thus we have

$$(1 - \lambda)a \in P \cap (-P) = \{\theta\}$$

and so $a = \theta$.

(2) Let $\theta \ll c$ be given. Choose a symmetric neighborhood V such that $c + V \subseteq P$. Since $a_n \to \theta$, there exists a positive integer n_0 such that $a_n \in V = -V$ for $n > n_0$. This means that $c \pm a_n \in c + V \subseteq P$ for $n > n_0$, that is, $a_n \ll c$. $\square$

Lemma 2.18. *([14]) Let A be a real Banach algebra with a unit e and P be a solid cone in A. We define the spectral radius $r(x)$ of $x \in A$ by*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n}.$$

(1) *If $0 \leq r(x) < 1$, then $e - x$ is invertible,*

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i \quad \text{and} \quad r((e-x)^{-1}) \leq \frac{1}{1-r(x)}.$$

(2) *If $r(x) < 1$, then $\|x^n\| \to 0$ as $n \to \infty$.*
(3) *If $x \in P$ and $r(x) < 1$, then $(e-x)^{-1} \in P$.*
(4) *If $k, u \in P$, $r(k) < 1$ and $u \preceq ku$, then $u = \theta$.*
(5) *$r(x) \leq \|x\|$ for all $x \in A$.*
(6) *If $x, y \in A$ and x, y commute, then the following holds:*
  (a) *$r(xy) \leq r(x)r(y)$*
  (b) *$r(x+y) \leq r(x) + r(y)$ and*
  (c) *$|r(x) - r(y)| \leq r(x-y)$.*

Lemma 2.19. *([10], [14]) Let (X, d) be a complete cone b-metric space over Banach algebra A and let P be a solid cone in A. Let $\{x_n\}$ be a sequence in X. Then*

(1) *If $\|x_n\| \to 0$ as $n \to \infty$, then $\{x_n\}$ is a $c-$sequence.*
(2) *If $k \in P$ is any vector and $\{x_n\}$ is $c-$sequence in P, then $\{kx_n\}$ is a $c-$sequence.*
(3) *If $x, y \in A$, $a \in P$ and $x \preceq y$, then $ax \preceq ay$.*
(4) *If $\{x_n\}$ converges to $x \in X$, then $\{d(x_n, x)\}$, $\{d(x_n, x_{n+p})\}$ are c-sequences for any $p \in \mathbb{N}$.*

3. Common coupled coincidence point for two maps on the product space

In this section, we give common coincidence point results and coupled fixed point results for two mappings $S, T : X \times X \to X$ satisfying some natural and more general contractive condition given by fixed mapping g defined on a complete cone b-metric space X over Banach algebra. Our main results generalize the results of Yang([19], [20]) and Song [12] by giving the weak radius condition (3.2), and modify many exciting results in the literature.

Definition 3.1. ([3], [14]) Let (X, d) be a cone b-metric space over Banach algebra A.

(1) An element $(x, y) \in X \times X$ is called a *coupled fixed point* of $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

(2) An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of mappings $F : X \times X \to X$and $g : X \to X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$, and (gx, gy) is called coupled point of coincidence;

(3) An element $(x, y) \in X \times X$ is called a *common coupled fixed point* of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

(4) The mappings $F : X \times X \to X$ and $g : X \times X$ are called *weakly compatible* if $g(F(x, y)) = F(gx, gy)$ whenever $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Theorem 3.2. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Suppose that $S, T : X \times X \to X$ and $g : X \to X$ are mappings satisfying the contractive condition*

$$\begin{aligned} d(S(x,y), T(u,v)) \quad &\preceq \quad a_1 d(gx, gu) + a_2 d(gy, gv) \\ &+ \quad a_3 d(S(x,y), gx) + a_4 d(S(y,x), gy) \\ &+ \quad a_5 d(T(u,v), gu) + a_6 d(T(v,u), gv) \\ &+ \quad a_7 d(S(x,y), gu) + a_8 d(S(y,x), gv) \\ &+ \quad a_9 d(T(u,v), gx) + a_{10} d(T(v,u), gy), \end{aligned} \tag{3.1}$$

for all $x, y, u, v \in X$, where $a_i \in P$ commute for $i = 1, 2, \cdots, 10$ and

$$sr(\sum_{i=1}^{4} a_i) + sr(a_5 + a_6) + r(a_7 + a_8) + (s^2 + s) r(a_9 + a_{10}) < 1. \tag{3.2}$$

If $S(X \times X), T(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then S, T and g have a common coupled coincidence point in X, that is, there exist $x, y \in X$ such that

$$gx = S(x,y) = T(x,y) \quad \textit{and} \quad gy = S(y,x) = T(y,x).$$

Also, if S, T and g are weakly compatible, then S, T and g have a unique coupled fixed point, that is, there is a unique $u \in X$ such that $u = gu = S(u, u) = T(u, u)$.

Proof. Let x_0 and y_0 be two arbitrary elements in X. Since $S(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = S(x_0, y_0)$ and $gy_1 = S(y_0, x_0)$. Again noting $T(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = T(x_1, y_1)$ and $gy_2 = T(y_1, x_1)$. Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_{2n+1} = S(x_{2n}, y_{2n})$, $gy_{2n+1} = S(y_{2n}, x_{2n})$, $gx_{2n+2} = T(x_{2n+1}, y_{2n+1})$ and $gy_{2n+2} = T(y_{2n+1}, x_{2n+1})$.

For each $k \in \mathbb{N}$, by the given contractive condition (3.1), we have

$$\begin{aligned}
d(gx_{2k+1}, gx_{2k+2}) &= d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \\
&\preceq a_1 d(gx_{2k}, gx_{2k+1}) + a_2 d(gy_{2k}, gy_{2k+1}) \\
&\quad + a_3 d(S(x_{2k}, y_{2k}), gx_{2k}) + a_4 d(S(y_{2k}, x_{2k}), gy_{2k}) \\
&\quad + a_5 d(T(x_{2k+1}, y_{2k+1}), gx_{2k+1}) + a_6 d(T(y_{2k+1}, x_{2k+1}), gy_{2k+1}) \\
&\quad + a_7 d(S(x_{2k}, y_{2k}), gx_{2k+1}) + a_8 d(S(y_{2k}, x_{2k}), gy_{2k+1}) \\
&\quad + a_9 d(T(x_{2k+1}, y_{2k+1}), gx_{2k}) + a_{10} d(T(y_{2k+1}, x_{2k+1}), gy_{2k}) \\
&= a_1 d(gx_{2k}, gx_{2k+1}) + a_2 d(gy_{2k}, gy_{2k+1}) + a_3 d(gx_{2k+1}, gx_{2k}) \\
&\quad + a_4 d(gy_{2k+1}, gy_{2k}) + a_5 d(gx_{2k+2}, gx_{2k+1}) + a_6 d(gy_{2k+2}, gy_{2k+1}) \\
&\quad + a_7 d(gx_{2k+1}, gx_{2k+1}) + a_8 d(gy_{2k+1}, gy_{2k+1}) \\
&\quad + a_9 d(gx_{2k+2}, gx_{2k}) + a_{10} d(gy_{2k+2}, gy_{2k}),
\end{aligned}$$

which implies that

$$\begin{aligned}
(e - a_5 - sa_9) d(gx_{2k+1}, gx_{2k+2}) &\preceq (a_1 + a_3 + sa_9) d(gx_{2k}, gx_{2k+1}) \\
&\quad + (a_2 + a_4 + sa_{10}) d(gy_{2k}, gy_{2k+1}) \\
&\quad + (a_6 + sa_{10}) d(gy_{2k+1}, gy_{2k+2}).
\end{aligned} \tag{3.3}$$

since $d(gx_{2k+2}, gx_{2k}) \preceq sd(gx_{2k}, gx_{2k+1}) + sd(gx_{2k+1}, gx_{2k+2})$ and

$$d(gy_{2k+2}, gy_{2k}) \preceq sd(gy_{2k+2}, gy_{2k+1}) + sd(gy_{2k+1}, gy_{2k}).$$

Similarly, we have

$$
\begin{aligned}
d(gy_{2k+1}, gy_{2k+2}) &= d(S(y_{2k}, x_{2k}), T(y_{2k+1}, x_{2k+1})) \\
&\preceq a_1 d(gy_{2k}, gy_{2k+1}) + a_2 d(gx_{2k}, gx_{2k+1}) \\
&+ a_3 d(S(y_{2k}, x_{2k}), gy_{2k}) + a_4 d(S(x_{2k}, y_{2k}), gx_{2k}) \\
&+ a_5 d(T(y_{2k+1}, x_{2k+1}), gy_{2k+1}) + a_6 d(T(x_{2k+1}, y_{2k+1}), gx_{2k+1}) \\
&+ a_7 d(S(y_{2k}, x_{2k}), gy_{2k+1}) + a_8 d(S(x_{2k}, y_{2k}), gx_{2k+1}) \\
&+ a_9 d(T(y_{2k+1}, x_{2k+1}), gy_{2k}) + a_{10} d(T(x_{2k+1}, y_{2k+1}), gx_{2k}) \\
&= a_1 d(gy_{2k}, gy_{2k+1}) + a_2 d(gx_{2k}, gx_{2k+1}) + a_3 d(gy_{2k+1}, gy_{2k}) \\
&+ a_4 d(gx_{2k+1}, gx_{2k}) + a_5 d(gy_{2k+2}, gy_{2k+1}) + a_6 d(gx_{2k+2}, gx_{2k+1}) \\
&+ a_9 d(gy_{2k+2}, gy_{2k}) + a_{10} d(gx_{2k+2}, gx_{2k}) \\
&\preceq a_1 d(gy_{2k}, gy_{2k+1}) + a_2 d(gx_{2k}, gx_{2k+1}) + a_3 d(gy_{2k+1}, gy_{2k}) \\
&+ a_4 d(gx_{2k+1}, gx_{2k}) + a_5 d(gy_{2k+2}, gy_{2k+1}) + a_6 d(gx_{2k+2}, gx_{2k+1}) \\
&+ sa_9[d(gy_{2k+2}, gy_{2k+1}) + d(gy_{2k+1}, gy_{2k})] \\
&+ sa_{10}[d(gx_{2k+2}, gx_{2k+1}) + d(gx_{2k+1}, gx_{2k})],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(e - a_5 - sa_9) d(gy_{2k+1}, gy_{2k+2}) &\preceq (a_2 + a_4 + sa_{10}) d(gx_{2k}, gx_{2k+1}) \\
&+ (a_6 + sa_{10}) d(gx_{2k+1}, gx_{2k+2}) \qquad (3.4) \\
&+ (a_1 + a_3 + sa_9) d(gy_{2k}, gy_{2k+1}).
\end{aligned}
$$

Adding both inequalities, we have

$$
\begin{aligned}
& (e - a_5 - sa_9)[d(gx_{2k+1}, gx_{2k+2}) + d(gy_{2k+1}, gy_{2k+2})] \\
\preceq\ & (a_1 + a_2 + a_3 + a_4 + sa_9 + sa_{10})[d(gx_{2k}, gx_{2k+1}) + d(gy_{2k}, gy_{2k+1})] \\
+\ & (a_6 + sa_{10})[d(gx_{2k+1}, gx_{2k+2}) + d(gy_{2k+1}, gy_{2k+2})]
\end{aligned}
$$

That is,

$$
\begin{aligned}
& (e - a_5 - a_6 - sa_9 - sa_{10})[d(gx_{2k+1}, gx_{2k+2}) + d(gy_{2k+1}, gy_{2k+2})] \\
\preceq\ & (a_1 + a_2 + a_3 + a_4 + sa_9 + sa_{10})[d(gx_{2k}, gx_{2k+1}) + d(gy_{2k}, gy_{2k+1})].
\end{aligned}
$$

Since $r(a_5 + a_6) + sr(a_9 + a_{10}) < 1$ by hypothesis, $e - (a_5 + a_6) - s(a_9 + a_{10})$ is invertible and so the above relation implies that

$$
d(gx_{2k+1}, gx_{2k+2}) + d(gy_{2k+1}, gy_{2k+2}) \preceq h[d(gx_{2k}, gx_{2k+1}) + d(gy_{2k}, gy_{2k+1})]
$$

where $h = (e - a_5 - a_6 - sa_9 - sa_{10})^{-1}(a_1 + a_2 + a_3 + a_4 + sa_9 + sa_{10})$. Similarly we have

$$d(gx_{2k+2}, gx_{2k+3}) + d(gy_{2k+2}, gy_{2k+3}) \preceq h[d(gx_{2k+1}, gx_{2k+2}) + d(gy_{2k+1}, gy_{2k+2})].$$

Therefore

$$\begin{aligned} d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) &\preceq h[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)] \\ &\preceq h^2[d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})] \\ &\vdots \\ &\preceq h^n[d(gx_0, gx_1) + d(gy_0, gy_1)] \end{aligned}$$

By hypothesis and Lemma 2.18, we have

$$\begin{aligned} r(h) &\leq r((e - a_5 - a_6 - sa_9 - sa_{10})^{-1})r(a_1 + a_2 + a_3 + a_4 + sa_9 + sa_{10}) \\ &\leq \frac{r(a_1 + a_2 + a_3 + a_4) + sr(a_9 + a_{10})}{1 - r(a_5 + a_6) - sr(a_9 + a_{10})} < \frac{1}{s} \leq 1 \end{aligned}$$

which means that $e - h$ is invertible, $(e-h)^{-1} = \sum_{i=0}^{\infty} h^n$ and $\|h^n\| \to 0$ as $n \to \infty$.

Now if $\delta_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})$, then the above relation implies

$$\delta_n \preceq h\delta_{n-1} \preceq \cdots \preceq h^n\delta_0.$$

For $m > n$, we have

$$\begin{aligned} d(gx_n, gx_m) + d(gy_n, gy_m) &\preceq s[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\ &+ s[d(gy_n, gy_{n+1}) + d(gy_{n+1}, dy_m)] \\ &\preceq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + s^2 d(gx_{n+2}, gx_m) \\ &+ sd(gy_n, gy_{n+1}) + s^2 d(gy_{n+1}, gy_{n+2}) + s^2 d(gy_{n+2}, gy_m) \\ &\preceq \\ &\vdots \\ &\preceq s\delta_n + s^2\delta_{n+1} + \cdots + s^{m-n}\delta_{m-1} \\ &\preceq s(h^n + sh^{n+1} + \cdots + s^{m-n-1}h^{m-1})\delta_0 \\ &= sh^n[e + sh + (sh)^2 + \cdots + (sh)^{m-n-1}]\delta_0 \\ &\preceq sh^n(\sum_{i=0}^{\infty}(sh)^i)\delta_0 \\ &= (e - sh)^{-1}sh^n\delta_0 \end{aligned}$$

since $r(sh) < 1$ and P is closed. Since $r(h) < 1$, $\|h^n\| \to 0$ as $n \to \infty$ and so

$$\|(e - sh)^{-1}sh^n\delta_0\| \leq \|(e - sh)^{-1}s\|\|h^n\|\|\delta_0\| \to 0.$$

Thus for any $c \in A$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that for any $m > n > N$, we have

$$d(gx_n, gx_m) + d(gy_n, gy_m) \preceq (e-h)^{-1} s h^n \delta_0 \ll c.$$

Thus $\{d(gx_n, gx_m) + d(gy_n, gy_m)\}$ is a c-sequence in P. Since

$$\theta \preceq d(gx_n, gx_m), d(gy_n, gy_m) \preceq d(gx_n, gx_m) + d(gy_n, gy_m),$$

we note that $\{d(gx_n, gx_m)\}$ and $\{d(gy_n, gy_m)\}$ are c-sequences and so Cauchy sequence in X. Since X is complete, there exists $x \in X$ and $y \in X$ such that $gx_n \to gx$ and $gy_n \to gy$ as $n \to \infty$.

Now we show that $gx = S(x, y)$ and $gy = S(y, x)$. Then

$$\begin{aligned} d(gx, S(x,y)) &\preceq sd(gx, gx_{2k+2}) + sd(gx_{2k+2}, S(x,y)) \\ &= sd(gx, gx_{2k+2}) + sd(T(x_{2k+1}, y_{2k+1}), S(x,y)) \end{aligned}$$

and

$$\begin{aligned} sd(T(x_{2k+1}, y_{2k+1}), S(x,y)) \quad &\preceq \quad sa_1 d(gx_{2k+1}, gx) + sa_2 d(gy_{2k+1}, gy) \\ &+ \quad sa_3 d(T(x_{2k+1}, y_{2k+1}), gx_{2k+1}) \\ &+ \quad sa_4 d(T(y_{2k+1}, x_{2k+1}), gy_{2k+1}) \\ &+ \quad sa_5 d(S(x,y), gx) + sa_6 d(S(y,x), gy) \\ &+ \quad sa_7 d(T(x_{2k+1}, y_{2k+1}), gx) + sa_8 d(Ty_{2k+1}, x_{2k+1}), gy) \\ &+ \quad sa_9 d(S(x,y), gx_{2k+1}) + sa_{10} d(S(y,x), gy_{2k+1}) \\ &= \quad sa_1 d(gx_{2k+1}, gx) + sa_2 d(gy_{2k+1}, gy) \\ &+ \quad sa_3 d(gx_{2k+2}, gx_{2k+1}) + sa_4 d(gy_{2k+2}, gy_{2k+1}) \\ &+ \quad sa_5 d(S(x,y), gx) + sa_6 d(S(y,x), gy) \\ &+ \quad sa_7 d(gx_{2k+2}, gx) + sa_8 d(gy_{2k+2}, gy) \\ &+ \quad sa_9 d(S(x,y), gx_{2k+1}) + sa_{10} d(S(y,x), gy_{2k+1}). \end{aligned}$$

Since $d(S(x,y), gx_{2k+1}) \preceq sd(S(x,y), gx) + sd(gx, gx_{2k+1})$ and

$$d(S(y,x), gy_{2k+1}) \preceq sd(S(y,x), gy) + sd(gy, gy_{2k+1}),$$

taking $n \to \infty$, we have

$$\begin{aligned} d(gx, S(x,y)) \quad &\preceq \quad sa_5 d(S(x,y), gx) + sa_6 d(S(y,x), gy) \\ &+ \quad s^2 a_9 d(S(x,y), gx) + s^2 sa_{10} d(S(y,x), gy). \end{aligned} \tag{3.5}$$

Also since

$$\begin{aligned} d(gy, S(y,x)) &\preceq sd(gy, gy_{2k+2}) + sd(gy_{2k+2}, S(y,x)) \\ &= sd(gy, gy_{2k+2}) + sd(T(y_{2k+1}, gx_{2k+1}), S(y,x)) \end{aligned}$$

by the given contractive condition and the similar calculation, we have

$$\begin{aligned} d(gy, S(y,x)) &\preceq sa_5 d(S(y,x), gy) + sa_6 d(S(x,y), gx) \\ &+ s^2 a_9 d(S(y,x), gy) + s^2 s a_{10} d(S(x,y), gx). \end{aligned} \quad (3.6)$$

Adding both inequalities, we have

$$d(gx, S(x,y)) + d(gy, S(y,x) \preceq (sa_5 + sa_6 + s^2 a_9 + s^2 a_{10})[d(S(x,y), gx) + d(S(y,x), gy)].$$

Since

$$r(sa_5 + sa_6 + s^2 a_9 + s^2 a_{10}) \leq sr(a_5 + a_6) + s^2 r(a_9 + a_{10}) < 1,$$

by Lemma 2.18, we obtain

$$d(gx, S(x,y)) + d(gy, S(y,x)) = \theta$$

and so $gx = S(x,y), gy = S(y,x)$. Similarly we obtain that $gx = T(x,y)$ and $gy = T(y,x)$. Therefore (x,y) is a common coupled coincidence point of S, T and g.

In order to prove the uniqueness, let $(x', y') \in X \times X$ be another common coupled coincidence point of S, T and g. Then by the given contractive condition,

$$\begin{aligned} d(gx, gx') &= d(S(x,y), T(x',y')) \\ &\preceq a_1 d(gx, gx') + a_2 d(gy, gy') + a_3 d(S(x,y), gx) \\ &+ a_4 d(S(y,x), gy) + a_5 d(T(x',y'), gx') + a_6 d(T(y',x'), gy') \\ &+ a_7 d(S(x,y), gx') + a_8 d(S(y,x), gy') \\ &+ a_9 d(T(x',y'), gx) + d(T(y',x'), gy) \\ &= a_1 d(gx, gx') + a_2 d(gy, gy') + a_7 d(gx, gx') \\ &+ a_8 d(gy, gy') + a_9 d(gx', gx) + a_{10} d(gy', gy) \end{aligned}$$

which implies that

$$d(gx, gx') \preceq (a_1 + a_7 + a_9) d(gx, gx') + (a_2 + a_8 + a_{10}) d(gy, gy'). \quad (3.7)$$

Similarly we have

$$d(gy, gy') \preceq (a_1 + a_7 + a_9) d(gy, gy') + (a_2 + a_8 + a_{10}) d(gx, gx'). \quad (3.8)$$

Adding both inequalities, we have

$$\begin{aligned} d(gx, gx') + d(gy, gy') &\preceq (a_1 + a_2 + a_7 + a_8 + a_9 + a_{10})[d(gx, gx') + d(gy, gy')] \\ &\preceq (\sum_{i=1}^{4} a_i + \sum_{i=7}^{10} a_i)[d(gx, gx') + d(gy, gy')] \end{aligned}$$

Since

$$r(\sum_{i=1}^{4} a_i + \sum_{i=7}^{10} a_i) \leq r(\sum_{i=1}^{4} a_i) + r(a_7 + a_8) + r(a_9 + a_{10}) < 1$$

by hyphothesis, by Lemma 2.18, we have

$$d(gx, gx') + d(gy, gy') = \theta.$$

Therefore $gx = gx'$ and $gy = gy'$.

Now, let $gx = u$. Then we have $u = gx = S(x, x) = T(x, x)$. By weak compatibility of S, T and g, we have

$$\begin{aligned} gu &= g(gx) = g(S(x, x)) = S(gx, gx) = S(u, u), \\ gu &= g(gx) = g(T(x, x)) = T(gx, gx) = T(u, u). \end{aligned}$$

Then (gu, gu) is a coupled point of coincidence of S, T and g. Consequently $gu = gx$. Therefore $u = gu = S(u, u) = T(u, u)$. Hence (u, u) is unique common coupled fixed point of S, T and g. □

The above Theorem 3.2 improve Theorem 2.1 of Song [12] by giving the weak radius condition (3.2), that is, Theorem 3.2 implies Theorem 2.1 of Song [12] . But the converse is not valid.

Corollary 3.3. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Suppose that $S : X \times X \to X$, $g : X \to X$ are mappings satisfying the contractive condition*

$$\begin{aligned} d(S(x, y), S(u, v)) &\preceq a_1 d(gx, gu) + a_2 d(gy, gv) \\ &+ a_3 d(S(x, y), gx) + a_4 d(S(y, x), gy) \\ &+ a_5 d(S(u, v), gu) + a_6 d(S(v, u), gv) \\ &+ a_7 d(S(x, y), gu) + a_8 d(S(y, x), gv) \\ &+ a_9 d(S(u, v), gx) + a_{10} d(S(v, u), gy), \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i \in P$ commute for $i = 1, 2, \cdots, 10$ and

$$sr(\sum_{i=1}^{4} a_i) + sr(a_5 + a_6) + r(a_7 + a_8) + (s^2 + s)r(a_9 + a_{10}) < 1.$$

If $S(X\times X)\subseteq g(X)$ and $g(X)$ is a complete subspace of X, then S and g have a common coupled coincidence point in X.

Also, if S and g are weakly compatible, then S and g have a unique coupled fixed point.

Proof. It follows from Theorem 3.2 by taking $S=T$. □

The above Corollary improve Corollary 2.2 of Song [12].

Corollary 3.4. *Let (X,d) be a complete cone b-metric space over Banach algebra A with the coefficient $s\geq 1$ and the underlying solid cone P. Suppose that $S,T: X\times X\to X$ are mappings satisfying the condition*

$$\begin{aligned} d(S(x,y),T(u,v)) &\preceq a_1d(x,u)+a_2d(y,v)\\ &+ a_3d(S(x,y),x)+a_4d(S(y,x),y)\\ &+ a_5d(T(u,v),u)+a_6d(T(v,u),v)\\ &+ a_7d(S(x,y),u)+a_8d(S(y,x),v)\\ &+ a_9d(T(u,v),x)+a_{10}d(T(v,u),y), \end{aligned}$$

for all $x,y,u,v\in X$, where $a_i\in P$ commute for $i=1,2,\cdots,10$ and

$$sr(\sum_{i=1}^{4}a_i)+sr(a_5+a_6)+r(a_7+a_8)+(s^2+s)r(a_9+a_{10})<1.$$

Then S and T have a coupled fixed point in X.

Proof. It follows from Theorem 3.2 by taking $g=I$ the identity mapping. □

The above Corollary improve Corollary 2.2 of Song [12]

Corollary 3.5. *Let (X,d) be a complete cone metric space over Banach algebra A with the underlying solid cone P. Suppose that $S,T: X\times X\to X$, $g: X\to X$ are mappings satisfying the condition*

$$\begin{aligned} d(S(x,y),T(u,v)) &\preceq a_1d(gx,gu)+a_2d(gy,gv)\\ &+ a_3d(S(x,y),gx)+a_4d(S(y,x),gy)\\ &+ a_5d(T(u,v),gu)+a_6d(T(v,u),gv)\\ &+ a_7d(S(x,y),gu)+a_8d(S(y,x),gv)\\ &+ a_9d(T(u,v),gx)+a_{10}d(T(v,u),gy), \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i \in P$ commute for $i = 1, 2, \cdots, 10$ and

$$r(\sum_{i=1}^{4} a_i) + r(a_5 + a_6) + r(a_7 + a_8) + 2r(a_9 + a_{10}) < 1.$$

If $S(X \times X), T(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then S, T and g have a common coupled coincidence point in X, that is, there exist $x, y \in X$ such that $gx = S(x, y) = T(x, y)$ and $gy = S(y, x) = T(y, x)$.

Also, if S, T and g are weakly compatible, then S, T and g have a unique coupled fixed point.

Proof. It follows from Theorem 3.2 by taking $s = 1$. □

The above Corollary improve Corollary 2.3 of Song [12].

Corollary 3.6. *Let (X, d) be a complete metric space. Suppose that $S, T : X \times X \to X$ and $g : X \to X$ are mappings satisfying the condition*

$$\begin{aligned} d(S(x,y), T(u,v)) \quad \preceq \quad & a_1 d(gx, gu) + a_2 d(gy, gv) \\ + \quad & a_3 d(S(x,y), gx) + a_4 d(S(y,x), gy) \\ + \quad & a_5 d(T(u,v), gu) + a_6 d(T(v,u), gv) \\ + \quad & a_7 d(S(x,y), gu) + a_8 d(S(y,x), gv) \\ + \quad & a_9 d(T(u,v), gx) + a_{10} d(T(v,u), gy), \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i \in P$ commute for $i = 1, 2, \cdots, 10$ and

$$\sum_{i=1}^{8} a_i + 2(a_9 + a_{10}) < 1.$$

If $S(X \times X), T(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then S, T and g have a common coupled coincidence point in X, that is, there exist $x, y \in X$ such that $gx = S(x, y) = T(x, y)$ and $gy = S(y, x) = T(y, x)$.

Also, if S, T and g are weakly compatible, then S, T and g have a unique coupled fixed point.

Proof. It follows from Theorem 3.2 by noting that d is a metric space with $A = \mathbb{R}$. □

The above Corollary improve Corollary 2.4 of Song [12].

Corollary 3.7. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Let $S, T : X \times X \to X$ and*

$g : X \to X$ *be mappings satisfying*

$$d(S(x,y),T(u,v)) \preceq a_1 d(gx,gu) + a_2 d(gy,gv)$$

for all $x,y,u,v \in X$, *where* $a_1, a_2 \in P$ *commute and* $r(a_1+a_2) < \frac{1}{s}$. *If* $S(X\times X), T(X\times X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X, *then* S,T *and* g *have a coupled coincidence point, that is, there exist* $x,y \in X$ *such that* $gx = S(x,y) = T(x,y)$ *and* $gy = S(y,x) = T(y,x)$.

Also, if S,T *and* g *are weakly compatible, then* S,T *and* g *have a unique coupled fixed point.*

Proof. It follows from Theorem 3.2 by taking $a_i = \theta$ $(i = 3,4,\cdots,10)$. □

The above Corollary improve Corollary 2.5 of Song [12].

Corollary 3.8. *Let* (X,d) *be a complete cone b-metric space over Banach algebra* A *with the coefficient* $s \geq 1$ *and the underlying solid cone* P. *Let* $S,T : X\times X \to X$ *and* $g : X \to X$ *be mappings satisfying*

$$\begin{aligned} d(S(x,y),T(u,v)) \preceq\ & k_1 d(gx,gu) + k_2 d(gy,gv) \\ +\ & k_3 d(S(x,y),gx) + k_4 d(T(u,v),gu) \\ +\ & k_5 d(S(x,y),gu) + k_6 d(T(u,v),gx) \end{aligned}$$

for all $x,y,u,v \in X$, *where* $k_i \in P$ *commute for* $i = 1,2,\cdots,6$ *and*

$$sr(k_1+k_2+k_3) + sr(k_4) + r(k_5) + (s^2+s)r(k_6) < 1.$$

If $S(X\times X), T(X\times X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X, *then* S,T *and* g *have a coupled coincidence point, that is, there exist* $x,y \in X$ *such that* $gx = S(x,y) = T(x,y)$ *and* $gy = S(y,x) = T(y,x)$. *Also, if* S,T *and* g *are weakly compatible, then* S,T *and* g *have a unique coupled fixed point.*

Proof. It follows from Theorem 3.2 by taking $k_1 = a_1, k_2 = a_2$, $k_3 = a_3$, $k_4 = a_5, k_5 = a_7$, $k_6 = a_9$ and $a_4 = a_6 = a_8 = a_{10} = \theta$. □

The above Corollary is a generalization of the main result obtained by Song [12] and Yang ([19], [20]).

Corollary 3.9. *Let* (X,d) *be a complete cone b-metric space over Banach algebra* A *with the coefficient* $s \geq 1$ *and the underlying solid cone* P. *Let* $S,T : X\times X \to X$ *be* *two mappings satisfying*

$$\begin{aligned} d(S(x,y),T(u,v)) \quad &\preceq \quad k_1 d(x,u) + k_2 d(y,v) \\ &+ \quad k_3 d(S(x,y),x) + k_4 d(T(u,v),u) \\ &+ \quad k_5 d(S(x,y),u) + k_6 d(T(u,v),x) \end{aligned}$$

for all $x, y, u, v \in X$, where $k_i \in P$ commute for $i = 1, 2, \cdots, 6$ and

$$sr(k_1 + k_2 + k_3) + sr(k_4) + r(k_5) + (s^2 + s)r(k_6) < 1.$$

Then S, T have a common coupled fixed point.

Proof. It follows from Corollary 3.8 by taking $g = I$. □

Corollary 3.10. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Let $S, T : X \times X \to X$ and $g : X \to X$ be mappings satisfying*

$$\begin{aligned} d(S(x,y),T(u,v)) \quad &\preceq \quad a_1 d(gx,gu) + a_2 d(gy,gv) \\ &+ \quad a_3[d(S(x,y),gx) + d(T(u,v),gu)] \\ &+ \quad a_4[d(S(x,y),gu) + d(T(u,v),gx)] \end{aligned}$$

for all $x, y, u, v \in X$, where $a_1, a_2, a_3, a_4 \in P$ commute and either

$$(1) \qquad sr(a_1 + a_2 + a_3) + sr(a_3) + (s^2 + s + 1)r(a_4) < 1$$

or

$$(2) \qquad sr(a_1 + a_2) + (s + 1)r(a_3 + sa_4) < 1.$$

If $S(X \times X), T(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then S, T and g have a coupled coincidence point, that is, there exist $x, y \in X$ such that $gx = S(x, y) = T(x, y)$ and $gy = S(y, x) = T(y, x)$.

Also, if S, T and g are weakly compatible, then S, T and g have a unique coupled fixed point.

Proof. Assume that (1) holds. It follows from Theorem 3.2 by taking $a_3 = a_5$, $a_7 = a_9$ and $a_4 = a_6 = a_8 = a_{10} = \theta$

Assume that (2) holds. Let x_0, y_0 be two arbitrary elements in X. Since $S(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = S(x_0, y_0)$ and $gy_1 = S(y_0, x_0)$. Again noting $T(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = T(x_1, y_1)$ and

$gy_2 = T(y_1, x_1)$. Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{2n+1} = S(x_{2n}, \quad y_{2n}), gy_{2n+1} = S(y_{2n}, \quad x_{2n}), \quad gx_{2n+2} = T(x_{2n+1}, y_{2n+1})$$

and $gy_{2n+2} = T(y_{2n+1}, x_{2n+1})$.

For each $n \in \mathbb{N}$, by the given conditions, we have

$$\begin{aligned} d(gx_{2n+1}, gx_{2n+2}) &= d(S(x_{2n}, y_{2n}), T(x_{2n+1}, y_{2n+1})) \\ &\preceq a_1 d(gx_{2n}, gx_{2n+1}) + a_2 d(gy_{2n}, gy_{2n+1}) \\ &+ a_3(d(gx_{2n+1}, gx_{2n}) + d(gx_{2n+2}, gx_{2n+1})) \\ &+ a_4(d(gx_{2n+1}, gx_{2n+1}) + d(gx_{2n+2}, gx_{2n})). \end{aligned}$$

Since $d(gx_{2n+2}, gx_{2n}) \preceq sd(gx_{2n+2}, gx_{2n+1}) + sd(gx_{2n+1}, gx_{2n})$,

$$\begin{aligned} (e - a_3 - sa_4)d(gx_{2n+1}, gx_{2n+2}) &\preceq (a_1 + a_3 + sa_4)d(gx_{2n}, gx_{2n+1}) \\ &+ a_2 d(gy_{2n}, gy_{2n+1}). \end{aligned} \tag{3.9}$$

Similarly, we get

$$\begin{aligned} (e - a_3 - sa_4)d(gy_{2n+1}, gy_{2n+2}) &\preceq (a_1 + a_3 + sa_4)d(gy_{2n}, gy_{2n+1}) \\ &+ a_2 d(gx_{2n}, gx_{2n+1}). \end{aligned} \tag{3.10}$$

By the given condition, we have $r(a_3 + sa_4) < 1$ and so by Lemma 2.18, $e - a_3 - sa_4$ is invertible. Let

$$\lambda = (e - a_3 - sa_4)^{-1}(a_1 + a_2 + a_3 + sa_4).$$

From the inequalities (3.9) and (3.10), we obtain that

$$d(gx_{2n+1}, gx_{2n+2}) + d(gy_{2n+1}, gy_{2n+2}) \preceq \lambda[d(gx_{2n}, gx_{2n+1}) + d(gy_{2n}, gy_{2n+1})]. \tag{3.11}$$

On the other hand, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} d(gx_{2n+1}, gx_{2n}) &= d(S(x_{2n}, y_{2n}), T(x_{2n-1}, y_{2n-1})) \\ &\preceq a_1 d(gx_{2n}, gx_{2n-1}) + a_2 d(gy_{2n}, gy_{2n-1}) \\ &+ a_3(d(gx_{2n+1}, gx_{2n}) + d(gx_{2n}, gx_{2n-1})) \\ &+ a_4(d(gx_{2n+1}, gx_{2n-1}) + d(gx_{2n}, gx_{2n})), \end{aligned}$$

which implies that

$$\begin{aligned} (e - a_3 - sa_4)d(gx_{2n}, gx_{2n+1}) &\preceq (a_1 + a_3 + sa_4)d(gx_{2n}, gx_{2n-1}) \\ &+ a_2 d(gy_{2n}, gy_{2n-1}). \end{aligned} \tag{3.12}$$

By the similar arguments as above, we can get

$$\begin{aligned}(e-a_3-sa_4)d(gy_{2n},gy_{2n+1}) &\preceq (a_1+a_3+sa_4)d(gy_{2n},gy_{2n-1})\\ &+ a_2d(gx_{2n},gx_{2n-1}).\end{aligned} \tag{3.13}$$

Adding the inequalities (3.12) and (3.13), we get

$$\begin{aligned}d(gx_{2n},gx_{2n+1})+d(gy_{2n},gy_{2n+1}) &\preceq \lambda(d(gx_{2n},gx_{2n-1})\\ &+ d(gy_{2n},gy_{2n-1})).\end{aligned} \tag{3.14}$$

Then the inequality (3.11) together with (3.14) implies that

$$\begin{aligned}d(gx_{2n+1},gx_{2n+2})+d(gy_{2n+1},gy_{2n+2}) &\preceq \lambda(d(gx_{2n},gx_{2n+1})+d(gy_{2n},gy_{2n+1}))\\ &\preceq \lambda^2(d(gx_{2n},gx_{2n-1})+d(gy_{2n},gy_{2n-1}))\\ &\vdots\\ &\preceq \lambda^{2n+1}(d(gx_0,gx_1)+d(gy_0,gy_1)).\end{aligned}$$

Let $\{w_n\}_{n=0}^{\infty}=(gx_0,gx_1,gx_2,\cdots)$ and $\{z_n\}_{n=0}^{\infty}=(gy_0,gy_1,gy_2,\cdots)$. Then for $n\in\mathbb{N}$, we have

$$d(w_n,w_{n+1})+d(z_n,z_{n+1})\preceq\lambda^n(d(w_0,w_1)+d(z_0,z_1)). \tag{3.15}$$

We need only to consider the following two cases:

Case 1: $d(w_0,w_1)+d(z_0,z_1)=\theta$. This case yields that $w_0=w_1$ and $z_0=z_1$. By the formula (3.15), we get that $w_0=w_n$ and $z_0=z_n$ for each $n\in\mathbb{N}$. Hence

$$gx_0=gx_1=S(x_0,y_0)\quad\text{and}\quad gy_0=gy_1=S(y_0,x_0).$$

Now we show that $T(x_0,y_0)=gx_0$ and $T(y_0,x_0)=gy_0$. For that, we have

$$\begin{aligned}d(gx_0,T(x_0,y_0)) &= d(S(x_0,y_0),T(x_0,y_0))\\ &\preceq a_1d(gx_0,gx_0)+a_2d(gy_0,gy_0)+a_3[d(gx_0,gx_0)+d(T(x_0,y_0),gx_0)]\\ &+ a_4(d(gx_0,gx_0)+d(T(x_0,y_0),gx_0)).\end{aligned}$$

Hence

$$\begin{aligned}d(gx_0,T(x_0,y_0)) &\preceq (a_3+a_4)d(gx_0,T(x_0,y_0))\\ &\preceq (a_3+sa_4)d(gx_0,T(x_0,y_0)).\end{aligned}$$

Since $r(a_3+sa_4)<1$ by the given condition, $d(gx_0,T(x_0,y_0))=\theta$ by Lemma 2.18(4) and so $gx_0=T(x_0,y_0)$. Similarly, we can show that $gy_0=T(y_0,x_0)$. Therefore we get that (x_0,y_0) is a common coupled coincidence point of S,T and g.

Case 2: $d(w_0, w_1) + d(z_0, z_1) \neq \theta$. Indeed let $m > n$, then

$$
\begin{aligned}
d(w_n, w_m) &\preceq s[d(w_n, w_{n+1}) + d(w_{n+1}, w_m)] \\
&\preceq sd(w_n, w_{n+1}) + s^2[d(w_{n+1}, w_{n+2}) + d(w_{n+2}, w_m)] \\
&\vdots \\
&\preceq sd(w_n, w_{n+1}) + s^2 d(w_{n+1}, w_{n+2}) + \cdots + s^{m-n} d(w_{m-1}, w_m).
\end{aligned}
$$

Similarly we have

$$
d(z_n, z_m) \preceq sd(z_n, z_{n+1}) + s^2 d(z_{n+1}, z_{n+2}) + \cdots + s^{m-n} d(z_{m-1}, z_m).
$$

In order to prove the following conclusion, we firstly verify the fact that $r(s\lambda) < 1$. In fact since $sr(a_1 + a_2) + (s+1)r(a_3 + sa_4) < 1$, then $r(a_3 + sa_4) < 1$ which together with Lemma 2.18 implies that $(e - a_3 - sa_4)^{-1}$ exists. Then from Lemma 2.18 again,

$$
\begin{aligned}
r(s\lambda) = sr(\lambda) &= sr((e - a_3 - sa_4)^{-1}(a_1 + a_2 + a_3 + a_4)) \\
&\leq sr(e - a_3 - sa_4)^{-1} r(a_1 + a_2 + a_3 + sa_4) \\
&\leq \frac{sr(a_1 + sa_2) + sr(a_3 + sa_4)}{1 - r(a_3 + sa_4)} \\
&< 1.
\end{aligned}
$$

By (3.15) and the fact of $r(s\lambda) < 1$, we have

$$
d(w_n, w_m) + d(z_n, z_m) \preceq s\lambda^n (e - s\lambda)^{-1}(d(w_0, w_1) + d(z_0, z_1)).
$$

Since $r(\lambda) < \frac{1}{s} \leq 1$, by Lemma 2.18(2) and Lemma 2.19, $\|\lambda^n\| \to 0$ as $n \to \infty$ and so $\{\lambda^n\}$ is a c-sequence. Since $(e - s\lambda)^{-1} \in P$, for each $c \in P$ with $\theta \ll c$, we can find a sufficiently large natural number $k \in \mathbb{N}$ such that

$$
s\lambda^n (e - s\lambda)^{-1}(d(w_0, w_1) + d(z_0, z_1)) \ll c,
$$

which gives, for all $n \geq k$,

$$
d(w_n, w_m) + d(z_n, z_m) \ll c.
$$

So $\{w_n\}$ and $\{z_n\}$ are Cauchy sequences in $g(X)$. As $g(X)$ is complete, there exist x, y in X such that $w_n = gx_n \to gx$ and $z_n = gy_n \to gy$ as $n \to \infty$. These give that

$$
gx_{2n+1} \to gx, \quad gx_{2n} \to gx, \quad gy_{2n+1} \to gy \quad \text{and} \quad gy_{2n} \to gy
$$

as $n \to \infty$.

Now we prove that $S(x, y) = T(x, y) = gx$ and $S(y, x) = T(y, x) = gy$. Clearly,

$$
d(S(x, y), gx) \preceq s[d(S(x, y), gx_{2n+2}) + d(gx_{2n+2}, gx)]. \tag{3.16}
$$

So the given conditions yield that

$$\begin{aligned} d(S(x,y), gx_{2n+2}) &= d(S(x,y), T(x_{2n+1}, y_{2n+1})) \\ &\preceq a_1 d(gx, gx_{2n+1}) + a_2 d(gy, gy_{2n+1}) \\ &+ a_3(d(S(x,y), gx) + d(gx_{2n+2}, gx_{2n+1})) \\ &+ a_4(d(S(x,y), gx_{2n+1}) + d(gx_{2n+2}, gx)). \end{aligned}$$

Then the formula (3.16) turns to

$$\begin{aligned} (e - sa_3 - s^2a_4)d(S(x,y), gx) &\preceq (sa_1 + s^2a_3 + s^2a_4)d(gx_{2n+1}, gx) \\ &+ (se + s^2a_3 + sa_4)d(gx, gx_{2n+2}) + sa_2 d(gy, gy_{2n+1}). \end{aligned}$$

Since $(s+1)r(a_3 + sa_4) < 1$, we have $sr(a_3 + sa_4) < 1$ and so $e - sa_3 - s^2a_4$ is invertible. Thus

$$d(S(x,y), gx) \preceq \lambda_1 d(gx_{2n+2}, gx) + \lambda_2 d(gx, gx_{2n+1}) + \lambda_3 d(gy, gy_{2n+1})$$

where

$$\begin{aligned} \lambda_1 &= (e - sa_3 - s^2a_4)^{-1}(se + s^2a_3 + sa_4), \\ \lambda_2 &= (e - sa_3 - s^2a_4)^{-1}(sa_1 + s^2a_3 + s^2a_4), \\ \lambda_3 &= (e - sa_3 - s^2a_4)^{-1}sa_2. \end{aligned}$$

Since $gx_{2n+2} \to gx$, $gx_{2n+1} \to gx$ and $gy_{2n+1} \to gy$ as $n \to \infty$, by Lemma 2.19,

$$\{d(gx_{2n+2}, gx)\}, \quad \{d(gx, gx_{2n+1})\} \quad \text{and} \quad \{d(gy, gy_{2n+1})\}$$

are c-sequences. Since $\lambda_1, \lambda_2, \lambda_3 \in P$,

$$\{\lambda_1 d(gx_{2n+2}, gx)\}, \quad \{\lambda_2 d(gx, gx_{2n+1})\} \quad \text{and} \quad \{\lambda_3 d(gy, gy_{2n+1})\}$$

are c-sequences by Lemma 2.19. Thus for $c \gg \theta$ there is $N_0 \in \mathbb{N}$ such that

$$\lambda_1 d(gx_{2n+2}, gx) \ll \frac{c}{3}, \quad \lambda_2 d(gx_{2n+1}, gx) \ll \frac{c}{3} \quad \text{and} \quad \lambda_3 d(gy_{2n+2}, gy) \ll \frac{c}{3}$$

for all $n \geq N_0$. So $d(S(x,y), gx) \ll c$, that is, $S(x,y) = gx$. By the similar arguments as above and the following inequality

$$\begin{aligned} d(gx, T(x,y)) &\preceq s[d(gx, gx_{2n+1}) + d(gx_{2n+1}, T(x,y))] \\ &= sd(gx, gx_{2n+1}) + sd(S(x_{2n}, y_{2n}), T(x,y)), \end{aligned}$$

we get $T(x,y) = gx$. Hence $S(x,y) = T(x,y) = gx$. Similarly we can get $S(y,x) = T(y,x) = gy$. Therefore (x,y) is a common coupled coincidence point of S, T and g. □

Remark 3.11. The above Corollary implies that there is a difference between the radius condition (1) obtained from Theorem 3.2 and the radius condition (2) obtained from the direct proof

Corollary 3.12. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Let $S, T : X \times X \to X$ and $g : X \to X$ be mappings satisfying*

$$\begin{aligned} d(S(x,y), T(u,v)) &\preceq k_1 d(gx, gu) + k_2 d(gy, gv) \\ &+ k_3 d(S(y,x), gx) + k_4 d(T(v,u), gv) \\ &+ k_5 d(S(y,x), gv) + k_6 d(T(v,u), gy) \end{aligned}$$

for all $x, y, u, v \in X$, where $k_i \in P$ commute for $i = 1, 2, \cdots, 6$ and

$$sr(k_1 + k_2 + k_3) + sr(k_4) + r(k_5) + (s^2 + s)r(k_6) < 1.$$

If $S(X \times X), T(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then S, T and g have a coupled coincidence point, that is, there exist $x, y \in X$ such that $gx = S(x, y) = T(x, y)$ and $gy = S(y, x) = T(y, x)$. Also, if S, T and g are weakly compatible, then S, T and g have a unique coupled fixed point.

Proof. It follows from Theorem 3.2 by taking $k_1 = a_1, k_2 = a_2$, $k_3 = a_4$, $k_4 = a_6, k_5 = a_8$, $k_6 = a_{10}$ and $a_3 = a_5 = a_7 = a_9 = \theta$.

□

Corollary 3.13. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Let $S, T : X \times X \to X$ and $g : X \to X$ be mappings satisfying*

$$\begin{aligned} d(S(x,y), T(u,v)) &\preceq k_1 d(gx, gu) + k_2 d(gy, gv) \\ &+ k_3[d(S(y,x), gy) + d(T(v,u), gv)] \\ &+ k_4[d(S(y,x), gv) + d(T(v,u), gy)] \end{aligned}$$

for all $x, y, u, v \in X$, where $k_i \in P$ commute for $i = 1, 2, 3, 4$ and

$$sr(k_1 + k_2 + k_3) + sr(k_3) + (s^2 + s + 1)r(k_4) < 1.$$

If $S(X \times X), T(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then S, T and g have a coupled coincidence point, that is, there exist $x, y \in X$ such that $gx = S(x, y) = T(x, y)$ and $gy = S(y, x) = T(y, x)$.

Also, if S, T and g are weakly compatible, then S, T and g have a unique coupled fixed point.

Proof. It follows from Theorem 3.2 by taking $k_1 = a_1, k_2 = a_2$, $k_3 = a_4 = a_6, k_4 = a_8 = a_{10}$ and $a_3 = a_5 = a_7 = a_9 = \theta$ □

Corollary 3.14. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Let $S, T : X \times X \to X$ and $g : X \to X$ be mappings satisfying*

$$\begin{aligned} d(S(x,y), T(u,v)) &\preceq k_1 d(gx, gu) + k_2 d(gy, gv) \\ &+ k_3[d(S(x,y), gx) + d(S(y,x), gy)] \\ &+ k_4[d(T(u,v), gu) + d(T(v,u), gv)] \\ &+ k_5[d(S(x,y), gu) + d(S(y,x), gv)] \\ &+ k_6[d(T(u,v), gx) + d(T(v,u), gy)] \end{aligned}$$

for all $x, y, u, v \in X$, where $k_i \in P$ commute for $i = 1, 2, \cdots, 6$ and

$$sr(k_1 + k_2 + 2k_3) + 2sr(k_4) + 2r(k_5) + 2(s^2 + s)r(k_6) < 1.$$

If $S(X \times X), T(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then S, T and g have a coupled coincidence point, that is, there exist $x, y \in X$ such that $gx = S(x, y) = T(x, y)$ and $gy = S(y, x) = T(y, x)$.

Also, if S, T and g are weakly compatible, then S, T and g have a unique coupled fixed point.

Proof. It follows from Theorem 3.2 by taking $k_1 = a_1, k_2 = a_2$, $k_3 = a_3 = a_4, k_4 = a_5 = a_6, k_5 = a_7 = a_8$, and $k_6 = a_9 = a_{10}$.

□

Corollary 3.15. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Let $S : X \times X \to X$ and $g : X \to X$ be mappings satisfying*

$$\begin{aligned} d(S(x,y), S(u,v)) &\preceq k_1 d(gx, gu) + k_2 d(gy, gv) \\ &+ k_3 d(S(x,y), gx) + k_4 d(S(u,v), gu) \\ &+ k_5 d(S(x,y), gu) + k_6 d(S(u,v), gx) \end{aligned}$$

for all $x, y, u, v \in X$, where $k_i \in P$ commute for $i = 1, 2, \cdots, 6$ and

$$sr(k_1 + k_2 + k_3) + sr(k_4) + r(k_5) + (s^2 + s)r(k_6) < 1.$$

If $S(X \times X), T(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then S and g have a coupled coincidence point.

Also, if S and g are weakly compatible, then S and g have a unique coupled fixed point.

Proof. It follows from Corollary 3.8 by taking $S = T$. $\square$

Remark 3.16. (1) Taking $S = T$ in Corollary 3.4 $-$ 3.14, we obtain several corresponding results.

(2) Taking $s = 1$ in Corollary 3.6 $-$ 3.14, we obtain several corresponding results.

(3) Corollary 3.12- 3.15 are generalizations of results obtained by Song [12] and Yang ([19], [20]).

Example 3.17. Let $A = \mathbb{R}^2$ and define a norm on A by $\|(x_1, x_2)\| = |x_1| + |x_2|$ for $x = (x_1, x_2) \in A$. Define the multiplication in A by

$$(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2).$$

Put $P = \{x = (x_1, x_2) \in A : x_1, x_2 \geq 0\}$. Then P is a normal cone and A is a real Banach algebra with unit $e = (1, 1)$. Let $X = [0, \infty)$. Define a mapping $d : X \times X \to A$ by $d(x, y) = (|x-y|^2, |x-y|^2)$ for each $x, y \in X$. Then (X, d) is a complete cone b-metric space over Banach algebra with the coefficient $s = 2$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality.

Consider the mappings $S : X \times X \to X$ and $g : X \to X$ defined by

$$S(x, y) = x + \frac{|\sin y|}{2} \quad \text{and} \quad g(x) = 3x.$$

Then $S(X \times X) \subseteq g(X) = X$. Let $a_i \in P$ be defined with

$$\begin{aligned}
a_1 &= (\frac{2}{9}, \frac{2}{9}),\ a_2 = (\frac{1}{18}, \frac{1}{18}),\ a_3 = (\frac{1}{108}, \frac{1}{108}),\\
a_4 &= (\frac{1}{109}, \frac{1}{109}),\ a_5 = (\frac{1}{54}, \frac{1}{54}),\ a_6 = (\frac{1}{55}, \frac{1}{55}),\\
a_7 &= (\frac{1}{12}, \frac{1}{12}),\ a_8 = (\frac{1}{13}, \frac{1}{13}),\ a_9 = (\frac{1}{84}, \frac{1}{84}),\ a_{10} = (\frac{1}{85}, \frac{1}{85}).
\end{aligned}$$

Then, by definition of spectral radius,

$$\begin{aligned} r(a_1+a_2+a_3+a_4) &= \frac{3,487}{11,772}, \ \ r(a_5+a_6)=\frac{109}{2,970}, \\ r(a_7+a_8) &= \frac{25}{156}, \quad r(a_9+a_{10})=\frac{169}{7,140}, \end{aligned}$$

and so

$$sr(\sum_{i=1}^{4} a_i)+sr(a_5+a_6)+r(a_7+a_8)+(s^2+s)r(a_9+a_{10})=0.9680<1.$$

By careful calculations, it is easy to verify that for any $x, y, u, v \in X$, S and g satisfy the contractive condition of Theorem 3.2. Thus by Theorem 3.2, S and g have a coupled coincidence point in a complete cone b-metric space X over Banach algebra $A=\mathbb{R}^2$. Since $S(0,0)=g0=0$, $(0,0)$ is the common coupled coincidence point of S and g.

Example 3.18. Let $A=\mathbb{R}, P=[0,\infty)$ and $X=[0,\infty)$ or $[0,1]$. Define $d: X\times X\to P$ by $d(x,y)=|x-y|^2$. Then (X,d) is a complete cone b-metric space over $A=\mathbb{R}$ with the coefficient $s=2$. Let

$$S(x,y)=\frac{x+3y}{9}, \ \ g=i_X \quad \text{and} \quad a_1=\frac{2}{9^2}, \ a_2=\frac{2}{9}.$$

Then $r(a_1)=\frac{2}{81}, r(a_2)=\frac{2}{9}$ and so $r(a_1)+r(a_2)=\frac{19}{81}<\frac{1}{2}=\frac{1}{s}$. By Corollary 3.7, S has a unique coupled fixed point $(0,0)$.

4. **Coupled fixed point results for three self-maps**

Many researchers proved the existences of coupled coincidence point and coupled fixed point for two self-mappings $f, h : X \to X$ in a cone metric space (X, d) satisfying the contractive conditions with at most six terms of (4.3)([1], [2], [3], [4], [10], [15], etc). Recently many researchers proved the existences of coupled coincidence point and coupled fixed point for three self-mappings $f, h, g : X \to X$ in a cone metric space (X, d) over Banach algebra satisfying the contractive conditions with at most six terms of (4.3)([6], [8], [14], etc).

In this section, by applying Theorem 3.2(not giving direct proof), we shall prove some common coupled coincidence point results and coupled fixed point results for three self-mappings $f, h, g : X \to X$ satisfying the generalized contractive condition with ten terms in a cone b-metric space (X, d) over Banach algebra without assumption of normality. Our main results generalize the results of Yang([19], [20]), and Song [12] by giving the weak radius condition (4.2),

Theorem 4.1. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Suppose $f, h, g : X \to X$ are three mappings satisfying the contractive condition*

$$\begin{aligned} d(fx, hu) \quad &\preceq \quad a_1 d(gx, gu) + a_2 d(gy, gv) \\ &+ \quad a_3 d(fx, gx) + a_4 d(fy, gy) \\ &+ \quad a_5 d(hu, gu) + a_6 d(hv, gv) \\ &+ \quad a_7 d(fx, gu) + a_8 d(fy, gv) \\ &+ \quad a_9 d(hu, gx) + a_{10} d(hv, gy), \end{aligned} \tag{4.1}$$

for all $x, y, u, v \in X$, where $a_i \in P$ commute for $i = 1, 2, \cdots, 10$ and

$$sr(\sum_{i=1}^{4} a_i) + sr(a_5 + a_6) + r(a_7 + a_8) + (s^2 + s)r(a_9 + a_{10}) < 1. \tag{4.2}$$

If $f(X), h(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f, h and g have a common coupled coincidence point in X, that is, there exist $x \in X$ such that $gx = fx = hx$.

Also, if f, h and g are weakly compatible, then f, h and g have a unique coupled fixed point.

Proof. Let $f, h : X \to X$ be mappings satisfying the hypotheses. Define the mapping $S, T : X \times X \to X$ by

$$S(x,y) = fx, \quad T(x,y) = hx, \quad x, y \in X.$$

From (4.1), we get

$$\begin{aligned} d(S(x,y), T(u,v)) \quad &\preceq \quad a_1 d(gx, gu) + a_2 d(gy, gv) \\ &+ \quad a_3 d(S(x,y), gx) + a_4 d(S(y,x), gy) \\ &+ \quad a_5 d(T(u,v), gu) + a_6 d(T(v,u), gv) \\ &+ \quad a_7 d(S(x,y), gu) + a_8 d(S(y,x), gv) \\ &+ \quad a_9 d(T(u,v), gx) + a_{10} d(T(v,u), gy), \end{aligned}$$

for all $x, y, u, v \in X$. Thus the contractive condition (3.1) is satisfied.

On the other hand, from the definition of S, T, we have $S(X \times X) = f(X) \subseteq g(X)$ and $T(X \times X) = h(X) \subseteq g(X)$. Also, $g(X)$ is a complete subspace of (X, d). Now, applying Theorem 3.1, we obtain that S, T and g have a coupled coincidence point in X, that is, there exists $(x, y) \in X \times X$ such that $gx = S(x,y) = T(x,y)$ and $gy = S(y,x) = T(y,x)$. From the definition of S and T, this implies that $gx = fx = hx$, that is, x is a common coincidence point of f, h and g.

If f, h and g are weakly compatible, then S, T and g are weakly compatible. By Theorem 3.2, S, T and g have a unique common coupled fixed point and so f, h and g have a unique common coupled fixed point. □

Corollary 4.2. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Suppose $f, g : X \to X$ are two mappings satisfying the contractive condition*

$$\begin{aligned} d(fx, fu) \quad &\preceq \quad a_1 d(gx, gu) + a_2 d(gy, gv) \\ &+ \quad a_3 d(fx, gx) + a_4 d(fy, gy) \\ &+ \quad a_5 d(fu, gu) + a_6 d(fv, gv) \\ &+ \quad a_7 d(fx, gu) + a_8 d(fy, gv) \\ &+ \quad a_9 d(fu, gx) + a_{10} d(fv, gy), \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i \in P$ commute for $i = 1, 2, \cdots, 10$ and

$$sr(\sum_{i=1}^{4} a_i) + sr(a_5 + a_6) + r(a_7 + a_8) + (s^2 + s) r(a_9 + a_{10}) < 1.$$

If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a common coupled coincidence point.

Proof. The proof follows by taking $h = f$ in Theorem 4.1. $\square$

Corollary 4.3. *Let (X, d) be a complete cone b-metric space over Banach algebra A and let P be a solid cone in A. Suppose $f, h : X \to X$ are two mappings satisfying the contractive condition*

$$\begin{aligned} d(fx, hu) \quad &\preceq \quad a_1 d(x, u) + a_2 d(y, v) \\ &+ \quad a_3 d(fx, x) + a_4 d(fy, y) \\ &+ \quad a_5 d(hu, u) + a_6 d(hv, v) \\ &+ \quad a_7 d(fx, u) + a_8 d(fy, v) \\ &+ \quad a_9 d(hu, x) + a_{10} d(hv, y), \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i \in P$ commute for $i = 1, 2, \cdots, 10$ and

$$sr(\sum_{i=1}^{4} a_i) + sr(a_5 + a_6) + r(a_7 + a_8) + (s^2 + s) r(a_9 + a_{10}) < 1.$$

If $f(X), h(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f, h and g have a common coupled coincidence point in X, that is, there exist $x \in X$ such that $gx = fx = hx$.

Also, if f, h and g are weakly compatible, then f, h and g have a unique coupled fixed point.

Proof. The proof follows by taking $g = I$ in Theorem 4.1. $\square$

Corollary 4.4. *Let (X, d) be a complete cone metric space over Banach algebra A with the underlying solid cone P. Suppose $f, g, h : X \to X$ are three mappings satisfying the contractive condition*

$$\begin{aligned} d(fx, hu) \quad &\preceq \quad a_1 d(gx, gu) + a_2 d(gy, gv) \\ &+ \quad a_3 d(fx, gx) + a_4 d(fy, gy) \\ &+ \quad a_5 d(hu, gu) + a_6 d(hv, gv) \\ &+ \quad a_7 d(fx, gu) + a_8 d(fy, gv) \\ &+ \quad a_9 d(hu, gx) + a_{10} d(hv, gy), \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i \in P$ commute for $i = 1, 2, \cdots, 10$ and

$$r(\sum_{i=1}^{4} a_i) + r(a_5 + a_6) + r(a_7 + a_8) + 2r(a_9 + a_{10}) < 1.$$

If $f(X), h(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f, h and g have a common coupled coincidence point, that is, there exist $x \in X$ such that $gx = fx = hx$.

Also, if f, h and g are weakly compatible, then f, h and g have a unique coupled fixed point.

Proof. The proof follows by taking $s = 1$ in Theorem 4.1. □

Corollary 4.5. *Let (X, d) be a complete metric space. Suppose $f, g, h : X \to X$ are three mappings satisfying the contractive condition*

$$\begin{aligned} d(fx, hu) \quad \preceq \quad & a_1 d(gx, gu) + a_2 d(gy, gv) \\ + \quad & a_3 d(fx, gx) + a_4 d(fy, gy) \\ + \quad & a_5 d(hu, gu) + a_6 d(hv, gv) \\ + \quad & a_7 d(fx, gu) + a_8 d(fy, gv) \\ + \quad & a_9 d(hu, gx) + a_{10} d(hv, gy), \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i \in P$ commute for $i = 1, 2, \cdots, 10$ and

$$\sum_{i=1}^{8} a_i + 2(a_9 + a_{10}) < 1.$$

If $f(X), h(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f, h and g have a common coupled coincidence point, that is, there exist $x \in X$ such that $gx = fx = hx$.

Also, if f, h and g are weakly compatible, then f, h and g have a unique coupled fixed point.

Proof. The proof follows by taking $A = \mathbb{R}$ in Corollary 4.4. □

The following Corollary is a generalization of Theorem 2.1 of Liu et al [8] or Theorem 3.1 of Xu et al [14].

Corollary 4.6. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Suppose $f, g, h : X \times X \to X$ are three mappings satisfying the contractive condition*

$$d(fx, hy) \quad \preceq \quad kd(gx, gy)$$

for all $x, y \in X$, where $k \in P$ and $r(k) < \frac{1}{s}$. If $f(X), h(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f, h and g have a common coupled coincidence point, that is, there exist $x \in X$ such that $gx = fx = hx$.

Also, if f, h and g are weakly compatible, then f, h and g have a unique coupled fixed point.

Proof. The proof follows by taking $a_1 = k, a_2 = a_3 = \cdots = a_{10} = \theta$ in Theorem 4.1. $\square$

The following Corollary is a generalization of Theorem 2.3 of Liu et al [8] or Theorem 3.3 of Xu et al [14].

Corollary 4.7. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Suppose $f, g, h : X \to X$ are three mappings satisfying the contractive condition*

$$d(fx, hy) \preceq k[d(fx, gx) + d(hy, gy)]$$

for all $x, y \in X$, where $k \in P$ and $r(k) < \frac{1}{2s}$. If $f(X), h(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f, h and g have a common coupled coincidence point.

Also, if f, h and g are weakly compatible, then f, h and g have a unique coupled fixed point.

Proof. The proof follows by taking $a_3 = a_4 = k, a_1 = a_2 = a_5 = \cdots = a_{10} = \theta$, in Theorem 4.1. $\square$

The following Corollary is a generalization of Theorem 2.2 of Liu et al [8] or Theorem 3.2 of Xu et al [14].

Corollary 4.8. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Suppose $f, g, h : X \to X$ are three mappings satisfying the contractive condition*

$$d(fx, hy) \preceq k[d(fx, gy) + d(hy, gx)]$$

for all $x, y \in X$, where $k \in P$ and $(s^2 + s + 1)r(k) < 1$. If $f(X), h(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f, h and g have a common coupled coincidence point, that is, there exist $x \in X$ such that $gx = fx = hx$.

Also, if f, h and g are weakly compatible, then f, h and g have a unique coupled fixed point.

Proof. The proof follows by taking $a_7 = a_9 = k, a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_8 = a_{10} = \theta$ in Theorem 4.1. $\square$

Corollary 4.9. *Let (X,d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Suppose $f : X \to X$ and $g : X \to X$ are two mappings satisfying the contractive condition*

$$\begin{aligned} d(fx,fu) &\preceq k_1 d(gx,gu) + k_2 d(gy,gv) \\ &+ k_3 d(fx,gx) + k_4 d(fu,gu) \\ &+ k_5 d(fx,gu) + k_6 d(fu,gx) \end{aligned}$$

for all $x,y,u,v \in X$, where $a_i \in P$ commute for $i = 1,2,\cdots,6$ and

$$sr(\sum_{i=1}^{3} k_i) + sr(k_4) + r(k_5) + (s^2+s)r(k_6) < 1.$$

If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a common coupled coincidence point.

Also, if f and g are weakly compatible, then f and g have a unique coupled fixed point.

Proof. The proof follows by taking $h = f$, $k_1 = a_1, k_2 = a_2, k_3 = a_3, k_4 = a_5, k_5 = a_7, k_6 = a_9$ and $a_4 = a_6 = a_8 = a_{10} = \theta$ in Theorem 4.1. □

Corollary 4.10. *Let (X,d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Suppose $f,g,h : X \to X$ are three mappings satisfying the contractive condition*

$$\begin{aligned} d(fx,hu) &\preceq k_1 d(gx,gu) + k_2 d(gy,gv) \\ &+ k_3 d(fy,gy) + k_4 d(hv,gv) \\ &+ k_5 d(fy,gv) + k_6 d(hv,gy), \end{aligned} \tag{4.3}$$

for all $x,y,u,v \in X$, where $a_i \in P$ commute for $i = 1,2,\cdots,6$ and

$$sr(\sum_{i=1}^{3} k_i) + sr(k_4) + r(k_5) + (s^2+s)r(k_6) < 1.$$

If $f(X), h(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f,h and g have a common coupled coincidence point in X, that is, there exist $x \in X$ such that $gx = fx = hx$.

Also, if f,h and g are weakly compatible, then f,h and g have a unique coupled fixed point.

Proof. The proof follows by taking $k_1 = a_1, k_2 = a_2, k_3 = a_4, k_4 = a_6, k_5 = a_8, k_6 = a_{10}$ and $a_3 = a_5 = a_7 = a_9 = \theta$ in Theorem 4.1. □

Corollary 4.11. *Let (X,d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Suppose $f,g,h : X \to X$ are three mappings satisfying the contractive condition*

$$\begin{aligned} d(fx,hu) &\preceq a_1 d(gx,gu) + a_2 d(gy,gv) + a_3(d(fx,gx) + d(hu,gu)) \\ &+ a_4(d(fx,gu) + d(hu,gx)) \end{aligned}$$

for all $x,y,u,v \in X$, where $a_i \in P$ commute for $i = 1,2,3,4$ and either

$$(1) \qquad sr(a_1 + a_2 + a_3) + sr(a_3) + (s^2 + s + 1)r(a_4) < 1$$

or

$$(2) \qquad sr(a_1 + a_2) + (s+1)r(a_3 + sa_4) < 1.$$

If $f(X), h(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f,h and g have a common coupled coincidence point, that is, there exist $x \in X$ such that $gx = fx = hx$.

Also, if f,h and g are weakly compatible, then f,h and g have a unique coupled fixed point.

Proof. It follows from Corollary 3.10 and Theorem 4.1. □

Remark 4.12. Taking $h = f$ in Corollary $4.3 - 4.11$, we obtain several corresponding results.

Remark 4.13. Taking $s = 1$ in Corollary $4.6 - 4.11$, we obtain several corresponding results on a complete cone b-metric space (X,d) over Banach algebra A with the underlying solid cone P.

5. **Applications**

In this section, we shall apply the obtained conclusions to deal with the existence and uniqueness of solution for some equations. First of all, we refer to the following coupled equations:

$$\begin{cases} F(x,y) = 0, \\ G(x,y) = 0 \end{cases} \tag{5.1}$$

where $F, G : \mathbb{R}^2 \to \mathbb{R}$ are two mappings.

Theorem 5.1. *If there exists $0 < L < \frac{1}{2}$ such that for all the pairs $(x_1.y_1), (x_2, y_2) \in \mathbb{R}^2$, it satises that*

$$|F(x_1,y_1) - F(x_2,y_2) + x_1 - x_2| \leq \sqrt{L}|x_1 - x_2|$$

$$|G(x_1,y_1) - G(x_2,y_2) + y_1 - y_2| \leq \sqrt{L}|y_1 - y_2|.$$

Then the coupled equation (5.1) has a unique common solution in $\mathbb{R}^2$.

Proof. Let $A = \mathbb{R}^2$ with the norm $\|(u_1, u_2)\| = |u_1| + |u_2|$ and the multiplication by

$$uv = (u_1, u_2)(v_1, v_2) = (u_1v_1, u_2v_2).$$

Let $P = \{u = (u_1, u_2) \in A : u_1, u_2 \geq 0\}$. It is clear that P is a normal cone and A is a Banach algebra with a unit $e = (1, 1)$. Put $X = \mathbb{R}^2$ and define a mapping $d : X \times X \to A$ by

$$d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2|^2, |y_1 - y_2|^2).$$

It is easy to see that (X, d) is a complete cone b-metric space over Banach algebra A with the coefficient $s = 2$, but (X, d) is not a complete cone metric space.

Now define the mappings $S, T : X \to X$ by

$$S(x,y) = (x,y), \quad T(x,y) = (F(x,y) + x, G(x,y) + y).$$

Then

$$\begin{aligned}
d(T(x_1,y_1),T(x_2,y_2)) &= d((F(x_1,y_1)+x_1, G(x_1,y_1)+y_1), \\
&\quad (F(x_2,y_2)+x_2, G(x_2,y_2)+y_2)) \\
&= (|F(x_1,y_1)-F(x_2,y_2)+x_1-x_2|^2, \\
&\quad |G(x_1,y_1)-G(x_2,y_2)+y_1-y_2|^2) \\
&\preceq (L|x_1-x_2|^2, L|y_1-y_2|^2) \\
&= (L,L)(|x_1-x_2|^2, |y_1-y_2|^2) \\
&= (L,L)d(S(x_1,y_1),S(x_2,y_2)).
\end{aligned}$$

Since

$$\|(L,L)^n\|^{1/n} = \|(L^n,L^n)\|^{1/n} = (L^n+L^n)^{1/n} = 2^{1/n}L \to L < \frac{1}{2}$$

as $n \to \infty$, we have $r((L,L)) < \frac{1}{2}$. Now choose $a_1 = (L,L), a_2 = a_3 = \cdots = a_{10} = \theta$, then all conditions of Theorem 4.1 or Corollary 4.6 are satisfied. Hence, by Theorem 4.1 or Corollary 4.6, S and T have a unique common fixed point in X. In other words, the coupled equation (5.1) has a unique common solution in $\mathbb{R}^2$. $\square$

Let $X = \mathbb{R}^2$ and define a mapping $d : X \times X \to A = \mathbb{R}^2$ by

$$d((x_1,y_1),(x_2,y_2)) = (|x_1-x_2|, |y_1-y_2|).$$

Then (X,d) is clearly a complete cone b-metric space over Banach algebra $A = \mathbb{R}^2$ with the coefficient $s = 1$, and so (X,d) is a complete cone metric space over Banach algebra $A = \mathbb{R}^2$.

Using this cone metric(not cone b-metric with $s \neq 1$) and the similar proof of Theorem 5.1, Huang and Radenovic proved the following Theorem:

Theorem 5.2. *([7]) If there exists $0 < L < 1$ such that for all the pairs $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$, it satises that*

$$|F(x_1,y_1)-F(x_2,y_2)+x_1-x_2| \leq L|x_1-x_2|$$

$$|G(x_1,y_1)-G(x_2,y_2)+y_1-y_2| \leq L|y_1-y_2|.$$

Then the coupled equation (5.1) has a unique common solution in $\mathbb{R}^2$.

Secondly, we shall study the existence of solution to a class of system of nonlinear integral equations. We consider the following system of integral equations

$$\begin{cases} x(t) = \int_a^t f(s, x(s))ds, \\ x(t) = \int_a^t x(s)ds. \end{cases} \tag{5.2}$$

where $t \in [a, b]$ and $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

By applying our results, we shall prove the following theorem again.

Theorem 5.3. *([7]) Let $L_p[a, b] = \{x = x(t) : \int_a^b \ |x(t)|^p \ dt < \infty\}$ $(0 < p < 1)$. For (5.2), assume that the following hypotheses hold:*

(i) *if $f(s, x(s)) = x(s)$ for all $s \in [a, b]$, then*

$$f(s, \int_a^s x(w)dw) = \int_a^s f(w, x(w))dw$$

for all $s \in [a, b]$.

(ii) *if there exists a constant $M \in (0, 2^{1-\frac{1}{p}}]$ such that the partial derivative f_y of f with respect to y exists and $|f_y(x, y)| \leq M$ for all the pairs $(x, y) \in [a, b] \times \mathbb{R}$.*

Then the integral equation (5.2) has a unique common solution in $L_p[a, b]$.

Proof. Let $A = \mathbb{R}^2$ with the norm $\|(u_1, v_2)\| = |u_1| + |u_2|$ and the multiplication by

$$uv = (u_1, u_2)(v_1, v_2) = (u_1v_1, u_1v_2 + u_2v_1).$$

Let $P = \{u = (u_1, u_2) \in A : u_1, u_2 \geq 0\}$. It is clear that P is a normal cone and A is a Banach algebra with a unit $e = (1, 0)$. Let $X = L_p[a, b]$. We endow X with the cone b-metric

$$d(x, y) = \Big(\{\int_a^b |x(t) - y(t)|^p dt\}^{\frac{1}{p}}, \{\int_a^b |x(t) - y(t)|^p dt\}^{\frac{1}{p}}\Big)$$

for all $x, y \in X$. It is clear that (X, d) is a complete cone b-metric space over Banach algebra A with the coefficient $s = 2^{\frac{1}{p}-1}$. Define the mappings $S, T : X \to X$ by

$$Sx(t) = \int_a^t x(s)ds, \quad Tx(t) = \int_a^t f(s, x(s))ds$$

for all $t \in [a,b]$. Then the existence of a solution to (5.2) is equivalent to the existence of common fixed point of S and T. Indeed,

$$\begin{aligned} d(Tx,Ty) &= \Bigg(\Big\{\int_a^b \Big|\int_a^t f(s,x(s))ds - \int_a^t f(s,y(s))ds\Big|^p dt\Big\}^{\frac{1}{p}}, \\ &\qquad \Big\{\int_a^b \Big|\int_a^t f(s,x(s))ds - \int_a^t f(s,y(s))ds\Big|^p dt\Big\}^{\frac{1}{p}}\Bigg) \\ &= \Bigg(\Big\{\int_a^b \Big|\int_a^t [f(s,x(s)) - f(s,y(s))]ds\Big|^p dt\Big\}^{\frac{1}{p}}, \\ &\qquad \Big\{\int_b^a \Big|\int_a^t [f(s,x(s)) - f(s,y(s))]ds\Big|^p dt\Big\}^{\frac{1}{p}}\Bigg) \\ &\preceq \Bigg(M\Big\{\int_a^b \Big|\int_a^t [x(s) - y(s)]ds\Big|^p dt\Big\}^{\frac{1}{p}}, M\Big\{\int_a^b \Big|\int_a^t [x(s) - y(s)]ds\Big|^p dt\Big\}^{\frac{1}{p}}\Bigg) \\ &= (M,0)\Bigg(\Big\{\int_a^b |Sx(t) - Sy(t)|^p dt\Big\}^{\frac{1}{p}}, \Big\{\int_a^b |Sx(t) - Sy(t)|^p dt\Big\}^{\frac{1}{p}}\Bigg) \\ &= (M,0)d(Sx,Sy). \end{aligned}$$

Because

$$\|(M,0)^n\|^{\frac{1}{n}} = \|(M^n,0)\|^{\frac{1}{n}} = M \to M < 2^{1-\frac{1}{p}} \ (n \to \infty)$$

which means $r((M,0) < 2^{1-\frac{1}{p}}$. Now choose $a_1 = (M,0)$ and $a_2 = a_3 = \cdots = k_{10} = \theta$. Note that by (i), it is easy to see that the mappings S and T are weakly compatible. Therefore, all conditions of Theorem 4.1 are satisfied. As a result, S and T have a unique common fixed point $x^* \in X$. That is, x^* is the unique common solution of the system of integral equation (5.2). □

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<국문초록>

일반화 축약조건을 만족하는 세 함수의 공통 연계된 일치점 정리

송 미 혜

X를 공집합이 아닌 임의의 집합이고, 함수 $f : X \to X$, $fx = x$일 때, x를 f의 부동점이라 한다. 바나흐(Banach)는 1922년에 완비 거리공간에서 일변수 함수에 대한 부동점 정리를 증명하였다. 이를 바나흐 축약원리(Banach contration principle)라 한다. 그 후 많은 수학자들은 부동점 이론에 대한 연구를 하였다. 부동점 이론은 현대 수학의 가장 영향력 있는 분야이며, 바나흐에 의해 시작된 비선형 해석학의 핵심주제로 여겨진다.

2013년에 Liu와 Xu는 바나흐 대수의 원뿔 거리공간의 개념을 소개하고 이 공간에서 부동점 정리(fixed point theorems)를 얻었다. 그 후에 일부 연구자들이 이런 거리공간에서 어떤 축약사상에 대한 부동점의 존재문제 등을 연구하기 시작하였다. 이론은 많은 논문을 통하여 발전되어왔다.

본 논문에서는 스펙트럼 반지름(spectral radius)의 특성을 사용하여 바나흐 대수 위의 완비 원뿔b-거리공간(complete cone b-metric space) (X,d)에서 일반화된 축약조건을 만족하는 세 개의 함수 $S,T: X\times X\to X$, $g: X\to X$에 대하여 공통 연계된 일치점 결과(common coupled coincidence point results)와 연계된 부동점 결과(coupled fixed point results)가 존재하기 위한 충분조건을 얻는다. 그리고 주요 정리 3.2를 적용하여, 같은 공간 (X,d)에서 더 일반화된 축약조건을 만족하는 세 개의 자기 함수(self-mapping) $f,g,h: X\to X$에 대하여 공통 연계된 일치점(common coupled coincidence point)과 연계된 부동점(coupled fixed point)에 관한 결과들을 증명한다. 이러한 결과들은 바나흐대수 위의 원뿔b-거리공간에서 이미 잘 알려진 비교 가능한 결과들을 개선하고 확장하였을 뿐만 아니라, 바나흐대수 위의 원뿔거리공간에서 이전의 결과들을 융합하고 개선한다.

응용으로서 논문의 결과를 이용하여 방정식의 해의 존재와 유일성을 얻는다.

감사의 글

박사학위라는 벅찬 선물 앞에 감사의 글을 쓰자니, 도움을 주신 많은 분들이 떠오르며 마음이 숙연해집니다.

하나의 큰 열매를 맺기 위해서 얼마나 많은 분들이 제 주위에 있었는지 진심으로 깨닫게 되었습니다.

먼저, 학부시절부터 학문의 길을 제시해 주시고 오늘날까지 한결같은 모습으로 이끌어주신 지도교수님, 양영오교수님께 머리 숙여 감사의 말씀 드립니다.

교수님께서는 박사과정을 시작하는 저에게 큰 용기를 주셨습니다. 그 후 여러 가지 사유로 긴 시간동안 휴학을 하였고, 수료 후에도 또 긴 시간이 흘렀음에도 불구하고 학위 논문을 쓰도록 독려하시며, 제 부족한 논문을 논문답게 만들어주셨습니다.

오늘 이 시간, 박사학위 논문의 끝자락에서 무한한 감사의 마음을 다 담지도 못하는 이 미약한 글로 고마움을 전하는 것이 오히려 죄송스럽습니다.

그리고, 바쁜 일정에도 불구하고 심사를 맡아 주신 고윤희교수님, 정승달교수님, 유상욱교수님 그리고 김성운교수님께도 감사의 말씀 드립니다. 심사위원님들의 세심한 조언 덕분에 좀 더 완성도 높은 논문이 되었습니다.

마지막으로 부족한 딸, 부족한 며느리지만, 항상 믿어주시는 양가 부모님과 곁에서 언제나 든든하게 지원해 주시는 남편 그리고 더 많이 주지 못한 빈 자리를 엄마를 향한 응원으로 채워주는 세 아들 현민, 건, 준. 모두 감사합니다.

저의 박사학위가 아닌 고마운 모든 분들의 박사학위로 여기며, 온 정성을 다해 감사의 마음과 함께 이 논문을 바칩니다.

2018년 12월