A Note on the Nonholonomic Self-Adjoints in V_n

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V_n 공간에서의 Non-holonomic Self-Adjoint에 관한 소고

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Summary

The purpose of the present paper is to study some of the relationships between holonomic and nonholonomic components, and so derive some special prorerties of this frame.

INTRODUCTION

Let V_n be a n-dimensional Riemannian space referred to a real coordinate system X^ν and defined by a fundamental metric tensor $h_{\lambda\mu}$, whose determ inant

$$(1.\ 1)\quad h \overset{\text{def}}{=} \text{Det}\ ((h_{\lambda\mu})) \neq 0.$$

According to (1. 1), there is a unique tensor $h^{\lambda\nu}=h^{\nu\lambda}$ defined by

$$(1.\ 2)\quad h_{\lambda\mu}h^{\lambda\nu} \overset{\text{def}}{=} \delta^{\nu}_{\mu}$$

Let $\underset{i}{e}^{\nu}$, $(i=1,\ 2\cdots\cdots,\ n)$, be a set of n linearly independent vectors.
Then there is a unique reciprocal set of n linearly independent covariant vectors $\overset{i}{e}_{\lambda}$, $(i=1,\ 2,\cdots\cdots,\ n)$, satisfying

$$(1.\ 3)\text{a}\quad \underset{i}{e}^{\nu}\,\overset{i}{e}_{\lambda}=\delta^{\nu}_{\lambda}\ (*)$$

$$(1.\ 3)\text{b}\quad \underset{j}{e}^{\lambda}\,\overset{i}{e}_{\lambda}=\delta^{i}_{j}$$

DEFINITION 1. 1) With the vectors $\underset{i}{e}^{\nu}$ and $\overset{i}{e}_{\lambda}$ a nonholonomic frame of V_n is defined in the following way ; If $T^{\nu}_{\lambda}{}^{\cdots\cdots}_{\cdots\cdots}$ are holonomic components of a tensor, then its nonholonomic components are defined by

$$(1.\ 4)\text{a}\quad T^{i}_{j}{}^{\cdots\cdots}_{\cdots\cdots} \overset{\text{def}}{=} T^{\nu}_{\lambda}{}^{\cdots\cdots}_{\cdots\cdots}\,\overset{i}{e}_{\nu}\,\underset{j}{e}^{\lambda}\cdots\cdots.$$

An easy inspection of (1. 3)a and (1. 4)a show that

$$(1.\ 4)\text{b}\quad T^{\nu}_{\lambda}{}^{\cdots\cdots}_{\cdots\cdots} = T^{i}_{j}{}^{\cdots\cdots}_{\cdots\cdots}\,\underset{i}{e}^{\nu}\,\overset{j}{e}_{\lambda}\cdots\cdots.$$

* Throughout the present paper, ***Greek indices*** are used for the ***holonomic components*** of a tensor, while ***Roman indices*** are used for the ***nonholonomic components*** of a tensor.
Both indices take the values 1, 2,$\cdots\cdots$, n, and follow the summation convention.

PRELIMINARY RESULTS

In the present section, for our further discussions, results obtained in our previous paper will be introduced without proof.

THEOREM 2. 1) The product of two nonholonomic components of $h_{\lambda\mu}$ and $h^{\lambda\nu}$ is kronecker delta.

(2. 1)' $h_{ij}\, h^{ik} = \delta_j^h$

THEOREM 2. 2) We have

(2. 2) $\underset{i}{e}^{\nu} = \overset{j}{e}\, h_{ij} h^{\lambda\nu}, \quad \overset{j}{e}_{\lambda} = \underset{i}{e}^{\nu} h^{ij}\, h_{\lambda\nu}.$

The nonholonomic frame in V_n constructed by the unit vectors $\underset{i}{e}^{\nu}$ tangent to the n congruences of an orthogonal ennuple, will be termed an orthogonal nonholonomic frame of V_n.

THEOREM 2. 3) We have

(2. 3)a $\quad h_{ij} = \delta_{ij}, \qquad h^{ij} = \delta^{ij}.$

(2.3)b $\quad \underset{i}{e}^{\nu} = \overset{i\nu}{e}, \qquad \overset{j}{e}_{\lambda} = \underset{j}{e}_{\lambda}.$

MAIN THEOREMS

In this section, we will study some of the relationships between holonomic and nonholonomic components, and derive a useful representation of the nonholonomic components.

Our further discussions will be restricted to an orthogonal nonholonomic frames only.
First of all, we shall derive some special properties of this frame in the following teorem.

THEOREM 3. 1) We have

(3.1) $\quad \underset{i}{e}^{\nu} = \underset{i}{e}_{\nu}, \qquad \overset{j}{e}_{\lambda} = \overset{j}{e}^{\lambda}.$

Proof). By means of (2.3)b and $\underset{i}{e}^{\nu}$ are mutually orthogonal unit vectors, easily obtained the results.

THEOREM 3. 2) The nonholonomic components of the covariant $h_{\lambda\mu}$ and contravariant tensor $h^{\lambda\mu}$ expressed in terms of $\underset{i}{e}^{\lambda}$, as follows ;

(3.2) $\quad h^{\lambda\mu} = \underset{i}{e}^{\lambda} h^{ij} \underset{j}{e}^{\mu} = \overset{i}{e}_{\lambda} h_{ij} \overset{j}{e}_{\mu}.$

Proof). Using (1.4)b, (2.3)a and (3.1), easily obtained the results.

DEFINITION 3.3) A symmetric covariant tensor a whose determinant a $\underline{\underline{\text{def}}}$ Det $((a_{\lambda\mu})) \neq 0$ defined by

(3.3) $\quad a^{\lambda\nu} \overset{\text{def}}{=} \dfrac{A_{\lambda\nu}}{a}$ is a symmetric contravariant tensor satisfying $a_{\lambda\mu}\, a^{\lambda\nu} = \delta_\mu^\nu$,
where $A_{\lambda\nu}$ is the cofactor of $a_{\lambda\nu}$ in a.

THEOREM 3.4) The derivative of $\underset{i}{e}^{\lambda}$ is negative self-adjoint. That is,

(3.4)a $\quad \partial_K(\overset{i}{e}_{\lambda})\underset{j}{e}^{\mu} = -\partial_K(\underset{j}{e}^{\mu})\overset{i}{e}_{\lambda}.$

Proof). Take a coordinate system y^i for which we have at a point p of V_n.

(3.4)b $\quad \dfrac{\partial y^i}{\partial x^\lambda} = \overset{i}{e}_{\lambda}, \quad \dfrac{\partial x^\nu}{\partial y^i} = \underset{i}{e}^{\nu}$

$\partial_K(\overset{i}{e}_{\lambda})\, \underset{j}{e}^{\mu} = -(\overset{i}{e}_{\lambda})^2\, \partial_K(\underset{i}{e}^{\lambda})\underset{j}{e}^{\mu}$

$$= -\delta^\mu_\lambda \ (\overset{i}{e}_\mu)\overset{i}{e}_\lambda \underset{j}{e}^\mu \ \partial_K(\underset{i}{e}^\lambda)$$

$$= -\delta^i_j \ \overset{i}{e}_\lambda \ \partial_K(\underset{i}{e}^\mu)$$

$$= -\overset{i}{e}_\lambda \ \partial_K(\underset{j}{e}^\mu).$$

THEOREM 3.5) The derivative of the tensor $a_{\lambda\mu}$ is negative self-adjoint.

Proof). By means of (3.3), we derive the

$$(3.5) \quad a^{\lambda\mu} \ \partial_K(a_{\lambda\mu}) = -a_{\lambda\mu} \ \partial_K(a^{\lambda\mu}).$$

THEOREM 3.6) The derivative of the nonholonomic comiponents of $a_{\lambda\mu}$ is negative self-adjoint.

Proof). Using (1.4)a, (1.4)b, (3.3), (3.4)a, (3.5),

$$a^{ij} \ \partial_K(a_{ij}) + a_{ij} \ \partial_K(a^{ij})$$

$$= a^{ij} \ \partial_K(a_{\lambda\mu} \ \underset{i}{e}^\lambda \ \underset{j}{e}^\mu) + a_{ij} \ \partial_K(a^{\lambda\mu} \ \overset{i}{e}_\lambda \ \overset{j}{e}_\mu)$$

$$= a^{ij} \ \partial_K(a_{\lambda\mu})\underset{i}{e}^\lambda \ \underset{j}{e}^\mu + a^{\lambda\omega} \ \overset{i}{e}_\lambda \overset{j}{e}_\omega \ a_{\lambda\mu} \ \partial_K(\underset{i}{e}^\lambda)\underset{j}{e}^\mu$$

$$+ a^{\lambda\omega} \ \overset{i}{e}_\lambda \ \overset{j}{e}_\omega \ a_{\lambda\mu} \ \underset{i}{e}^\lambda \ \partial_K(\underset{j}{e}^\mu)$$

$$+ a_{ij} \ \partial_K(a^{\lambda\mu})\overset{i}{e}_\lambda \ \overset{j}{e}_\mu + a_{\lambda\omega} \ \underset{i}{e}^\lambda \ \underset{j}{e}^\omega \ a^{\lambda\mu} \ \partial_K(\overset{i}{e}_\lambda)\overset{j}{e}_\mu$$

$$+ a_{\lambda\omega} \ \underset{i}{e}^\lambda \ \underset{j}{e}^\omega \ a^{\lambda\mu} \ \overset{i}{e}_\lambda \ \partial_K(\overset{j}{e}_\mu)$$

$$= a^{\lambda\mu} \ \partial_K(a_{\lambda\mu}) + \overset{i}{e}_\lambda \ \partial_K(\underset{i}{e}^\lambda) + \overset{j}{e}_\mu \ \partial_K(\underset{j}{e}^\mu)$$

$$+ a_{\lambda\mu} \ \partial_K(a^{\lambda\mu}) + \underset{i}{e}^\lambda \ \partial_K(\overset{i}{e}_\lambda) + \underset{j}{e}^\mu \ \partial_K(\overset{j}{e}_\mu)$$

$$= a^{\lambda\mu} \ \partial_K(a_{\lambda\mu}) + a_{\lambda\mu} \ \partial_K(a^{\lambda\mu}).$$

Bythe theorem (3.4)b, we have the result.

COROLLARY 3.7) The negative self-adjoint of the derivative of the tensor $a_{\lambda\mu}$ is equal to its nonholonomic components.

LITERATURE CITEO

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〈국문초록〉

V_n 공간에서의 Nonholonomic Self-Adjoint에 관한 소고

본 논문은 Holonomic과 Nonholonomic Compoment 사이의 관계를 연구하고 이 구조에 대한 몇 가지 특수한 성질을 증명하였다.