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Kannan type fixed point theorems in modular metric spaces

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⟨**Abstract**⟩

Kannan type fixed point theorems in modular metric spaces

In 2010, V.V. Chistyakov introduced the concept of *modular metric* spaces. This concept generalizes modular linear spaces, modular function spaces and metric spaces. In this thesis, we prove the new existence theorem of a unique fixed point for Kannan type w-contractive mappings in modular metric spaces. Our result generalizes the results of Kannan on complete metric spaces.

In 2017, Aksoy et al. defined Bogin type w-contraction and Kannan type w-contraction in modular metric spaces, and proved the existence theorems of a fixed point under some conditions. Also, in 2019, Mitrovic et al. defined Reich type w-contraction and Kannan type w-contraction in the same spaces, and also proved the existence theorems of a fixed point under some conditions. But we give counterexamples that the above two results of Aksoy et al. and Mitrovic et al. do not hold.

Finally, we introduce the new concept of weak w-completeness, which generalizes w-completeness and show that the converse of our main theorem holds as a special case.

1 Introduction

In 2010, V.V. Chistyakov ([4]) introduced the notion of *modular metric* spaces (or *metric modular* spaces) and investigated properties of the spaces. This concept generalizes modular linear spaces ([15]), modular function spaces ([12]) and metric spaces. Moreover, in 2011, Chistyakov ([5]) defined w-contraction in modular metric spaces and proved the existence theorems of a fixed point under such contractive conditions. The main idea behind this new concept is physical interpretations. Informally speaking, a metric on a set represents nonnegative finite distances between any two points of the set. On the other hand, a modular on a set attributes a nonnegative (possibly, infinite valued) field of (generalized) velocities if we take λ as a parameter of time: If we set $w_\lambda(x,y) = \dfrac{d(x,y)}{\lambda}$, then w is average velocity which means that it takes time λ to cover the distance between x and y. What if x is disconnected with y? Clearly, it is impossible to reach from x to y regardless of given time λ, so it seems reasonable for us to assign $w_\lambda(x,y)$ to ∞. It is a reason that we adopt *extended real system* as a codomain of a modular metric.

In 1968, R. Kannan ([11]) proved the following theorem on complete metric spaces:

Theorem 1 ([11]) *Let (X,d) be a complete metric space. If $T : X \longrightarrow X$ and there exists $k \in (0,1/2)$ such that for all $x,y \in X$,*

$$d(Tx,Ty) \le k\{d(x,Tx) + d(y,Ty)\} \tag{1.1}$$

then T has a unique fixed point.

The main purposes of this thesis are to investigate fixed point results under *Kannan type* w-contractive condition in modular metric spaces, and to give counterexamples that two results of Aksoy et al. ([2]) and Mitrovic et al. ([13]) do not hold. This thesis is organized as follows:

In chapter 2, we briefly introduce the notions and examples in ([4]) and ([5]) concerning modular metric spaces.

In chapter 3, we not only compare a variety of definitions concerning *Kannan type* w-contraction in modular metric spaces but also show the new existence theorem of a unique fixed point under Kannan type w-contraction which is our main result. The result generalizes *Kannan contraction principle* in ([11]). We also give counterexamples that Theorem 3.6 of Aksoy et al. ([2]) and Theorem 2.1 of Mitrovic et al. ([13]) do not hold, respectively.

In chapter 4, we define the new concept, the *weak w-completeness* in modular metric spaces which is a generalization of w-completeness introduced in ([5]). Also, we show that the converse of the main result in chapter 3 also holds as a special case.

2 Preliminaries

We mainly recall some basic terms and notations in ([4]) and ([5]). From now on, X represents a nonempty set and $w : (0, \infty) \times X \times X \longrightarrow [0, \infty]$ will be written as $w_\lambda(x, y) = w(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

Definition 2.1. A function $w : (0, \infty) \times X \times X \longrightarrow [0, \infty]$ is called *modular metric* (or *modular*) on X if it satisfies the following three conditions:

(i) given $x, y \in X$, $x = y$ iff $w_\lambda(x, y) = 0$ for *all* $\lambda > 0$.

(ii) $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$.

(iii) $w_{\lambda+\mu}(x, y) \le w_\lambda(x, z) + w_\mu(y, z)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If, instead of (i), the function w satisfies only

(i′) $w_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$,

then w is said to be a *pseudomodular* on X.

Also, if w satisfies (i′) and

(i″) given $x, y \in X$, $x = y$ iff $w_\lambda(x, y) = 0$ for *some* $\lambda > 0$, then the function w is called a *strict modular* on X.

The relationships between them are as follows:

$$\textit{srtict modular} \Longrightarrow \textit{modular} \Longrightarrow \textit{pseudomodular}.$$

If, instead of (iii), the function w satisfies

(iii′) $w_{\lambda+\mu}(x, y) \le \dfrac{\lambda}{\lambda+\mu} w_\lambda(x, z) + \dfrac{\mu}{\lambda+\mu} w_\mu(y, z)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$, then it is said to be a *convex* modular.

Let w be a convex modular and $0 < \mu \leq \lambda$. If we put $z = y$ in Definition 2.1 (iii′), then the following inequality holds:

$$w_\lambda(x,y) \leq \frac{\mu}{\lambda} w_\mu(x,y) \leq w_\mu(x,y). \tag{2.1}$$

The relationship between them is also as follows:

$$\textit{convex modular} \implies \textit{modular}.$$

Example 2.2. Let $(X,\, d)$ be a metric space. Then two canonical strict modulars are given as follows:

(i) If we set $w_\lambda(x,y) = d(x,y)$, then it is a nonconvex modular on X.

(ii) If we set $w_\lambda(x,y) = \dfrac{d(x,y)}{\lambda}$, then it is a convex modular on X.

Remark 2.3. (1) w is nonincreasing on λ by (2.1).

(2) Let w be a (pseudo)modular. If we set $\tilde{w}_\lambda(x,y) = \dfrac{w_\lambda(x,y)}{\lambda}$, then $\tilde{w}$ is a convex (pseudo)modular.

(3) If w is convex, the following inequality holds:

$$(\lambda_1 + \lambda_2 + \cdots + \lambda_n) w_{\lambda_1+\lambda_2+\cdots+\lambda_n}(x_1, x_{n+1}) \leq \sum_{i=1}^{n} \lambda_i w_{\lambda_i}(x_i, x_{i+1}), \tag{2.2}$$

where $\lambda_i > 0$ and $x_i \in X$.

Example 2.4. We give a few of examples of (pseudo)modulars.

Let d be a (pseudo)metric on X.

(i) If we set $w_\lambda(x,y)$ as follows:

$$w_\lambda(x,y) = \begin{cases} 0, & x = y, \\ \infty, & x \neq y, \end{cases}$$

then w is a modular.

(ii) If we set $w_\lambda(x,y) = d(x,y)/\phi(\lambda)$, where $\phi : (0,\infty) \longrightarrow (0,\infty)$ is a nondecreasing function, then w is a (pseudo)modular.

(iii) If we set $w_\lambda(x,y)$ as follows:

$$w_\lambda(x,y) = \begin{cases} 0, & \lambda > d(x,y), \\ \infty, & \lambda \leq d(x,y), \end{cases}$$

then w is a (pseudo)modular.

(iv) If we set $w_\lambda(x,y)$ as follows:

$$w_\lambda(x,y) = \begin{cases} 0, & \lambda \geq d(x,y), \\ \infty, & \lambda < d(x,y), \end{cases}$$

then w is a (pseudo)modular.

Definition 2.5. Let w be a pseudomodular on X. Then the two sets

$$X_w \equiv X_w(x_0) = \{x \in X : w_\lambda(x,x_0) \longrightarrow 0 \ \text{ as } \ \lambda \longrightarrow \infty\},$$

$$X_w^* \equiv X_w^*(x_0) = \{x \in X : w_\lambda(x,x_0) < \infty \ \text{ for some } \ \lambda(x)\}$$

are called to be *modular spaces* (around x_0).

Definition 2.6. Let w be a pseudomodular on X. Then a sequence $\{x_n\}$ from X_w or X_w^* is said to be *modular convergent* (or *w-convergent*) to an element $x \in X$ if there exists a number $\lambda(\{x_n\}, x) > 0$, such that $w_\lambda(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$ for some $\lambda > 0$ and any such element x will be called a *modular limit* of the sequence $\{x_n\}$.

Definition 2.7. A pseudomodular w on X is said to satisfy the (sequential) $\triangle_2-$ *condition* on X_w^* if the following condition holds:
Given a sequence $\{x_n\} \subset X_x^*$ and $x \in X_w^*$, if there exists a number $\lambda > 0$, depending on $\{x_n\}$ and x, such that $w_\lambda(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$, then $w_{\frac{\lambda}{2}}(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$.

Remark 2.8. In fact, there is another version of definition concerning w-convergence in modular metric spaces. In ([6]), Cho et al. defined w-convergence in modular metric spaces as follows:

$$w_\lambda(x_n, x) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \quad \text{for} \quad all \quad \lambda > 0,$$

which is evidently stronger than Definition 2.6. Also, it is equivalent to Definition 2.6 under $\triangle_2$-condition. Here we take Definition 2.6 as definition of w-convergence.

Theorem 2.9. *([5]) Let w be a pseudomodular on X. We have:*

(1) *the modular spaces X_w and X_w^* are closed with respect to the modular convergence i.e., if $\{x_n\} \subset X_w$ (or X_w^*), $x \in X$ and $w_\lambda(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$, then $x \in X_w$ (or $x \in X_w^*$, respectively);*

(2) *if w is a strict modular on X, the modular limit is determined uniquely (if it exists).*

Remark 2.10. Remark 2.3 shows that if $w_\lambda(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$ for some $\lambda > 0$, then $w_\mu(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\mu > \lambda > 0$.

Definition 2.11. Let w be a modular on X. Then a sequence $\{x_n\}$ in X_w^* is said to be *modular Cauchy* (or w – *Cauchy*) if there exists a number $\lambda(\{x_n\}) > 0$ such that $w_\lambda(x_m, x_n) \longrightarrow 0$ as $m, n \longrightarrow \infty$ for some $\lambda > 0$.

Definition 2.12. Let w be a modular on X. Then the modular space X_w^* is said to be *modular complete* (or w – *complete*) if each modular Cauchy sequence from X_w^* is modular convergent in the following sense:

If $\{x_n\} \subset X_w^*$ and there exists a number $\lambda(\{x_n\}) > 0$ such that $w_\lambda(x_m, x_n) \longrightarrow 0$ as $m, n \longrightarrow \infty$, then there exists an $x_0 \in X_w^*$ such that $w_\lambda(x_n, x_0) \longrightarrow 0$ as $n \longrightarrow \infty$.

3 Fixed point results in modular metric spaces

Banach contraction principle ([3]) is the very well-known *metric* fixed point theorem, which asserts that if $T : X \longrightarrow X$, where X is a complete metric space, and there exists $k \in (0,1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y),$$

then T has a unique fixed point. In ([5]), the same problem was raised in the context of modular metric spaces: Is there a fixed point of T satisfying conditions like Banach contraction principle in modular metric spaces? The affirmative answer was also given in ([5]).

Definition 3.1. ([5]) Let w be a modular metric on X and let X_w^* be a modular space.

(i) A map $T : X_w^* \longrightarrow X_w^*$ is said to be *w-contractive* if there exist $0 < k < 1$ and $\lambda_0 > 0$ depending on k such that

$$w_{k\lambda}(Tx, Ty) \leq w_\lambda(x, y)$$

for all $0 < \lambda \leq \lambda_0$ and $x, y \in X_w^*$.

(ii) A map $T : X_w^* \longrightarrow X_w^*$ is said to be *strongly w-contractive* if there exist $0 < k < 1$ and $\lambda_0 > 0$ depending on k such that

$$w_{k\lambda}(Tx, Ty) \leq kw_\lambda(x, y)$$

for all $0 < \lambda \leq \lambda_0$ and $x, y \in X_w^*$.

The following fixed point theorems were proved in ([5]).

Theorem 3.2. *([5]) Let w be a strict convex modular metric on X such that the modular space X_w^* is w-complete and let $T : X_w^* \longrightarrow X_w^*$ be w-contractive such that*

$$for\ each\ \lambda > 0,\ there\ exists\ an\ x = x(\lambda) \in X_w^*\ such\ that\ w_\lambda(x, Tx) < \infty.$$

Then T has a fixed point, i.e., $Tx_* = x_*$ for some $x_* \in X_w^*$. If, in addition, the modular metric w assumes only finite values on X_w^*, then the fixed point of T is unique.

Theorem 3.3. *([5]) Let w be a strict modular metric on X such that the modular space X_w^* is w-complete and let $T : X_w^* \longrightarrow X_w^*$ be strongly w-contractive such that*

$$for\ each\ \lambda > 0,\ there\ exists\ an\ x = x(\lambda) \in X_w^*\ such\ that\ w_\lambda(x, Tx) < \infty.$$

Then T has a fixed point, i.e., $Tx_* = x_*$ for some $x_* \in X_w^*$. If, in addition, the modular metric w assumes only finite values on X_w^*, then the fixed point of T is unique.

Since Banach contraction principle appeared, many different types of contraction have been emerging in the framework of metric spaces. See ([17]) for more information on it. Huge amount of fixed point results have been acquired under such various contractions. Some of them are independent of Banach contraction principle while some of others include it. For example, Kannan contraction principle in ([11]) is one of them which is indepedent of Banach contraction principle. See ([16]) for independence. Like metric spaces, there are various versions of w-contraction, so called *Kannan type, Bogin type and Reich type*, etc.. At first, we look at some definitions.

Definition 3.4. Let w be a modular metric on X and let T be a self map on X_w^*.

(i) T is said to be *Kannan type w-contractive* ([14]) if there exist $0 < k < 1/2$ such that

$$w_\lambda(Tx, Ty) \leq k\{w_{2\lambda}(x, Tx) + w_{2\lambda}(y, Ty)\} \tag{3.1}$$

for all $\lambda > 0$ and $x, y \in X_w^*$.

(ii) T is said to be *Bogin type w-contractive* ([2]) if there exist $0 < k < 1$ and $\lambda_0 > 0$ depending on k such that

$$\begin{aligned} w_{k\lambda}(Tx, Ty) \leq\ & a w_\lambda(x, y) + b\{w_{2\lambda}(x, Tx) + w_{2\lambda}(y, Ty)\} \\ & + c\{w_{2\lambda}(x, Ty) + w_{2\lambda}(y, Tx)\} \end{aligned} \tag{3.2}$$

for all $0 < \lambda < \lambda_0$, $x, y \in X_w^*$ and $a, b, c \geq 0$ with $a + 2b + 2c = 1$. If we put $a = c = 0$ and $b = 1/2$, then it is Kannan type $w-$contractive in the sense of ([2]).

(iii) T is said to be *Reich type w-contractive* ([13]) if there exist $0 < k < 1$ and $\lambda_0 > 0$ such that

$$w_\lambda(Tx, Ty) \leq w_{\frac{\lambda}{a}}(x, y) + w_{\frac{\lambda}{b}}(x, Tx) + w_{\frac{\lambda}{c}}(y, Ty) \tag{3.3}$$

for all $0 < \lambda \leq \lambda_0$ and $x, y \in X_w^*$ with $a, b, c \in (0, 1)$ and $a + b + c < 1$. If a tends to 0, it is Kannan type $w-$contractive in the sense of ([13]).

Now it is quite natural to ask, in the context of modular metric spaces, whether there exists a result corresponding to Theorem 1 in ([11]), or not. we look at the following three examples.

Example 3.5. Set $X = \{0,1,2\}$ and define $w : (0,\infty) \times X \times X \longrightarrow [0,\infty]$ by

$$w_\lambda(x,y) = \begin{cases} 0, & x = y, \\ \dfrac{1}{\lambda}, & x + y = 1 \quad \text{or} \quad \lambda \geq 10, \\ \infty, & x \neq y, \quad x + y = 2 \text{ or } 3 \quad \text{and} \quad \lambda < 10. \end{cases}$$

Then w satisfies the following conditions:

(i) w is a strict convex modular on X.

(ii) $X_w^* = X$.

(iii) X_w^* is w-complete.

Proof. (i) It is clear that $w_\lambda(x,y) = 0$ for some $\lambda > 0 \Longleftrightarrow x = y$. Symmetry of w is obvious. If $w_{\lambda+\mu}(x,y) < \infty$, then its value is either 0 or $\dfrac{1}{\lambda+\mu}$. Since

$$\frac{\lambda}{\lambda+\mu} w_\lambda(x,z) \geq \frac{\lambda}{\lambda+\mu} \cdot \frac{1}{\lambda} = \frac{1}{\lambda+\mu} \quad \text{if} \quad x \neq z,$$

$$\frac{\mu}{\lambda+\mu} w_\mu(y,z) \geq \frac{\mu}{\lambda+\mu} \cdot \frac{1}{\mu} = \frac{1}{\lambda+\mu} \quad \text{if} \quad y \neq z,$$

we see that

$$w_{\lambda+\mu}(x,y) \leq \frac{\lambda}{\lambda+\mu} w_\lambda(x,z) + \frac{\mu}{\lambda+\mu} w_\mu(y,z)$$

if $w_{\lambda+\mu}(x,y) < \infty$.

Suppose $w_{\lambda+\mu}(x,y) = \infty$, which implies that $\lambda + \mu < 10$ and hence $\lambda,\ \mu < 10$.

Case(1): $w_{\lambda+\mu}(0,2) = \infty$. Then $\dfrac{\lambda}{\lambda+\mu} w_\lambda(0,z) + \dfrac{\mu}{\lambda+\mu} w_\mu(z,2) = \infty$ for any $z \in X$.

Case(2): $w_{\lambda+\mu}(1,2) = \infty$. Then $\dfrac{\lambda}{\lambda+\mu} w_\lambda(1,z) + \dfrac{\mu}{\lambda+\mu} w_\mu(z,2) = \infty$ for any $z \in X$.

In both cases, we also see that

$$w_{\lambda+\mu}(x,y) \leq \frac{\lambda}{\lambda+\mu} w_\lambda(x,z) + \frac{\mu}{\lambda+\mu} w_\mu(y,z).$$

Thus w is a strict convex modular.

(ii) Fix $x_0 \in X$. Then $w_\lambda(x, x_0) = 0$ or $1/\lambda$ if $\lambda \geq 10$, which implies that $X_w^*(x_0) = X$.

(iii) Fix $\lambda \geq 10$ and set $\epsilon > 0$ such that $0 < \epsilon < 1/\lambda$. If $x \neq y$, then $w_\lambda(x, y) = 1/\lambda > \epsilon$, which shows if $\{x_n\}$ is a w-Cauchy sequence, then there exists $N \in \mathbb{N}$ such that $m, n \geq N \Longrightarrow x_m = x_n$. Thus X_w^* is w-complete.

From (i),(ii) and (iii), we see that w satisfies all required conditions. □

Example 3.6. Let w be the same as in Example 3.5. and define $T : X_w^* \longrightarrow X_w^*$ by

$$T(0) = 1, \quad T(1) = 2, \quad T(2) = 0.$$

Then the following conditions hold:

(i) T is a Bogin w-contraction (or Kannan w-contraction) in the sense of ([2]), where $a = 0$, $b = 0.5$, $c = 0$, $k = 0.5$ and $\lambda_0 = 5$ in (3.2) of Definition 3.4.

(ii) $w_\lambda(0, T(0)) = 1/\lambda < \infty$ for any $\lambda > 0$.

From (i) and (ii), T satisfies all required conditions of Theorem 3.6 in ([2]), but it is fixed point free.

Proof. (i) It suffices to show that (3.2) holds when $x \neq y$. If we set $a = 0, b = 0.5, c = 0$ and $k = 0.5$ in (3.2), then we see that $w_{2\lambda}(x, Tx) + w_{2\lambda}(y, Ty) = \infty$ for any $0 < \lambda < 5$ and any $x, y \in X_w^*$, $(x \neq y)$. Hence putting $\lambda_0 = 5$, we see that (3.2) of Definition 3.4 holds for any $0 < \lambda < \lambda_0$, which means that T is a Bogin w-contraction (or Kannan w-contraction) in the sense of ([2]).

(ii) It is obvious that $w_\lambda(0, T(0)) = w_\lambda(0, 1) = 1/\lambda < \infty$ for any $\lambda > 0$.

From (i) and (ii), we see that T disproves the results of Theorem 3.6 of Aksoy et al. in ([2]). □

Example 3.7. Let w and T be the same as in Example 3.6. Then the following conditions hold:

(i) T is a Reich w-contraction in the sense of ([13]), where $a = 0.1$, $b = c = 0.4$ and $\lambda_0 = 3$ in (3.3) of Definition 3.4.

(ii) $w_\lambda(0, T(0)) = 1/\lambda < \infty$ for any $\lambda > 0$.

From (i) and (ii), T satisfies all required conditions of Theorem 2.1 in ([13]), but it is fixed point free as shown in Example 3.6.

Proof. If we set $a = 0.1$, $b = c = 0.4$ and $\lambda_0 = 3$, then $\lambda_0/b = \lambda_0/c = 7.5 < 10$ and $w_{7.5}(x, Tx) + w_{7.5}(y, Ty) = \infty$ for any $x, y \in X_w^*$, $(x \neq y)$. Hence putting $\lambda_0 = 3$, as in Example 3.6, we see that T satisfies all required conditions of Theorem 2.1 in ([13]) and so, T disproves the results of Theorem 2.1 of Mitrovic et al. in ([13]). □

Remark 3.8. In Examples 3.6 and 3.7, we note that if $\lambda < 10$, there does *not* exist $x \in X_w^*$ such that

$$w_{\frac{\lambda}{2}}(T^{n-1}x, T^n x) < \infty, \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

Remark 3.9. Examples 3.6 and 3.7 obviously show that T satisfies all required conditions of Theorem 3.6 in ([2]) and Theorem 2.1 in ([13]), but does *not* have a fixed point, respectively. Thus, we have to be careful when dealing with modulars. In ([2, 13]), *Picard iteration* is utilized for constructing $\{x_n\}_{n\in\mathbb{N}}$, i.e., $x_n = T^n x_0$. In this case, $w_\lambda(x_n, x_{n+1})$ may take an infinite value even if $w_\lambda(x_{n-1}, x_n)$ take a finite value. Then the inequality

$$w_\lambda(x_n, x_{n+1}) \leq a w_\lambda(x_{n-1}, x_n) + b w_\lambda(x_n, x_{n+1}), \text{ where } 0 < a, b < 1,$$

no longer implies that inequality

$$w_\lambda(x_n, x_{n+1}) \le \frac{a}{1-b} w_\lambda(x_{n-1}, x_n)$$

holds when $w_\lambda(x_n, x_{n+1}) = \infty$. This is the problem that the authors of ([2, 13]) did not pay attention to. As we have shown in Examples 3.6 and 3.7, most of the claims concerning properties of Reich or Kannan w-contractions are incorrect, nullifying the validity of many fixed points results obtained in modular metric spaces. As a result, unlike cases of Theorem 3.2 and 3.3, we can see that a slightly stronger condition like (3.4) is needed. Additionally, some of flaws in ([14]) were also pointed out in ([1]).

Motivated by these results, we define an *Kannan type w-contraction* as follows:

Definition 3.10. Let w be a modular metric on X.

(i) A map $T : X_w^* \longrightarrow X_w^*$ is said to be *Kannan type w-contractive* if there exist $0 < k < 1/2$ and $\lambda_0 > 0$ depending on k such that

$$w_{k\lambda}(Tx, Ty) \le \frac{1}{2}\{w_{\frac{\lambda}{2}}(x, Tx) + w_{\frac{\lambda}{2}}(y, Ty)\} \tag{3.5}$$

for all $0 < \lambda \le \lambda_0$ and $x, y \in X_w^*$.

(ii) A map $T : X_w^* \longrightarrow X_w^*$ is said to be *strongly Kannan type w-contractive* if there exist $0 < k < 1/2$ and $\lambda_0 > 0$ depending on k such that

$$w_{k\lambda}(Tx, Ty) \le k\{w_{\frac{\lambda}{2}}(x, Tx) + w_{\frac{\lambda}{2}}(y, Ty)\} \tag{3.6}$$

for all $0 < \lambda \le \lambda_0$ and $x, y \in X_w^*$.

Lemma 3.11. *Let w be a modular metric on X and let X_w^* be a modular space. Suppose that*

$$w_{k\lambda}(Tx,Ty) \le \frac{1}{2}\{w_{\frac{\lambda}{2}}(x,Tx) + w_{\frac{\lambda}{2}}(y,Ty)\}$$

for some $\lambda > 0$ and $x, y \in X_w^*$. If there exists $x(\lambda) \in X_w^*$ such that

$$w_{\frac{\lambda}{2}}(T^{n-1}x, T^n x) < \infty$$

for each $n \in \mathbb{N}$, then $w_{2^i k^{i+1}\lambda}(T^n x, T^{n+1}x) < \infty$ for each $i \in \mathbb{N} \cup \{0\}$ and $n \ge i+1$.

Proof. Suppose that $\lambda > 0$ satisfies (3.5). Then by assumption,

$$w_{k\lambda}(T^n x, T^{n+1}x) \le \frac{1}{2}\{w_{\frac{\lambda}{2}}(T^{n-1}x, T^n x) + w_{\frac{\lambda}{2}}(T^n x, T^{n+1}x)\} < \infty$$

for all $n \in \mathbb{N}$. In general, using induction on i, we obtain

$$w_{2^i k^{i+1}\lambda}(T^n x, T^{n+1}x) \le \frac{1}{2}\{w_{2^{i-1}k^i\lambda}(T^{n-1}x, T^n x) + w_{2^{i-1}k^i\lambda}(T^n x, T^{n+1}x)\} < \infty$$

for all $n \ge i+1$, and so our result holds. □

Lemma 3.12. *Let w be a convex modular metric on X. Then the following inequality holds: For all $i, j \in \mathbb{N}$ and $0 < k < 1/2$,*

$$w_{2^{j-1}k^j\lambda}(x,y) \le (2k)^i w_{2^{i+j-1}k^{i+j}\lambda}(x,y).$$

Proof. From (2.1), we obtain the following inequality:

$$\begin{aligned} w_{2^{j-1}k^j\lambda}(x,y) &\le \frac{2^{i+j-1}k^{i+j}\lambda}{2^{j-1}k^j\lambda} w_{2^{i+j-1}k^{i+j}\lambda}(x,y) \\ &= (2k)^i w_{2^{i+j-1}k^{i+j}\lambda}(x,y) \end{aligned}$$

and so our result holds. □

Theorem 3.13. *Let w be a strict convex modular metric on X such that the modular space X_w^* is w-complete. Let $T : X_w^* \longrightarrow X_w^*$ be Kannan type $w-$contractive with the property that*

$$for\ each\ \lambda > 0,\ there\ exists\ an\ x(\lambda) \in X_w^*\ such\ that\ w_{\frac{\lambda}{2}}(T^{n-1}x, T^n x) < \infty$$

for all $n \in \mathbb{N}$. Then there uniquely exists $x_* \in X_w^*$ such that

$$w_{(2k)^{j-1}\lambda_0}(T^n x, x_*) \longrightarrow 0 \ \ as \ \ n \longrightarrow \infty$$

for each $j \in \mathbb{N}$. Moreover, if $w_{\frac{(2k)^{j-1}\lambda_0}{1-2k}}(Tx_*, x_*) < \infty$ for some $j \in \mathbb{N}$, where $j > 1 + \log_{2k}(\frac{1}{2} - k)$, then x_* is the unique fixed point of T.

Proof. Put $\lambda_1 = \dfrac{(1-2k)}{1-k}\lambda_0$. Then there exists $x_0 \in X_w^*$, depending on λ_1, such that

$$w_{\frac{\lambda_1}{2}}(T^{n-1}x_0, T^n x_0) < \infty$$

for all $n \in \mathbb{N}$. At first, we show that

$$w_{2^{i+j-1}k^{i+j}\lambda_1}(T^{i+j}x_0, T^{i+j+1}x_0) \ \le\ \left\{\frac{1}{2(1-k)}\right\}^i w_{2^{j-1}k^j\lambda_1}(T^j x_0, T^{j+1}x_0). \qquad (3.7)$$

Using (3.5) and Lemma 3.12, for any $j \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} w_{2^j k^{j+1}\lambda_1}(T^{j+1}x_0, T^{j+2}x_0) &\le \frac{1}{2}\{w_{2^{j-1}k^j\lambda_1}(T^j x_0, T^{j+1}x_0) + w_{2^{j-1}k^j\lambda_1}(T^{j+1}x_0, T^{j+2}x_0)\} \\ &\le \frac{1}{2}w_{2^{j-1}k^j\lambda_1}(T^j x_0, T^{j+1}x_0) + k w_{2^j k^{j+1}\lambda_1}(T^{j+1}x_0, T^{j+2}x_0). \end{aligned}$$

Since $w_{2^j k^{j+1}\lambda_1}(T^{j+1}x_0, T^{j+2}x_0) < \infty$ by Lemma 3.11, we obtain

$$w_{2^j k^{j+1}\lambda_1}(T^{j+1}x_0, T^{j+2}x_0) \ \le\ \frac{1}{2(1-k)} w_{2^{j-1}k^j\lambda_1}(T^j x_0, T^{j+1}x_0).$$

In general, for any $i \in \mathbb{N}$,

$$\begin{aligned} w_{2^{i+j-1}k^{i+j}\lambda_1}(T^{i+j}x_0, T^{i+j+1}x_0) &\le \frac{1}{2}\{w_{2^{i+j-2}k^{i+j-1}\lambda_1}(T^{i+j-1}x_0, T^{i+j}x_0) \\ &\quad + w_{2^{i+j-2}k^{i+j-1}\lambda_1}(T^{i+j}x_0, T^{i+j+1}x_0)\} \\ &\le \frac{1}{2} w_{2^{i+j-2}k^{i+j-1}\lambda_1}(T^{i+j-1}x_0, T^{i+j}x_0) \\ &\quad + k w_{2^{i+j-1}k^{i+j}\lambda_1}(T^{i+j}x_0, T^{i+j+1}x_0). \end{aligned}$$

Keeping the fact that $w_{2^{i+j-1}k^{i+j}\lambda_1}(T^{i+j}x_0, T^{i+j+1}x_0) < \infty$ in mind, and using induction on i, we have

$$\begin{aligned} w_{2^{i+j-1}k^{i+j}\lambda_1}(T^{i+j}x_0, T^{i+j+1}x_0) &\le \frac{1}{2(1-k)} w_{2^{i+j-2}k^{i+j-1}\lambda_1}(T^{i+j-1}x_0, T^{i+j}x_0) \\ &\le \left\{\frac{1}{2(1-k)}\right\}^i w_{2^{j-1}k^j\lambda_1}(T^j x_0, T^{j+1}x_0). \end{aligned}$$

and so (3.7) is true. Let $m < n$ and set

$$\lambda(m,n) = 2^m k^{m+1}\lambda_1 + \cdots + 2^{n-1}k^n\lambda_1 = \frac{2^m k^{m+1}\{1-(2k)^{n-m}\}\lambda_1}{1-2k}.$$

Fix $j \in \mathbb{N}$. Then using (2.2) and (3.7), we have

$$\begin{aligned} & w_{(2k)^{j-1}\lambda}(T^{m+j}x_0, T^{n+j}x_0) \\ \le & \sum_{i=m}^{n-1} \frac{2^{i+j-1}k^{i+j}\lambda_1}{(2k)^{j-1}\lambda} w_{2^{i+j-1}k^{i+j}\lambda_1}(T^{i+j}x_0, T^{i+j+1}x_0) \\ \le & \sum_{i=m}^{n-1} \frac{2^i k^{i+1}\lambda_1}{\lambda}\left\{\frac{1}{2(1-k)}\right\}^i w_{2^{j-1}k^j\lambda_1}(T^j x_0, T^{j+1}x_0) \\ = & \frac{\lambda_1(1-k)k}{\lambda(1-2k)}\left(\frac{k}{1-k}\right)^m\left\{1-\left(\frac{k}{1-k}\right)^{n-m}\right\} w_{2^{j-1}k^j\lambda_1}(T^j x_0, T^{j+1}x_0). \end{aligned} \tag{3.8}$$

Since $1-(2k)^m k > 1/2$ and $k\{1-(2k)^n\} < 1/2$, we have

$$\begin{aligned} \lambda_0 - \lambda &= \left\{\frac{1-k}{1-2k} - \frac{2^m k^{m+1}\{1-(2k)^{n-m}\}}{1-2k}\right\}\lambda_1 \\ &= \frac{\{1-(2k)^m k\} - k\{1-(2k)^n\}}{1-2k}\lambda_1 \\ &> 0. \end{aligned}$$

Noting that $\dfrac{\lambda_1(1-k)}{\lambda_0(1-2k)} = 1$ and $\lambda < \lambda_0$, by (2.1) and (3.8),

$$\begin{aligned}
&w_{(2k)^{j-1}\lambda_0}(T^{m+j}x_0, T^{n+j}x_0)\\
&\le \frac{(2k)^{j-1}\lambda}{(2k)^{j-1}\lambda_0} w_{(2k)^{j-1}\lambda}(T^{m+j}x_0, T^{n+j}x_0)\\
&\le \frac{\lambda}{\lambda_0}\cdot\frac{\lambda_1(1-k)k}{\lambda(1-2k)}\Big(\frac{k}{1-k}\Big)^m\Big\{1-\Big(\frac{k}{1-k}\Big)^{n-m}\Big\} w_{2^{j-1}k^j\lambda_1}(T^jx_0, T^{j+1}x_0)\\
&= k\Big(\frac{k}{1-k}\Big)^m\Big\{1-\Big(\frac{k}{1-k}\Big)^{n-m}\Big\} w_{2^{j-1}k^j\lambda_1}(T^jx_0, T^{j+1}x_0)\\
&\le k\Big(\frac{k}{1-k}\Big)^m w_{2^{j-1}k^j\lambda_1}(T^jx_0, T^{j+1}x_0).
\end{aligned}$$

The last term of the above inequality tends to 0 as $m \longrightarrow \infty$. Hence for *any* $j \in \mathbb{N}$,

$$w_{(2k)^{j-1}\lambda_0}(T^{m+j}x_0, T^{n+j}x_0) \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty \quad (m < n). \tag{3.9}$$

Choose $j_1 \in \mathbb{N}$ such that $j_1 > 1 + \log_{2k}(\frac{1}{2}-k)$ which leads to $\dfrac{(2k)^{j_1-1}\lambda_0}{\frac{1}{2}-k} < \lambda_0$. By $w-$completeness and (3.9), there exists $x_* \in X_w^*$ such that

$$w_{(2k)^{j_1-1}\lambda_0}(T^nx_0, x_*) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.10}$$

Moreover, we assume that $w_{\frac{(2k)^{j_1-1}\lambda_0}{1-2k}}(Tx_*, x_*) < \infty$. We claim that x_* is the unique fixed point of T. If $j_2 > j_1 - \log_{2k}(1-2k)$ $(j_2 \in \mathbb{N})$, then from (3.9), the following inequality holds: Taking $n \longrightarrow \infty$,

$$w_{\frac{(2k)^{j_1-1}\lambda_0}{1-2k}}(T^{n-1}x_0, T^nx_0) \le w_{(2k)^{j_2-1}\lambda_0}(T^{n-1}x_0, T^nx_0) \longrightarrow 0. \tag{3.11}$$

Using Definition 2.1 (iii) and (3.5),

$$\begin{aligned}
w_{\frac{(2k)^{j_1-1}\lambda_0}{1-2k}}(x_*, Tx_*) &= w_{k\frac{(2k)^{j_1-1}\lambda_0}{\frac{1}{2}-k}+(2k)^{j_1-1}\lambda_0}(x_*, Tx_*)\\
&\le w_{k\frac{(2k)^{j_1-1}\lambda_0}{\frac{1}{2}-k}}(Tx_*, T^nx_0) + w_{(2k)^{j_1-1}\lambda_0}(T^nx_0, x_*)\\
&\le \frac{1}{2}w_{\frac{(2k)^{j_1-1}\lambda_0}{1-2k}}(x_*, Tx_*) + \frac{1}{2}w_{\frac{(2k)^{j_1-1}\lambda_0}{1-2k}}(T^{n-1}x_0, T^nx_0)\\
&\quad + w_{(2k)^{j_1-1}\lambda_0}(T^nx_0, x_*).
\end{aligned}$$

Thus, we obtain

$$\frac{1}{2} w_{\frac{(2k)^{j_1-1}\lambda_0}{1-2k}}(x_*, Tx_*) \ \leq \ \frac{1}{2} w_{\frac{(2k)^{j_1-1}\lambda_0}{1-2k}}(T^{n-1}x_0, T^n x_0) + w_{(2k)^{j_1-1}\lambda_0}(T^n x_0, x_*).$$

Applying (3.10) and (3.11), we see that the right hand side of the last inequality tends to 0 as $n \longrightarrow \infty$. Hence, by the strictness of w, we obtain that

$$w_{\frac{(2k)^{j_1-1}\lambda_0}{1-2k}}(x_*, Tx_*) = 0,$$

i.e., $Tx_* = x_*$.

If $Ty_* = y_*$, then

$$w_{k\lambda_0}(x_*, y_*) = w_{k\lambda_0}(Tx_*, Ty_*) \ \leq \ \frac{1}{2}\Big\{ w_{\frac{\lambda_0}{2}}(x_*, Tx_*) + w_{\frac{\lambda_0}{2}}(y_*, Ty_*) \Big\} = 0,$$

and so x_* is the unique fixed point of T. □

Now, we show that Theorem 1 is a Corollary of Theorem 3.13 as follows:

Corollary 3.14. *(Theorem 1 in ([11])) Let $(X,\ d)$ be a complete metric space. If $T : X \longrightarrow X$ and there exists $k \in (0, 1/2)$ such that (1.1) is satisfied for all $x, y \in X$, then T has a unique fixed point.*

Proof. If we set $w_\lambda(x, y) = \dfrac{d(x, y)}{\lambda}$, then Example 2.2 (ii) shows that w is a strict convex modular. And it is clear that $X_w^* = X$. Also, w-convergence and w-Cauchy sequence are equivalent to convergence and Cauchy sequence in $(X,\ d)$, respectively. Therefore, w-completeness is equivalent to completeness in $(X,\ d)$. Dividing both sides of (1.1) by $k\lambda$, we see that (1.1) implies (3.5). From which, the desired conclusion is derived. □

Remark 3.15. In many cases of dealing with modular metric or generalized modular metric ([21]), the condition

$$\delta_w(\lambda, x) = \sup\{w_\lambda(T^m(x), T^n(x)) : m, n \in \mathbb{N} \cup \{0\}\} < \infty$$

is often required for some $x \in X_w^*$ and for some $\lambda > 0$, which is stronger than the condition $w_{\frac{\lambda}{2}}(T^{n-1}x, T^n x) < \infty$ for each $n \in \mathbb{N}$. So, the results of Theorem 3.13 is independent of results in other papers, where $\delta_w(\lambda, x) < \infty$ for some $x \in X_w^*$ and for some $\lambda > 0$ is assumed. See ([1, 6, 8, 10]) for more details.

Corollary 3.16. *Let w be a strict modular metric on X such that the modular space X_w^* is w-complete. Let $T : X_w^* \longrightarrow X_w^*$ be strongly $Kannan\ type\ w-contractive$ with the property that*

$$for\ each\ \lambda > 0,\ there\ exists\ an\ x(\lambda) \in X_w^*\ such\ that\ w_{\frac{\lambda}{2}}(T^{n-1}x, T^n x) < \infty$$

for all $n \in \mathbb{N}$. Then there uniquely exists $x_* \in X_w^*$ such that

$$w_{(2k)^{j-1}\lambda_0}(T^n x, x_*) \longrightarrow 0\ \ as\, n \longrightarrow \infty$$

for each $j \in \mathbb{N}$. Moreover, if $w_{\frac{(2k)^{j-1}\lambda_0}{1-2k}}(Tx_*, x_*) < \infty$ for some $j \in \mathbb{N}$, where $j > 1 + \log_{2k}(\frac{1}{2} - k)$, then x_* is the unique fixed point of T.

Proof. Let us set $\tilde{w}_\lambda(x, y) = \dfrac{w_\lambda(x, y)}{\lambda}$. Remark 2.3 shows that $\tilde{w}$ is a strict convex modular metric with the property that $X_{\tilde{w}}^* = X_w^*$. It is easy to show that $\tilde{w}$ satisfies (3.5). Applying Theorem 3.13, we obtain the desired results. □

Corollary 3.17. *Even if (3.5) is replaced by*

$$w_{k\lambda}(T^N x, T^N y) \leq \frac{1}{2}\{w_{\frac{\lambda}{2}}(x, T^N x) + w_{\frac{\lambda}{2}}(y, T^N y)\}$$

for some $N \in \mathbb{N}$, then T still has the unique fixed point.

Proof. By Theorem 3.13, there exists an unique $x_* \in X_*$ such that $T^N x_* = x_*$. Hence we see that $T^N T x_* = T T^N x_* = T x_*$. Applying the uniqueness of a fixed point of T^N, we have $T x_* = x_*$. □

Remark 3.18. The contraction (3.5) with $k = 1/2$ in complete modular metric spaces does not guarantee the existence of a fixed point of T. See Examples 1.4 or 1.5 in ([9]), where we set $w_\lambda(x, y) = \dfrac{2d(x, y)}{\lambda}$.

Example 3.19. Let w be the same as in Example 3.5. Define $T_1 : X_w^* \longrightarrow X_w^*$ by

$$T_1(0) = 0, \quad T_1(1) = 0, \quad T_1(2) = 0.$$

It is easy to prove that T_1 satisfies all required conditions in Theorem 3.13. In particular, $w_{\frac{\lambda}{2}}(T_1^{n-1}(1), T_1^n(1)) < \infty$ for any $\lambda > 0$. Also, T_1 has the unique fixed point at $0 \in X_w^*$.

4 Characterization of completeness in modular metric spaces

P.V. Subrahmanyam ([18]) showed that fixed point properties under Kannan contraction can characterize completeness in metric spaces. Since a family of modular metric spaces is properly larger than a family of metric spaces, it is quite natural to ask whether fixed point properties under Kannan w-contraction can characterize completeness in modular metric spaces. In this context, as we show, we need a different kind of definition of completeness. At first, we look at the following example.

Example 4.1. Set $X = \mathbb{R}$ and define $w : (0, \infty) \times X \times X \longrightarrow [0, \infty]$ by

$$w_\lambda(x, y) = \begin{cases} \infty, & \lambda < 10 \quad \text{and} \quad \text{either } x = 0 \text{ or } y = 0, \\ \dfrac{|x - y|}{\lambda}, & \text{otherwise.} \end{cases}$$

It is easy to see that w is a strict (convex) modular and $X_w^*(0) = X$. Setting $x_n = 1/n$ and $\lambda = 5$, we have $w_5(x_m, x_n) \longrightarrow 0$ as $m, n \longrightarrow \infty$, which shows that $\{x_n\}_{n \in \mathbb{N}}$ is a w-Cauchy sequence in this modular metric space. But $w_5(x_l, 0) = \infty$ for any $l \in \mathbb{N}$. Hence X_w^* is *not* w−complete in the sense of Definition 2.12. However, if we put $\lambda = 10$, then $w_{10}(x_l, 0) \longrightarrow 0$ as $l \longrightarrow \infty$.

Example 4.1 motivates us to define so called *weak w-completeness* in modular metric spaces.

Definition 4.2. Given a modular w on X, the modular space X_w^* is said to be *weak modular complete* (or *weak w-complete*) if each modular Cauchy sequence from X_w^* is modular convergent in the following sense:

If $\{x_n\} \subset X_w^*$ and there exists a number $\lambda_1(\{x_n\}) > 0$, such that $w_{\lambda_1}(x_m, x_n) \longrightarrow 0$ as

$m, n \longrightarrow \infty$, then there exist an $x_0 \in X_w^*$ and $\lambda_2(\{x_n\}, x_0) > 0$ such that $w_{\lambda_2}(x_n, x_0) \longrightarrow 0$ as $n \longrightarrow \infty$.

Remark 4.3. Noting that w is nonincreasing on λ, we see that Definition 4.2 is equivalent to the following statement:
There exists a number $\lambda(\{x_n\}, x_0) > 0$ with the property that $w_\lambda(x_m, x_n) \longrightarrow 0$ as $m, n \longrightarrow \infty$ implies that there exists an $x_0 \in X_w^*$ such that $w_\lambda(x_l, x_0) \longrightarrow 0$ as $l \longrightarrow \infty$.

Remark 4.4. It is easy to see that X_w^* is weak w-complete in Example 4.1. Hence, it shows that Definition 4.2 is weaker than Definition 2.12, properly. Also, the former is equivalent to the latter under $\triangle_2$−condition.

We are ready to say about characterization of completeness in modular metric spaces without imposing $\triangle_2$−condition (See Definition 2.7.) on X_w^*.

Theorem 4.5. *Let w be a strict modular metric on X and let X_w^* be a modular space. Suppose that $T : X_w^* \longrightarrow X_w^*$ has a fixed point whenever the following two conditions are satisfied:*

(1) For fixed $c > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$,

$$w_{\lambda_1}(Tx, Ty) \leq c\, max\{w_{\lambda_2}(x, Tx),\ w_{\lambda_2}(y, Ty)\}.$$

(2) $T(X)$ is countable.

Then X_w^* is weak w-complete.

Proof. Suppose that X_w^* is not weak w-complete and fix $x \in X_w^*$. Then there exists a nonconvergent w-Cauchy sequence $\{x_n\} \subset X_w^*$, where, without loss of generality, we can assume $x_m \neq x_n$ if $m \neq n$. In other words, $w_{\lambda_1}(x_m, x_n) \longrightarrow 0$ as $m, n \longrightarrow \infty$ for some $\lambda_1 > 0$ but $w_{\lambda_2}(x_l, x) \not\longrightarrow 0$ as $l \longrightarrow \infty$ for any $x \in X_w^*$ and $\lambda_2 > 0$. Fix $\lambda_2 > 0$ and set

$$L(x) = \limsup_{l \to \infty} w_{\lambda_2}(x_l, x),$$

which may be possible to be ∞. Clearly, $L(x) > 0$. So, there exists a subsequence $\{x_{l_k}\}$ of $\{x_l\}$ such that $w_{\lambda_2}(x_{l_k}, x) \longrightarrow L(x)$ as $k \longrightarrow \infty$. For convenience, we put $\{x_{l_k}\} = \{x_l\}$. Then there exists $n_1 \in \mathbb{N}$ such that for any $l \geq n_1$, $w_{\lambda_2}(x_l, x) \geq \dfrac{L(x)}{2}$. Set

$$L_1(x) = min\{w_{\lambda_2}(x_l, x) : 1 \leq l \leq n_1 - 1,\ x_l \neq x\}.$$

By the strictness of w, $L_1(x) > 0$. Then there exists the *least* positive integer $n_2(x) \in \mathbb{N}$, which is not less than n_1, such that if $m, n \geq n_2$ and $x_l \neq x$, then the following inequality holds:

$$w_{\lambda_1}(x_m, x_n) \leq c\, min\Big\{\frac{L(x)}{2},\ L_1(x)\Big\} \leq c w_{\lambda_2}(x, x_l).$$

In particular, $w_{\lambda_2}(x_{n_2}, x) \geq \dfrac{L(x)}{2} > 0$ and hence $x_{n_2} \neq x$. Therefore, if $m \geq n_2$,

$$w_{\lambda_1}(x_m, x_{n_2}) \leq c w_{\lambda_2}(x, x_{n_2}). \tag{4.1}$$

Define $T : X_w^* \longrightarrow X_w^*$ by $Tx = x_{n_2}$. Clearly, T is fixed point free and $T(X)$ is countable. It remains to prove condition (1). Given $x, y \in X_w^*$, there exist $m, n \in \mathbb{N}$ such that $Tx = x_m$, $Ty = x_n$ by definition of T. From (4.1), if $m > n$,

$$w_{\lambda_1}(Tx, Ty) = w_{\lambda_1}(x_m, x_n) \ \leq\ c w_{\lambda_2}(y, x_n) = c w_{\lambda_2}(y, Ty).$$

In a similar way, if $m \leq n$, we obtain

$$w_{\lambda_1}(Tx, Ty) = w_{\lambda_1}(x_m, x_n) \ = w_{\lambda_1}(x_n, x_m) \ \leq\ c w_{\lambda_2}(x, x_m) = c w_{\lambda_2}(x, Tx).$$

From which, we obtain the following inequality:

$$w_{\lambda_1}(Tx, Ty) \le c\, max\{w_{\lambda_2}(x, Tx),\ w_{\lambda_2}(y, Ty)\}.$$

Consequently, we see that condition (1) also holds. This contradicts to the hypothesis that T must have a fixed point if conditions (1) and (2) are all satisfied. Hence X_w^* is weak w-complete. □

Remark 4.6. Setting $\lambda_1 = k\lambda$, $\lambda_2 = \lambda/2$, $c = 1/2$, it is clear that condition (1) in Theorem 4.5 is stronger than the Kannan type w-contraction (3.5). It means that the converse of the results of Theorem 3.13 still holds in modular metric spaces in the sense of weak $w-$completeness.

Remark 4.7. Let $\{x_n\}$ be a nonconvergent Cauchy sequence in a metric space (X, d). Then $\{d(x_n, x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for any $x \in X$ by virtue of triangle inequality. But in a modular metric space, due to deficiency of triangle inequality, even if $\{x_n\}$ is a nonconvergent w-Cauchy sequence in a modular metric space (X, w), $\{w_\lambda(x_n, x)\}_{n\in\mathbb{N}}$ need not be a Cauchy sequence in $\mathbb{R}$ for any $\lambda > 0$ and any $x \in X$ as shown in Example 4.1. Taking into account this fact, we can not directly apply the corresponding methods while possible in the framework of metric spaces. See ([18, 19, 20]) for more details.

Remark 4.8. In 1959, Connell (Example 3 in ([7])) showed that Banach contraction principle can *not* characterize completeness in metric spaces. Considering Example 2.2 together with this fact, it is clear that the converse of Theorem 3.2 does *not* hold in modular metric spaces.

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〈 국문초록 〉

모듈러 거리공간에서 Kannan 형태 축소사상의 부동점 정리

V.V. Chistyakov는 2010년에 모듈러 거리공간(modular metric space)에 관한 개념을 정립하였다. 이 개념은 모듈러 선형 공간, 모듈러 함수 공간, 그리고 거리공간을 일반화한 개념이다.

본 논문에서는 모듈러 거리공간에서 Kannan 형태 w-축소사상(contraction)의 부동점이 유일하게 존재한다는 주요 정리를 증명한다. 이 결과는 1968년에 얻은 완비 거리공간(complete metric space)에서 Kannan의 결과를 일반화한다. 또한 Aksoy 외 2인의 2017년 논문의 결과와 Mitrovic 외 4인의 2019년 논문의 결과를 반증하는 두 가지의 예를 제시한다. 마지막으로, w-완비성(completeness)을 일반화한 약한 w-완비성을 정의하고 특별한 경우로 앞의 주요 정리의 역이 성립함을 보인다.